Selected Topics in Algebraic Geometry: Notes on Xie Junyi's classes

BAO without will of writing papers

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Abstract

Try to take notes by LaTeX. And these are the notes on Prof. Xie Junyi's classes.

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2 Note on 20241003

2.1 Kobayashi hyperbolic

THEOREM 2.1 (Barth). *If X is Kobayashi hyperbolic, then d_X induces the standard topology on X.*

Sketch of the proof. For $p \in X$, define

$$\begin{aligned} \varphi_p \colon X &\to \mathbb{R}_{\geq 0} \\ x &\mapsto d_X(p, x), \end{aligned}$$

and one can prove that φ_p is continuous. ■

THEOREM 2.2. *Let $\pi\colon \widetilde{X} \to X$ be a covering space of X. Then $\forall p, q \in X$, $p' \in \pi^{-1}(p)$,*

$$d_X(p, q) = \inf_{\pi(q')=q} d_{X'}(p', q').$$

COROLLARY 2.3. *$\pi\colon \widetilde{X} \to X$ covering space. Then $\widetilde{X}$ is Kobayashi hyperbolic if and only if X is Kobayashi hyperbolic.*

EXAMPLE 2.4. If X is a quotient space of a bounded domain D in $\mathbb{C}^n$, then X is Kobayashi hyperbolic.

2.2 Brody's criterion

Suppose X is Kobayashi hyperbolic. Then any holomorphic $f\colon \mathbb{C} \to X$ is constant.

Indeed, for any $p, q \in \mathbb{C}$, DDP $\Rightarrow d_X(f(p), f(q)) \le d_{\mathbb{C}}(p, q) = 0 \Rightarrow f(p) = f(q)$.

THEOREM 2.5 (Brody). *If X is compact and not Kobayashi hyperbolic, then there exists an entire curve $f\colon \mathbb{C} \to X$.*

Sketch of the proof.

LEMMA 2.6. *There exists a distance function δ on X s.t.*

1. *δ induces the standard topology on X;*
2. *for every $x \in X$, there exists a hyperbolic neighborhood $x \in U_x$ s.t. $\delta \le d_{U_x}$ on U_x.*

Claim: Every holomorphic $f\colon \overline{\mathbb{D}} \to X$ satisfies the Lipschitz condition

$$\delta(f(a), f(b)) \le C_f \rho(a, b),\ \forall a, b \in \mathbb{D},$$

where C_f is the Lipschitz constant.

Suppose X is not Kobayashi hyperbolic. Then there exists a sequence of holomorphic maps $f_1, f_2, \ldots \in \mathrm{Hom}(\overline{\mathbb{D}}, X)$ s.t. $C_{f_i} \to +\infty$. ■

3 Note of the class on 20241008

3.1 Compact hyperbolic set

DEFINITION 3.1. For $F \subset X$ compact subset, we say F hyperbolic, if there exists no entire curve in F.

Brody's Lemma implies Openness:
If $F \subset X$ hyperbolic, then $\exists F \subset U \subset X$ with U open s.t. U is (Brody) hyperbolic.

THEOREM 3.2 (Green's Theorem). *Let L be a collection of 5 lines in general position in $\mathbb{P}^2(\mathbb{C})$. Then $\mathbb{P}^2(\mathbb{C}) \setminus L$ is hyperbolic.*

Proof. 5 lines defined by $l_0, \ldots, l_4 \in H^0(\mathcal{O}(1))$.

Embedd

$$\begin{aligned} \tau\colon \mathbb{P}^2 &\to \mathbb{P}^4, \\ z &\mapsto [l_0(z) : \cdots : l_4(z)], \end{aligned}$$

$P := \tau(\mathbb{P}^2)$. Then $\tau(\mathbb{P}^2 \setminus L) = P \setminus H =: P^*$. H coordinate hyperplanes.

Consider

$$\begin{aligned} F_n\colon \mathbb{P}^4 &\to \mathbb{P}^4,\ n \ge 1, \\ [z_0 : \cdots : z_4] &\mapsto [z_0^n : \cdots : z_4^n]. \end{aligned}$$

$F_n : \mathbb{P}^4 \setminus H$ to itself is an étale cover. Hyperbolicity of P^* is equivalent to the hyperbolicity of $F_n^{-1}(P^*)$. $F_n^{-1}(P^*) \to P^*$ is étale cover.

Idea: Show $F_n^{-1}(P^*)$ converges to some place which is hyperbolic as $n \to \infty$. Then apply openness.

General position of 5 lines $\Rightarrow P \cap \{z_i = z_j = z_k\} = \varnothing$, $\forall i, j, k$ distinct.

$$\Rightarrow P \subset X_\epsilon := \bigcap_{i,j,k \text{ distinct}} \Big\{ \max(|z_i|, |z_j|, |z_k|) \geq \epsilon \|z\| \Big\}$$

for some $\epsilon > 0$, where $\|z\|$ is the sup norm.

Observe that $F_n^{-1}(X_\epsilon) = X_{\epsilon^{1/n}}$, and $\bigcap_{n \geq 1} X_{\epsilon^{1/n}} = X_1$.

On X_1, $\|z\|$ is reached on one out of three arbitrary components of z.

Only need to shoe X_1 hyperbolic. If

$$\begin{aligned} f :=& [f_0 : \cdots : f_4], \\ & \mathbb{C} \to \mathbb{P}^4(\mathbb{C}) \end{aligned}$$

holomorphic with $f(\mathbb{C}) \subset X$. May assume

$$\{|f_0| = |f_1| = |f_2|\} \supset \text{ non empty open set.}$$

$\Rightarrow |f_0| = |f_1| = |f_2|$ on $\mathbb{C}$. But $\|f\| = \max(|f_0| = |f_1| = |f_2|) \Rightarrow |f_0| \geq |f_i|$, $\forall i$ on $\mathbb{C}$.

Write

$$\begin{aligned} f :& \mathbb{C} \to \{z_0 \neq 0\} \simeq \mathbb{C}^4, \\ & z \mapsto \left[1 : \frac{f_1}{f_0} : \cdots : \frac{f_4}{f_0} \right] \end{aligned}$$

which is bounded. Liouville's theorem shows f is constant, and thus X_1 is hyperbolic. ■

Remark 3.3. 4 *is not enough.*

3.2 Construct an example of hyperbolic sextic in $\mathbb{P}^3(\mathbb{C})$

Zaidenberg: Deformation method.

Brody's Lemma $\Rightarrow$ Sequences of entire curves: X = compact complex space. Then any sequence of entire curves in X can be made converging toward an entire curve after reparametrization and extraction.

Lemma 3.4 (Stability of intersections). *X complex manifold. $H \subset X$ analytic hypersurface. Suppose $f_n : \mathbb{C} \to X$ entire curves in X converge to an entire curve f (after ...). If $f(\mathbb{C}) \not\subset H$, then*

$$f(\mathbb{C}) \cap H \subset \lim f_n(\mathbb{C}) \cap H.$$

Proof. Assume $z \in f^{-1}(H)$. Near $f(z)$, $H = \{h = 0\}$. $f^* h \not\equiv 0$ near z.

By Rouché's Theorem, $h \circ f_n$ has a zero in any small neighborhood of z for $n \gg 0$. ■

Theorem 3.5. *Let $P_i := (p_i = 0)$ be 6 hyperplanes in $\mathbb{P}^3(\mathbb{C})$ in general position. Then $\exists S = (s = 0)$ of deg 6 s.t.*

$$\Sigma_\epsilon := \Big(\prod p_i = \epsilon s \Big)$$

is hyperbolic for $\epsilon \ll 1$ (S general position for P_i, $i = 1, \ldots, 6$).

Proof. **First Step:** reduce the problem to the hyperbolicity of complements.

If Σ_{ϵ_n} is not hyperbolic for $\epsilon_n \to 0$, we have entire curves $f_n : \mathbb{C} \to \Sigma_{\epsilon_n}$. Using Brody's Lemma, we get at the limit an entire curve $f : \mathbb{C} \to \Sigma_0 = \cup P_i$. Thus, $f(\mathbb{C}) \subset$ one of P_i.

A crucial observation: If $f(\mathbb{C}) \not\subset P_i$, then $f(\mathbb{C}) \cap P_i \subset S$.

Indeed, Stability if intersections $\Rightarrow$

$$f(\mathbb{C}) \cap P_i \subset \lim f_n(\mathbb{C}) \cap P_i \subset \lim \Sigma_{\epsilon_n} \cap P_i = S \cap P_i.$$

As a consequence, $f(\mathbb{C}) \not\subset P_i \cap P_j$, $(i \neq j)$. Otherwise, for example, if $f(\mathbb{C}) \subset P_1 \cap P_2$ (line), then $f(\mathbb{C}) \not\subset P_3, \dots, P_6$. S general position $\Rightarrow P_1 \cap P_2 \cap P_i \notin S$, $\forall i = 3, \dots, 6$. Observation $\Rightarrow$ $P_1 \cap P_2 \cap P_i \notin f(\mathbb{C}) \Rightarrow f(\mathbb{C}) \subset P_1 \cap P_2 \setminus \{P_1 \cap P_2 \cap P_i, i = 3, \dots, 6\}$, which contains 4 pts. It is impossible by Picard's Theorem.

Thus, $f(\mathbb{C}) \subset$ one plane and avoiding the others except at pts of S, $\Rightarrow$

$$f(\mathbb{C}) \subset P_i \setminus \left(\bigcup_{j \neq 1} P_i \setminus S \right),$$

which is P_i minus 5 lines then add a few pts.

Second Step: construct the sextic S.

Idea: to create S we proceed by deformation in order to remove these pts on more and more double lines.

By Green: $P_i \setminus \left(\bigcup_{j \neq i} P_j \right)$ hyperbolic.

Lemma 3.6. *Let Δ_k be a collection of k double lines (i.e. $P_i \cap P_j$, $i \neq j$), $D = P_{i_1} \cap P_{i_2}$ an extra one, and $\Delta_{k+1} = \Delta_k \cup D$. Assume $S_k = (s_k = 0)$ already constructed s.t.*

$$P_i \setminus \left(\bigcup_{j \neq i} P_j \setminus (\Delta_k \cap S_k) \right)$$

are hyperbolic. Then so are

$$P_i \setminus \left(\bigcup_{j \neq i} P_j \setminus (\Delta_{k+1} \cap S_{k+1}) \right),$$

where

$$S_{k+1} = \left(p_{i_3} p_{i_4} p_{i_5}^2 p_{i_6}^2 = \epsilon_k s_k \right)$$

for a small $\epsilon_k \neq 0$.

Note that S_{k+1} is still general position if S_k is. Moreover, $\Delta_k \cap S_{k+1} = \Delta_k \cap S_k$.

Lem+induction $\Rightarrow$ Thm.

Proof of the Lem:

Take $D = P_1 \cap P_2$ for simplicity. If we can not find such an ϵ_k, we have of sextic $S_{k+1,n}$ converging to $P_3 \cup P_4 \cup P_5 \cup P_6$ and entire curves $f_n(\mathbb{C})$ contained in one of the corresponding complements, say

$$P_1 \setminus \left(\bigcup_{j \geq 2} P_j \setminus \left((\Delta_k \cap S_k) \cup (D \cap S_{k+1,n}) \right) \right).$$

We get at the limit an entire curve $f(\mathbb{C})$ in P_1. As before, it can not be contained in a double line.
Stability of intersections for $j \geq 2$ $\Rightarrow$

$$\begin{aligned} f(\mathbb{C}) \cap P_j &\subset \lim f_n(\mathbb{C}) \cap P_j \\ &\subset (\Delta_k \cap S_{k+1}) \cup \lim(D \cap S_{k+1,n}). \end{aligned}$$

If $j \geq 3$, $P_j \cap D \cap S_{k+1,n} = \varnothing \Rightarrow f(\mathbb{C}) \cap P_j \subset \Delta_k \cap S_k$.
As $S_{k+1,n} \xrightarrow{n\to\infty} P_3 \cup \cdots \cup P_6$,

$$\lim D \cap S_{k+1,n} \subset \{\text{triple pts of } D\} = D \cap \Big(\bigcup_{j\geq 3} P_j\Big).$$

Then

$$f(\mathbb{C}) \cap P_2 \subset \Big(f(\mathbb{C}) \cap (\Delta_k \cap S_k) \cup (f(\mathbb{C}) \cap \lim(D \cap S_{k+1,n}))\Big).$$

Since

$$f(\mathbb{C}) \cap (\Delta_k \cap S_k) \supset f(\mathbb{C}) \cap \Big(\bigcup_{j\geq 3} P_j\Big),$$

we get

$$f(\mathbb{C}) \subset P_1 \setminus \Big(\bigcup_{j\geq 2} P_j \setminus (\Delta_k \cup S_k)\Big).$$

Contradiction. ■

4 Note on 20241015

A drawback of Brody Lemma is lack of information about the location of the entire curve it produces.

Example 4.1 (Winkelmann). Let $A = \mathbb{C}^2(\mathbb{Z} \oplus i\mathbb{Z})^2$ and $\pi\colon \mathrm{Bl}_p(A) \to A$, where p is a point. Take a dense injective line L in A (e.g. $L = (z_1 = \lambda z_2), \lambda \in \mathbb{R} \setminus \mathbb{Q}$). Consider

$$\begin{aligned} f_n\colon \mathbb{D} &\to L \subset A \\ z &\mapsto (nz, \lambda nz), \end{aligned}$$

$\tilde{f}_n\colon \mathbb{D} \to A$ the strict transform of f_n.
Then the Brody curve $\tilde{f}_n(\mathbb{C})$ produced by $\tilde{f}_n$ lands in $E = \pi^{-1}(p)$.

Proof. If $\tilde{f}_n(\mathbb{C}) \not\subset E$, then $f \coloneqq \pi \circ \tilde{f}\colon \mathbb{C} \to A$ is a Brody curve in A. Liouville Thm $\Rightarrow$ f linear, parallel to L by construction. Thus, $f(\mathbb{C})$ is a dense injective line L', and $\pi^{-1}|_{L'} = \tilde{f} \circ f^{-1}$ has bounded derivative.

On the other hand, consider an open cone of vertex p transversal to the direction of L'. By density of L', it cuts out a sequence of smaller and smaller discs d_n of L' converging to p. But $\pi^{-1}(d_n)$ converges to a non-constant disc in E. So $\mathrm{d}(\pi^{-1}|_{L'})$ is not bounded. Contradiction. ■

A more precise version of Brody Lemma due to Julien Duval.

4.1 Ahlfors current

DEFINITION 4.2. Let X be a complex Kähler manifold. $\dim X = d$. Let ω be a Kähler form on X. A current (of bidimension $(1,1)$) in X is a continuous linear form on the space of $(1,1)-$forms (smooth) equipped with the $C^\infty-$topology. It is positive if it is non-negative on positive $(1,1)-$forms; closed if it vanishes on exact forms.

EXAMPLE 4.3. (1) $\forall$ curve $C \subset X$,

$$[C] \colon \alpha \mapsto \int_C \alpha.$$

(2) ω^{d-1}.

Think: a positive $(1,1)$ current is "a form of bidegree $(d-1, d-1)$ with measure coefficients".

DEFINITION 4.4. For T positive current, mass of T $:= \langle T, \omega \rangle = \int_X T \wedge \omega$.

Fact: $\{\text{positive current } T \text{ of mass} \leq 1\}$ is compact.

Let $f_n \colon \mathbb{D} \to X$ be a sequence of holomorphic discs smooth up to $\overline{\mathbb{D}}$. Let

$$a_n := \text{area}(f_n(\mathbb{D})) := \int_{\mathbb{D}} f_n^* \omega.$$
$$l_n := \text{length}(f_n(\partial\mathbb{D})) = \text{length of } \partial\mathbb{D} \text{ for the metric induced by } f^*\omega.$$

Assumption:

(A) $\frac{l_n}{a_n} \to 0$, i.e. boundaries of discs are asymptotically negligible.

$$(\text{mass } 1)\ \frac{[f_n(\mathbb{D})]}{a_n} \xrightarrow{\text{taking subsequence}} T \text{ (positive current)},$$

(A)$\Rightarrow$ T is closed (called Ahlfors current).

5 Note on 20241017

5.1 Duval's theorem

THEOREM 5.1 (Duval). *Let T be an Ahlfors current in X and $K \subset X$ a compact subset charged by T. Then there exists an entire curve passing through K.*

Charged by T:

$$\int_K T \wedge \omega > 0.$$

Passing through K:

$$\text{Leb}(f^{-1}(K)) > 0,\ f \colon \mathbb{C} \to X.$$

COROLLARY 5.2. *If an Ahlfors current of X charges an analytic subset Y, then Y contains an entire curve.*

COROLLARY 5.3. *X is hyperbolic if and only if its holomorphic discs satisfy a linear isoperimetric inequality, i.e.*

$$\text{area}(f(\mathbb{D})) \leq C\, \text{length}(f(\partial\mathbb{D})).$$

$\Rightarrow$ A question of Gromov. Another proof given by Kleiner.

Proof. If there exists $f_n\colon \mathbb{D} \to X$, s.t.

$$\frac{\text{length}(f_n(\partial\mathbb{D}))}{\text{area}(f_n(\mathbb{D}))} \to 0,$$

then we can get an Ahlfors current T. Pick $K = X$. Theorem 5.1 implies there exists an entire curve on X. The converse is implied by Ahlfors Lemma. ■

LEMMA 5.4 (Ahlfors Lemma). *Let $f\colon \mathbb{C} \to X$ be an entire curve. Then there exists $r_n \to +\infty$ s.t. the corresponding sequence of holomorphic discs $f_n\colon \mathbb{D} \to X$, $z \mapsto f(r_n z)$ satisfies (A).*

Proof. $\mathbb{D}_r := \{z\colon |z| < r\}$, $l(r) := \text{length}(f(\partial\mathbb{D}_r))$, $a(r) := \text{area}((\mathbb{D}_r))$.

Write $f^*\omega = \frac{i}{2}\lambda^2 dz \wedge d\bar{z}$, $\lambda > 0$. Then

$$l(r) = \int_0^{2\pi} \lambda(r(\theta))r d\theta,$$

and

$$a'(r) = \int_0^{2\pi} \lambda^2(r(\theta))r d\theta.$$

By Cauchy-Schwarz,

$$l(r) \leq 2\pi r a'(r)$$

$\Rightarrow$

$$\int_1^{+\infty} \frac{l^2(r)}{a^2(r)}\frac{dr}{2\pi r} \leq \int_1^{+\infty} \frac{a'(r)}{a^2(r)} dr = \int_1^{+\infty} d\frac{1}{a(r)} \leq \frac{1}{a(1)} < +\infty$$

$\Rightarrow$

$$\exists r_n \to +\infty \text{ s.t. } \frac{l(r_n)}{a(r_n)} \to 0.$$

■

COROLLARY 5.5. *Let $f\colon \mathbb{C} \to X$ be an entire curve s.t.* $\text{area}(f(\mathbb{C})) < +\infty$. *Then f extends to a holomorphic map from $\mathbb{P}^1$ to X.*

Proof. By Riemann removable singularity theorem, it is enough to extend f continuously at ∞.

Bounded area of $f(\mathbb{C})$ and Ahlfors Lemma show there exists a sequence $r_n \to +\infty$ such that $\text{length}(f_n(\partial\mathbb{D})) \to 0$. Taking subsequence we may assume that $f(\partial\mathbb{D}_{r_n}) \to p$ a point.

It suffices to show that $A_n = f(\mathbb{D}_{r_n} \setminus \overline{\mathbb{D}_{r_{n-1}}})$ also converges to p. Note that $\text{area}(A_n) \to 0$. If a point $a_n \in A_n$ remains outside $B(p, \epsilon)$, then for $n \gg 0$, $A_n \cap B(q_n, \epsilon/2)$ is a proper holomorphic curve in $B(q_n, \epsilon)$ passing through q_n.

By Lelong inequality, we have

$$\text{area}(A_n \cap B(q_n, \epsilon/2)) > \frac{C\epsilon^2}{4},$$

C depending on (X, ω). Contradiction. ■

THEOREM 5.6 (Lelong inequality). *Let $C \subset \mathbb{C}^n$ (standard metric) be a closed holomorphic curve properly in $B(0, \epsilon)$. Then*

$$\text{area}(C \cap B(0, \epsilon)) \geq \pi\epsilon^2.$$

Proof. For $o < r < \epsilon$, $C_r := C \cap \overline{B(0,r)}$ and $a(r) := \mathrm{area}(C_r)$. Note that $\partial C_r = C \cap \partial B(0,r)$.

Claim: $\dfrac{a(r)}{\pi r^2}$ is decreasing.

Proof of Claim: For $r > s > 0$, we have

$$\begin{aligned}
\frac{a(r)}{r^2} - \frac{a(s)}{s^2} =& \frac{1}{r^2}\int_{C_r} dd^c\|z\|^2 - \frac{1}{s^2}\int_{C_s} dd^c\|z\|^2 \\
=& \frac{1}{r^2}\int_{\partial C_r} d^c\|z\|^2 - \frac{1}{s^2}\int_{\partial C_s} d^c\|z\|^2 \\
=& \int_{\partial C_r} \frac{1}{\|z\|^2} d^c\|z\|^2 - \int_{\partial C_s} \frac{1}{\|z\|^2} d^c\|z\|^2 \\
=& \int_{\partial C_r} d^c \log\|z\|^2 - \int_{\partial C_s} d^c \log\|z\|^2 \\
=& \int_{\partial(C_r \setminus C_r)} d^c \log\|z\|^2 \\
=& \int_{C_r \setminus C_s} dd^c \log\|z\|^2 \geq 0.
\end{aligned}$$

So $\forall r > 0$,

$$\frac{a(r)}{\pi r^2} \geq \lim_{s \to 0} \frac{a(s)}{\pi s^2} = \nu([C], 0),$$

which is the Lelong number of $[C]$. Pick $r = \epsilon$, only need to show $\nu([C], 0) \geq 1$.

If C is smooth at 0, then $\nu([C], 0) \geq 1$ holds. For general case, $\forall x \in B(0, \epsilon)$, $a(x, r)$ ($\nu([C], x)$) at x is similar to $a(r)$ ($\nu([C], 0)$) at 0. For any fixed $r > 0$, easy to check that $a(x, r)$ is upper-semicontinuous for x, i.e. for $\forall c$, $\{a(x, r) \geq c\}$ is closed. When $r \to 0$, $\dfrac{a(x, r)}{\pi r^2} \to \nu([C], x)$. Thus, $\nu([C], x)$ is u.s.c. As C^{smooth} is dense in C, $\nu([C], x) \geq 1$ for all $x \in C$. ■

THEOREM 5.7 (Gromov, ...). *Let $g_n : \mathbb{D} \to X$ be holomorphic maps. Assume* $\mathrm{area}(g_n(\mathbb{D}))$ *is bounded. Then after taking subsequence, we have*

(i) g_n locally uniformly converges to g on $\mathbb{D} \setminus E$, where E is a finite set (called points d'explosion);

(ii) g extends to a holomorphic map $g : \mathbb{D} \to X$;

(iii) for $e \in E$, $\exists$ a sequence of discs $d_n \subset \mathbb{D}$, $d_n \to e$, s.t. $g_n(d_n)$ converges in the sense of Hausdorff and in area to a finite union of rational curves (called bubbles).

Proof. Left to the readers. ■

For g_n above, we say g_n tends to g **in the sense of Gromov**.

THEOREM 5.8 (Variant 1). *Let d_n be a sequence of discs in $\mathbb{D}$ converges to 0, and $g_n : \mathbb{D} \setminus d_n \to X$ a sequence of holomorphic maps, smooth to ∂d_n. Suppose that* $\mathrm{area}(g_n(\mathbb{D} - d_n))$ *is bounded and the length of $g_n(\partial d_n) \to 0$. Then after taking subsequence, g_n tends to a holomorphic map $g : \mathbb{D} \to X$ in the sense of Gromov.*

In particular, if $\mathbb{D}_1 \Subset \mathbb{D}$ is a disc, then $g_n(\mathbb{D}_1 - d_n)$ converges in the sense of Hausdorff and in area to the union of $g(\mathbb{D}_1)$ and a finite union of rational curves.

Proof. Left to the readers. ■

THEOREM 5.9 (Variant 2). *Let $\mathbb{D}^+$ be a half disc and $g_n\colon \mathbb{D}^+ \to X$ a sequence of holomorphic maps, smooth to $(-1,1)$. Suppose that* $\operatorname{area}(g_n(\mathbb{D}^+))$ *is bounded and* $\operatorname{length}(g_n(-1,1)) \to 0$. *Then after taking subsequence, g_n converges to a point in the sense of Gromov.*

In particular, for $\mathbb{D}_1^+$ a little bit smaller than $\mathbb{D}^+$, then $g_n(\mathbb{D}_1^+)$ converges in the sense of Hausdorff and in area to a finite union of rational curves.

Key idea: reparametrization.

6 Note on 20241022

6.1 Proof of Variant 2

Proof of Variant 2. Taking subsequence, we may assume $g_n((-1,1)) \to p$ a point.

We may assume any rational curve in X has area ≥ 1. Set $B \coloneqq \sup \operatorname{area}(g_n(\mathbb{D}^+))$.

Fix $\epsilon > 0$ small w.r.t. the diameter of a chart U of X at p. Assume the metric on U is the standard metric.

We call $x \in (-1,1)$ a blowup point if any half disc d^+ centered at x, we have $\limsup \operatorname{area}(g_n(d^+)) > \epsilon^2$.

Let d^+ be a small half disc centered at a non blowup point.

Claim: $\forall n$, $\exists \beta_n \in (e^{-1}, 1)$, s.t.

$$\operatorname{length}(g_n(\partial \beta_n d^+ \cap (\Im(z) > 0))) \leq 2\epsilon.$$

Proof. Length-area argument

Write

$$g_n^*\omega = \frac{i}{2}\lambda_n^2 dz \wedge d\bar{z}$$

(omit n). $R \coloneqq$radius of d^+.

For $r \in [0, R)$, write

$$L(r) \coloneqq \operatorname{length}(g_n(\partial(\text{half disc with radius } r \text{ centered at the center of } d^+))) = \int_0^\pi \lambda \cdot r d\theta,$$

$$A(r) \coloneqq \operatorname{area}(g_n(\ldots)).$$

Then

$$A'(r) = \int_0^\pi \lambda^2 r d\theta \Rightarrow (L(r))^2 \leq A'(r) \cdot r.$$

Only need to show $\exists r \in (R/e, R)$ s.t. $L(r) \leq 2\epsilon$. Otherwise,

$$4\epsilon^2 \leq (L(r))^2 \leq A'(r) \cdot r.$$

Now

$$\int_{\frac{R}{e}}^R A'(r) dr \leq \int_0^R A'(r) dr = A(R) = \operatorname{area}(g(d^+)) \leq \epsilon^2,$$

but

$$\int_{\frac{R}{e}}^{R} A'(r)dr = \int_{\frac{R}{e}}^{R} A'(r) \cdot r\frac{dr}{r} > 4\epsilon^2 \int_{\frac{R}{e}}^{R} d\log r = 4\epsilon^2,$$

contradiction. ■

Then $g_n(\partial\beta_n d^+) \subset B(p, 100\epsilon) \subset U$.

Lelong inequality $\Rightarrow g_n(\beta_n d^+) \subset U$.

Otherwise, $\exists x \in g_n(\beta_n d^+) \cap B(p, 300\epsilon) \setminus B(p, 200\epsilon) \Rightarrow g_n(\beta_n d^+) \cap B(x, \epsilon)$ properly. Then Lelong inequality $\Rightarrow$

$$\epsilon^2 > \mathrm{area}(g_n(\beta_n d^+)) \geq \mathrm{area}(g_n(\beta_n d^+) \cap B(x, \epsilon)) \geq \pi\epsilon^2,$$

contradiction.

As all $\beta_n \geq 1/e$, after replacing d^+ by $\frac{1}{e}d^+$, we may assume that

$$g_n(d^+) \subset B(p, 100\epsilon) \subset U, \ \forall n.$$

Two constants theorem $\Rightarrow g_n|_{d^+} \to p$ uniformly (taking α to be the line in the half disc).

Claim: After taking subsequence, $g_n|_{\mathbb{D}^+ \setminus (-1,1)} \to p$ in the sense of Gromov.

Proof. After taking subsequence assume $g_n \to g$ Gromov. Pick $N > 10B/\epsilon^2$, $x_1, \ldots, x_N \in (-1, 1)$ distinct points. After taking subsequence, $\exists i \in \{1, \ldots, N\}$ s.t. x_i is non-blowup.

Let d^+ be a half disc at x_i as above. Then

$$g_n|_{d^+} \to p \text{ uniformly } \Rightarrow g|_{d^+ \setminus (-1,1)} = p,$$

$\Rightarrow g = p$. ■

Next we consider blowup points. For example, assume o is a blowup point. Then $\exists a_n \to o$ s.t. $g_n(a_n) \notin B(p, 100\epsilon)$. Otherwise, $\exists d^+$ half disc at o s.t. $g_n(d^+) \subset B(p, 100\epsilon)$, $\forall n$.

Then the above argument shows that $g_n(d^+) \to p$ uniformly $\Rightarrow$ not blowup at o, contradiction.

Let $d_n^+ :=$ half disc centered at $\Re(a_n)$ radius $2\Im(a_n)$.

$r_n \colon \mathbb{D}^+ \to d_n^+$, $z \mapsto 2\Im(a_n)z + \Re(a_n)$.

$h_n := g_n \circ r_n \colon \mathbb{D}^+ \to X$. Then

$$\mathrm{area}(h_n(\mathbb{D}^+)) \leq \mathrm{area}(g_n(\mathbb{D}^+)) \leq B.$$

$$h_n|_{(-1,1)} \to p.$$

Then after taking subsequence, $h_n(\mathbb{D}^+ \setminus (-1, 1)) \to p$ Gromov. But as $r_n^{-1}(a_n) = \frac{i}{2} \in \mathbb{D}^+ \setminus (-1, 1)$ and $h_n(r_n^{-1}(a_n)) \notin B(p, 100\epsilon)$, we get a bubble at $\frac{i}{2}$. Thus, $\mathrm{area}(h_n(\mathbb{D}^+)) \geq 1$.

Let d_n be small discs associated to a_n which converges to the bubbles, and we may assume $\mathrm{length}(\partial d_n) \to 0$. Now "**remove** d_n".

Repeat the previous process with $\mathrm{area}(g_n(\mathbb{D}))$ go done at least 1. Stop in finite steps. ■

THEOREM 6.1 (Two constants theorem). *Let D be a domain in $\mathbb{C}$ bounded by finite Jordan curves, and α an arc $\subset \partial D$. Let $z \in D$, and λ the harmonic measure of α w.r.t. D and z (which is positive). If $h \colon \overline{D} \to \mathbb{C}$ holomorphic, s.t.*

(1) $|h| \leq M$ on D;

(2) $|h(w)| \leq m < M$, $\forall w \in \alpha$,

then $|h(z)| \leq m^\lambda M^{1-\lambda}$.

Proof. Find in complex analysis books. ■

EXAMPLE 6.2. Assume no rational curve in X. Want to construct an entire curve.

Let $f_n(\mathbb{D}(2^n)) \to X$, $n \geq 0$ holomorphic maps. Assume

(1) $\mathrm{area}(f_n(\mathbb{D})) \geq 1$, $\forall n$;

(2) $\forall k \geq 1$, $\exists C_k \geq 1$ s.t. $\mathrm{area}(f_n(\mathbb{D}(2^k))) \leq C_k$.

Then Gromov's theorem $\Rightarrow$ there exists subsequence $f_i^0 \coloneqq f_{n_i}|_{\mathbb{D}} \colon \mathbb{D} \to X$ locally uniformly converges.

There exists subsequence $f_i^1 \coloneqq f_{n_1}^0 : \mathbb{D}(2) \to X$ converges.

Diagonal method gives $f^\infty \colon \mathbb{C} \to X$, and $\mathrm{area}(f^\infty(\mathbb{D})) \geq 1$. It follows that f^∞ is non-constant, and thus f^∞ provides an entire curve.

7 Note on 20241029

Proof of Theorem 5.1. $K \subset X$ compact subset,

$$T \coloneqq \lim_{n\to\infty} \frac{[f_n(\mathbb{D})]}{\mathrm{area}(f_n(\mathbb{D}))}$$

Ahlfors current. Assume $T \wedge \omega(K) > 0$. Set $a_n = \mathrm{area}(f_n(\mathbb{D}))$, $l_n = \mathrm{length}(f_n(\partial\mathbb{D}))$.

(A) $l_n = o(a_n)$.

We may assume

(H) No rational curve passing through K.

Let U_n be open neighborhood of K s.t. $\overline{U_{n+1}} \subset U_n$ and $\bigcap_{n\geq 0} U_n = K$. $T \wedge \omega(K) > 0 \Rightarrow \exists \delta > 0$ s.t. $\mathrm{area}(f_n(\mathbb{D}) \cap U_n) \geq \delta a_n$.

Case: a_n bounded.

Then $l_n \to 0$. Variant 2 (cover disc by half discs) $\Rightarrow$ up to taking subsequence, $f_n \to$ a point in the sense of Gromov $\Rightarrow$ $f_n(\mathbb{D}) \to$ finite union of rational curves in area $\Rightarrow$ $\exists$ rational curve passing through K. ■

8 Lost Notes

There are two days' notes lost, since I did not take the classes that days.

9 Note on 20241105

9.1 Continue to prove Theroem 5.1

Continue the proof. Double the anneaux.

If $a = d - d' \in \mathbf{a}_n$, $2a = d - d_1'$.

Most of $a \in \mathbf{a}_n$ can be double. Only when 2 anneaux with distance$\leq$ 2, their double can meet. So remove about 2/3 anneaux, we get a consistent subfamily $\mathbf{a}_n'$ s.t. $2a$, $a \in \mathbf{a}_n'$ still disjoint.

As in the disc case, we may extract a consistent subfamily of $\mathbf{a}_n'$ to further ask $\mathbf{a}_n'$ has

(1) $\mathrm{area}(f_n(2a \cap \mathbb{D})) \leq C_1$, C_1 is a constant;

(2) $\sup_{a\in\mathbf{a}_n'} \mathrm{length}(f_n(2a \cap \partial\mathbb{D})) \to 0$.

As in the disc case, we will show that for $n \gg 0$, $\forall a_n \in \mathbf{a}_n'$, $a_n \subset \mathbb{D}$.

Otherwise, let $a = d - d' \not\subset \mathbb{D}$. As $\frac{1}{4}d > \frac{1}{16}d > d'$, $\mathbb{D} \not\subset d' \Rightarrow a_n \cap \partial\mathbb{D} \neq \varnothing$.

So $\partial\mathbb{D}$ divides $b_n := d_1 - d$ to 2 topological discs.

There are 2 cases:

(a) Size of d_1, d comparable;

(b) $\dfrac{\text{size}\, d}{\text{size}\, d_1} \to 0$.

Case (a):

Apply Gromov's theorem to 2 half discs and 1 topological disc. Then $f_n(\text{top. disc}) \to$ point in the sense of Gromov.

μ_n mass ≥ 1. Keep to the limit $\Rightarrow \exists$ rational curve passing through K.

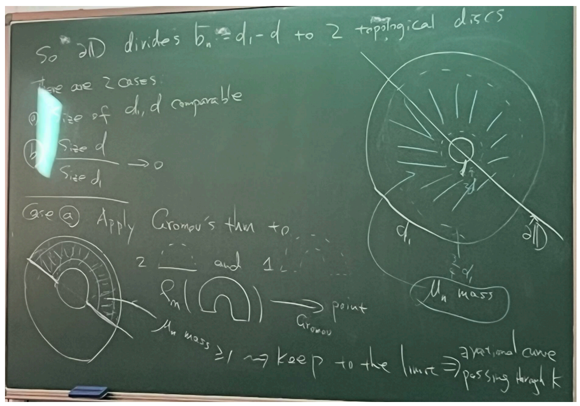

Figure 1: Fig.1

Case (b):

Modulus of $b \to +\infty$.

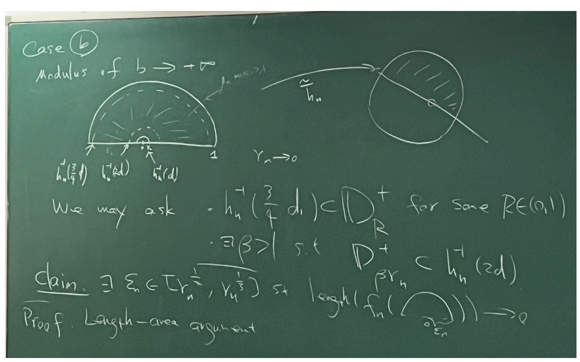

Figure 2: Fig.2

We may ask

$h_n^{-1}(\frac{3}{4}d_1) \subset \mathbb{D}_R^+$ for some $R \in (0,1)$;

$\exists \beta > 1$ s.t. $\mathbb{D}_{\beta r_n}^+ \subset h_n^{-1}(2d)$.

Claim: $\exists \epsilon_n \in [r_n^{1/2}, r_n^{1/3}]$ s.t. length(half disc radius ϵ_n) $\to 0$
$f_n^*\omega = \frac{i}{2}\lambda_n dz \wedge d\bar{z}$,
$l_n(r)$, $A_n'(r)$, $l_n^2(r) \le A_n'(r)\cdot r$. $c_n := \min_{r_n^{1/2}\le r \le r_n^{1/3}}$.

$$M \ge \text{area}(\cdot) = \int_{r_n^{1/2}}^{r_n^{1/3}} A_n'(r)\cdot r\frac{dr}{r} \ge c_n^2 \int \frac{dr}{r} = c_n^2 \times \frac{1}{6}(-\log r_n) \to +\infty.$$

$\Rightarrow c_n \to 0$. Pick ϵ_n s.t. $l_n(\epsilon_n) = c_n$.

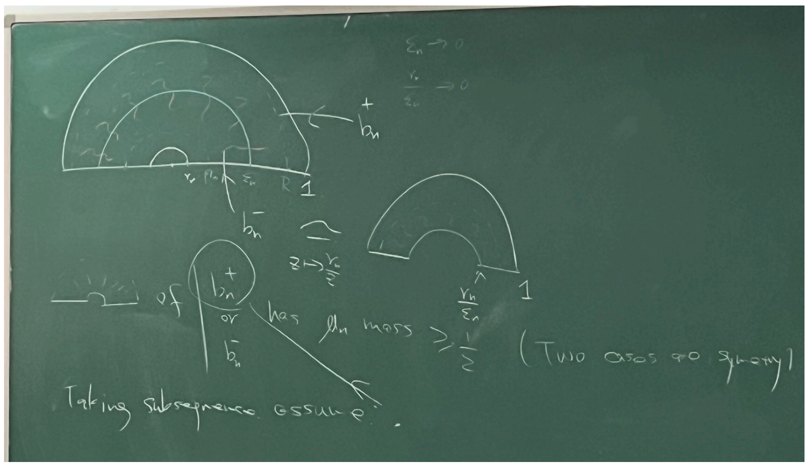

Figure 3: Fig.3

The topological disc with b_n^+ or b_n^- has μ_n mass greater than $\frac{1}{2}$ (two cases are symmetry). Taking subsequence, assume b_n^+.

Variant 2 $\Rightarrow$ $f_n(b_n^+) \to$ a point (Gromov). μ_n mass of b_n^+ in the topological disc keep to the limit. $\Rightarrow \exists$ rational curve passing through K.

Now assume $\forall a \in \mathbf{a}_n'$, $a \subset \mathbb{D}$.

Finish the proof.

This time we can do doubling infinitely many times. We get consistent families $\mathbf{a}_n^k (n \ge k)$ decreasing in k s.t.

(i) if $a \in \mathbf{a}_n^k$, $2^{k-1}a \subset \mathbb{D}$;

(ii) $\text{area}(f_n(2^k a)) \le C_k$.

$\exists$ anneaux a_n, $n \ge 0$, s.t. $2^n a_n \subset \mathbb{D}$, $\text{area}(f_n(a_n) \cap U_n) \ge 1$, and $\text{area}(f_n(2^k a_n)) \le C_k$, $\forall 1 \le k \le n$.

$a_n = d(n) - d'(n)$. (omit n)

Up to taking subsequence, 2 cases:

(i) $\dfrac{\text{size } d_n'}{\text{size } d_n}$ bounded;

(ii) $\dfrac{\text{size } d_n'}{\text{size } d_n} \to 0$.

First treat (i). h_n affine $\mathbb{C} \to \mathbb{C}$ s.t. $\mathbb{D} \to d$. Then $h_n^{-1}(2^k a_n) \to \mathbb{C} \setminus \{\text{a point}\} \simeq \mathbb{C}^*$, for k.

$$f_n \circ h_n \xrightarrow{\text{Gromov}} f_\infty \colon \mathbb{C} \setminus \{\text{a point}\} \to X.$$

μ_n mass ≥ 1 in $f_\infty(\mathbb{C}\backslash\{\text{a point}\})\cup\bigcup_{\text{finite}}(\text{rational curve})$, where $f_\infty(\mathbb{C}\backslash\{\text{a point}\})$ pass through K, <u>Assumption (H)</u> $\Rightarrow \bigcup_{\text{finite}}(\text{rational curve})$ not pass through K.

$\mathbb{C} \xrightarrow{\exp} \mathbb{C}^* \xrightarrow{f_\infty} X$ entire curve pass through K.

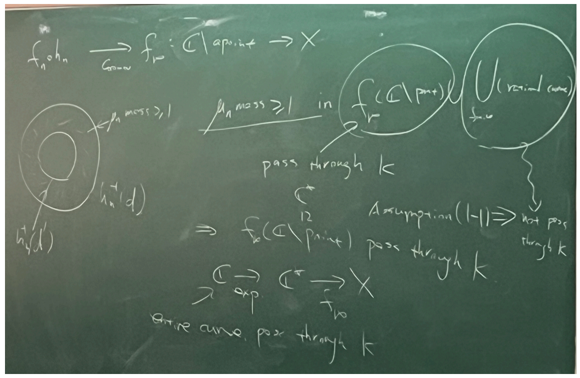

Figure 4: Fig.4

Case (ii):

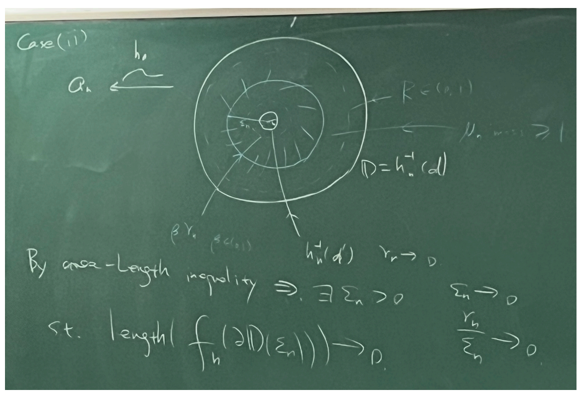

Figure 5: Fig.5

<u>Area-Length inequality</u> $\Rightarrow \exists \epsilon_n > 0$, $\epsilon_n \to 0$, $r_n.\epsilon_n \to 0$, s.t. $\text{length}(f_n(\partial\mathbb{D}(\epsilon_n))) \to 0$.

$$a_n^- \xrightarrow{z \mapsto \frac{r_n}{z}} \ldots.$$

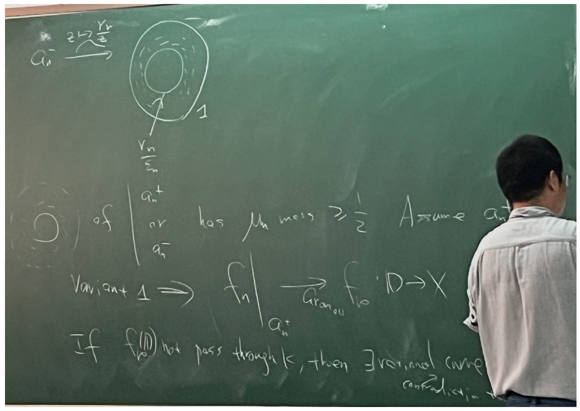

Figure 6: Fig.6

Anneaux of a_n+ or a_n^- has μ_n mass $\geq \frac{1}{2}$. Assume a_n^+. Variant 1 $\Rightarrow$

$$f_n|_{a_n^+} \xrightarrow{\text{Gromov}} f_\infty \colon \mathbb{D} \to X.$$

If $f_\infty(\mathbb{D})$ not pass through K, then $\exists$ rational curve pass through K, contradicting to (H).

So $f_\infty \colon \mathbb{D} \to X$ pass through K. Do Gromov for $2^k(a_n) \cap a_n^+$, where the modulus of $2^k(a_n) \cap a_n^+$ tends to ∞.

$$f_n \xrightarrow{\text{Gromov}} F_\infty \colon \mathbb{C} \to X,$$

where $f_\infty = F_\infty|_{\mathbb{D}}$. Get an entire curve passing through K.

■

9.2 Green-Griffiths Conjecture

X: proj. var. over $\mathbb{C}$.

Definition 9.1. We say X is of general type if $\exists$ birational map $\tilde{X} \to X$ with $\tilde{X}$ smooth s.t. $K_{\tilde{X}}$ is big.

Conjecture 9.2 (Green-Griffiths Conjecture). *If X is proj. var. of general type, then $\forall$ holomorphic curve $f \colon \mathbb{C} \to X$, $f(\mathbb{C})$ is not Zariski dense.*

This conjecture says: If $\forall$ subvariety of X is general type, then X is hyperbolic.

Conjecture 9.3 (Kobayashi-Ochiai). *If X is hyperbolic, then it is of general type.*

Conjecture 9.4 (Conj. 1). *A projective variety X is Kobayashi hyperbolic iff all subvarieties are of general type.*

Example 9.5. (1) Conj. 1 is true when $\dim X = 1$.

(2) When $\dim X = 2$,

$$X \text{ hyperbolic} \xRightarrow{\text{classification of surfaces } + K_3 \text{ are not hyperbolic}} \text{general type}.$$

(3) Kawamata: for X with $X \xrightarrow{\text{finite}} A$ abelian variety, then up to a finite étale cover, $X \simeq B \times X'$ with B abelian var. and X' general type. Apply above for all subvarieties of X $\Rightarrow$ Conj. 1 holds for X.

(4) Campana+Verbitsky: $\forall$ hyperKähler manifold is not hyperbolic.

hyperKähler: simply connected + $H^0(X,\Omega^2) = \mathbb{C}\omega$, where ω nowhere degenerate (we have $K_X = 0$).

Sketch of the proof of Campana+Verbitsky's theorem.

1. Campana: In any twisted family X_t, $t \in \mathbb{P}^1$, $\exists t_0 \in \mathbb{P}^1$ s.t. X_{t_0} is not hyperbolic.

   Deformation+Brody Lemma $\Rightarrow$ $\exists$ entire curve in some fiber X_t.

2. Verbitsky: X hyperKähler manifold.

   $$\mathrm{Teich}_b = \text{“birational Teichmüler space”}$$

   $\Gamma \circlearrowleft \mathrm{Teich}_b$, Γ monodromy group. $\mathrm{Teich}_b/\Gamma =$“moduli space”, which is very non-Hausdorff.

   Should study dynamics of $\Gamma \circlearrowleft \mathrm{Teich}_b$. $x \in \mathrm{Teich}_b$ is ergodic if Γ_x dense in its connected component. Show $\{\text{non ergodic } x \in \mathrm{Teich}_b\}$ is countable. Then there exists $I \subset \mathrm{Teich}_b$ with $I \simeq \mathbb{P}^1$ s.t. all points in I are ergodic. Conclude by Campana+Brody Lemma.

■

10 Note on 20241112

Lang: Special subset.

$$S_p(X) := \overline{\bigcup_{\varphi\colon A \dashrightarrow X} \varphi(A)}^{\mathrm{Zar}},$$

where A is abelian variety, and φ is non-compact rational map.

Conjecture 10.1 (Lang conj.). *X general type iff $S_p(X) \neq X$.*

Conj. 1+Lang conj. $\Rightarrow$

$$\underline{\text{Conj. 2}}\colon X \text{ hyperbolic } \Leftrightarrow S_p(X) = \varnothing.$$

Conjecture 10.2 (Green-Griffiths-Lang conj.). *$\forall$ entire curve $f\colon \mathbb{C} \to X$, we have $f(\mathbb{C}) \subset S_p(X)$.*

Implied by GG+L: GG $\Rightarrow$ $Z := \overline{f(\mathbb{C})}^{\mathrm{Zar}}$ not general type $\Rightarrow$ $S_p(Z) = Z$ by Lang, $\Rightarrow$ $f(\mathbb{C}) \subset S_p(X)$.

10.1 Known cases for GGL

- McQuillan (1998, Publ. IHES): X sm. proj. surface with $c_1^2(X) > c_2(X)$ (i.e. Ω_X big).
- Yamanoi (André Bloch, 1926): X with $X \to A$ finite map to an abelian variety.
- Kobayashi conj.: For $\forall N \geq 1$, $\exists D(N) \geq 1$ s.t. a hypersurface of $\mathbb{P}^{N+1}$ of degree $d \geq D(N)$, X is hyperbolic.

10.1.1 History

Siu: Survey [Hyperbolicity in complex geometry, 2004], Claim proof + strategy (brief). Preprint for the details in arXiv 2012, published 2015.

[Hyperbolicity of generic high-degree hypersurfaces in complex projective space, 2015, Invent. Math.]

(In principle, effective upper bound for $D(N)$)

$\Rightarrow$ Non degeneracy (Piverio-Mevker-Pousseau, 2010): $D_{\text{non-deg}}(N) \le 2^{n^5}$.

Brotbek 2017: [On the hyperbolicity of general hypersurfaces, Publ. IHES] (not effective. effective by Ya Deng).

Eric-Yang (2017, 12 pages)

$$D(N) \le D_{\text{non-deg}}(2N-1).$$

World Record: Bérci-Kirwan: [Invent. Math. 2024] $D(N) \le 16(2n-1)^3(10n-1)$, (before $\ge e^{O(n \log n)}$).

Siu + Demailly + Non-reductive GIT.

Cadorel: Without Non-reductive GIT, $D(N) \le \frac{153}{4}(2n-1)^5$?

10.2 McQuillan's theorem

THEOREM 10.3 (McQuillan). *X sm. proj. surface, $c_1(X)^2 > c_2(X)$. Then GGL holds for X, i.e. $\forall$ entire curve $f\colon \mathbb{C} \to X$, $f(\mathbb{C}) \subset S_p(X)$.*

May assume X general type. Otherwise, $S_p(X) = X$ (classification of surfaces).

LEMMA 10.4 (Bogomolov). *X sm. proj. surface of general type. If $c_1(X)^2 > c_2(X)$, then Ω_X big, i.e. $\mathcal{O}_{\mathbb{P}(T_X)}(1)$ big.*

Proof. Write $\pi\colon \mathbb{P}(T_X) \to X$. Set $L \coloneqq \mathcal{O}_{\mathbb{P}(T_X)}(1)$. Then

$$(L^2) + \pi^* c_1(X) \cdot L + \pi^* c_2(X) = 0.$$

Intersecting with L,

$$\begin{aligned}(L^3) =& -\pi^* c_1(X) \cdot L^2 - \pi^* c_2(X) \cdot L \\ =& \pi^* c_1(X)^2 \cdot L + \pi^*(c_1(X) \cdot c_2(X)) - \pi^* c_2(X) \cdot L \\ =& c_1(X)^2 - c_2(X) > 0.\end{aligned}$$

By Riemann-Roch, we have

$$h^0(nL) + h^2(nL) \ge \chi(nL) = \frac{(L^3)}{6} n^3 + O(n^2).$$

L is $\pi-$ample $\Rightarrow R^i\pi_*(nL) = 0$, $i = 1, 2$, $n > 0$. Then

$$H^2(\mathbb{P}(T_X), nL) = H^2(X, \pi_* nL) = H^2(X, S^n \Omega_X).$$

By the non-degenerate pairing

$$\Omega_X \oplus \Omega_X \to \Lambda^2 \Omega_X = K_X \Rightarrow T_X = \Omega_X \otimes K_X^{-1}.$$

We have

$$H^2(X, S^n\Omega_X) = H^0(X, K_X \otimes S^n T_X) = H^0(X, K_X^{1-n} \otimes S^n\Omega_X).$$

X general type $\Rightarrow K_X^{n-1}$ effective for $n \gg 0$. Thus,

$$h^2(nL) = h^0(K_X^{1-n} \otimes S^n\Omega_X) \leq h^0(X, S^n\Omega_X) = h^0(nL).$$

We now get

$$h^0(nL) \geq \frac{(L^3)}{12} n^3 + O(n^2)$$

with $(L^3) > 0$, which implies that L is big. ■

THEOREM 10.5 (Bogomolov). *X sm. proj. surface of general type. If $c_1(X)^2 > c_2(X)$, then there exists Zariski closed $Z \subsetneq X$ and $\epsilon > 0$ s.t. for any curve $C \subset X$ not in Z, we have*

$$\chi(\tilde{C}) = 2g(\tilde{C}) - 2 \geq \epsilon(C \cdot A),$$

where A is a fixed ample line bundle, and $\tilde{C}$ denotes the normalization of C.

$\Rightarrow$ at most finitely many curves $C \subset X$ with $g(\tilde{C}) \leq 1$.

Idea: Positivity of Ω_X + Foliation on surface.

11 Note on 20241114

11.1 Foliations

DEFINITION 11.1 (Foliation). X smooth surface, a foliation $\mathcal{F}$ is defined by rank 1 locally free subsheaf $N_{\mathcal{F}}^{\vee}$ of Ω_X such that $\Omega_X / N_{\mathcal{F}}^{\vee}$ is torsion free.

In other words, there exists a Zariski open cover U_i of X, $\omega_i \in \Omega_X(U_i) \setminus \{0\}$ s.t. on $U_i \cap U_j$,

$$\omega_i = g_{ij}\omega_j, \; g_{ij} \in \mathcal{O}^*(U_i \cap U_j).$$

$$N_{\mathcal{F}}^{\vee}\Big|_{U_i} = \mathcal{O}_X \cdot \omega_i, \; \mathrm{Sing}(\mathcal{F}|_X) = \bigcup_i \mathrm{Zero}(\omega_i).$$

THEOREM 11.2 (Jouanolou). *X variety, $\mathcal{F}$ foliation codim 1. If $\mathcal{F}$ has infinitely many algebraic leaves, then there exists a first integration i.e. there exists $h\colon X \dashrightarrow \mathbb{P}^1$ non-constant rational function, s.t. for any $x \in X \setminus I(h)$, $\mathcal{F}$ is defined by the fibers locally.*

Proof. May assume X smooth. Define

$$\mathrm{Div}_{\mathcal{F}} \coloneqq \bigoplus_{D = \text{ alg. leaves}} \mathbb{Z}D.$$

Consider exact sequence on X^{an},

$$0 \to \Omega_f \to \mathcal{M}_f \to Q \to 0,$$

where Ω_f is the sheaf of closed holomorphic $1-$forms, and $\mathcal{M}_f$ is the sheaf of closed meromorphic $1-$forms, thus giving

$$0 \to H^0(\Omega_f) \to H^0(\mathcal{M}_f) \to H^0(Q) \xrightarrow{\Phi} H^1(\Omega_f).$$

Claim: $\dim_{\mathbb{C}} H^1(\Omega_f) < \infty$.

Consider

$$0 \to \underline{\mathbb{C}} \to \mathcal{O}_X \xrightarrow{d} \Omega_f \to 0$$

$\Rightarrow$

$$H^1(\mathcal{O}_X) \to H^1(\Omega_f) \to H^2(\underline{\mathbb{C}}),$$

where $H^1(\mathcal{O}_X), H^2(\underline{\mathbb{C}})$ are finite dimensional.

Define a morphism

$$\tau\colon \operatorname{Div}_{\mathcal{F}} \to H^0(Q),$$

$$(U, f_i \in \mathcal{O}(U_i)) \mapsto \left(U_i, \left[\frac{df_i}{f_i}\right]\right),$$

where $[\frac{df_i}{f_i}] \in \mathcal{M}_f(U_i)/\Omega_f(U_i)$ (well-defined as for $g \in \mathcal{O}^*(U)$, $\frac{dg}{g} \in \Omega_f(U)$).

For $\sum \lambda_i D_i \in \operatorname{Div}_{\mathcal{F}}$, locally at a point $x \in D_i = (f_i = 0)$, $\tau(\lambda_i D_i) = [\eta]$. Then $\lambda_i =$ unique number in $\mathbb{Z}$ s.t. $\eta - \lambda_i \frac{df_i}{f_i}$ holomorphic. This does not depend on the choice of η.

Thus, τ is injective.

$V_f := \tau(\operatorname{Div}_{\mathcal{F}}) \subset H^0(Q)$. Note that $\Phi\colon H^0(Q) \to H^1(\Omega_f)$, where $H^1(\Omega_f)$ is finite dimensional. Set

$$V_f^0 := V_f \cap \ker(\Phi),$$

where $\ker\Phi \simeq H^0(\mathcal{M}_f)/H^0(\Omega_f)$. Then $\operatorname{codim} V_{\mathcal{F}}^0 \subset \dim H^1(\Omega_f)$ finite.

Only need to show either there exists first integration or $\dim V_{\mathcal{F}}^0 < \infty$.

$\mathcal{F}$ is defined by $0 \to N_{\mathcal{F}}^\vee \to \Omega_X \Rightarrow 0 \to \mathcal{O}_X \to \Omega_X \otimes N_{\mathcal{F}}$, which gives a section $\omega \in H^0(\Omega_X \otimes N_{\mathcal{F}})$.

For $\tau(D) \in V_{\mathcal{F}}^0$, write it as $\xi_D \mod H^0(\Omega_f)$, where $\xi_D \in H^0(\mathcal{M}_f)$. Define

$$\theta := V_{\mathcal{F}}^0 \to \frac{H^0(\Omega^2 \otimes N_{\mathcal{F}})}{H^0(\Omega_f) \wedge \omega},$$

$$\tau(D) \mapsto \xi_D \wedge \omega.$$

Locally, $D = \{z_1 = 0\}$, $\omega = dz_1$, $\xi_D = \frac{dz_1}{z_1} + du$ (u holomorphic), $\xi_D \wedge \omega = du \wedge dz_1$.

Only need to show that either $\dim\ker\theta < \infty$ (actually $\dim\ker\theta \le 1$) or there exists first integration.

For $\tau(D) \in \ker\theta$, we may choose ξ_D s.t. $\xi_D \wedge \omega = 0$. Then ξ_D defines $\mathcal{F}$ (in a non-empty Zariski open subset).

Assume that $\tau(D_1), \tau(D_2) \in \ker\theta$ linearly independent. Pick ξ_{D_1}, ξ_{D_2} as above. Then there exists non-constant rational function h s.t. $\xi_{D_1} = h \cdot \xi_{D_2}$. As ξ_{D_1} is closed, $d\xi_{D_1} = 0 \Rightarrow dh \wedge \xi_{D_2} = 0$. Thus, dh also defines $\mathcal{F}$. Now, h is a first integration. ∎

Remark 11.3. *If there does not exist first integration, then the dimension of the algebraic leaf is at most*

$$\dim \frac{H^0(\Omega^2 \otimes N_{\mathcal{F}})}{H^0(\Omega_f) \wedge \omega} + 1 + \dim(H^1(\Omega_f)).$$

11.2 Proof of Bogomolov's theorem

Now we prove Bogomolov's theorem.

Proof. Consider

$$\begin{tikzcd} & \mathbb{P}(T_X) \arrow[d, "\pi"] \\ \tilde{C} \arrow[ur, "f'"] \arrow[r, "f"] & C \subset X \end{tikzcd}$$

$L \coloneqq \mathcal{O}_{\mathbb{P}(T_X)}(1)$, L big by Lemma 10.4. Then there exists $m \in \mathbb{Z}_{\geq 1}$ s.t. $mL - \pi^*A$ effective. Pick $s \in H^0(mL - \pi^*A) \setminus \{0\}$, $Z \coloneqq \mathrm{Zero}(s) \subset \mathbb{P}(T_X)$.

Two cases:

1. $f'(\tilde{C}) \not\subset Z$;
2. $f'(\tilde{C}) \subset Z$.

Case (1):

$$0 \neq (f')^*s \in H^0(m(f')^*L - (f')^*\pi^*A),$$

where $(f')^*\pi^*A = f^*A$, and $(f')^*L = \mathrm{Im}(f^*\Omega_X \to K_{\tilde{C}}) \Rightarrow \deg(f')^*L \leq \deg K_{\tilde{C}} = 2g - 2$. Since there is non-zero section, we have

$$\deg m(f')^*L - \deg f^*A \geq 0,$$

where $\deg m(f')^*L \leq m(2g-2)$, and $\deg f^*A = (C \cdot A)$, so $2g - 2 \geq (C \cdot A)/m$. OK in case 1.

Case (2): Write $Z = Z_1 \cup Z_2$, with Z_1 horizontal and Z_2 vertical.

- $\pi(Z_2) \subseteq X$ (contains at most finite many curves);
- any irreducible component S of Z_1, $\pi(S) = X$.

For $y \in S$ general, think $y \in (x, [v_y])$, where $x = \pi(y)$, $[v_y] \in \mathbb{P}(T_X)$.

$$d\pi(y) \colon T_S(y) \simeq T_X(x),$$

we get the direction $[d\pi(y)^{-1}v_y]$. $\Rightarrow$ We get the tautological foliation $\mathcal{F}_S$ on S.

If $f'(\tilde{C}) \subset S$, then $f'(\tilde{C})$ is invariant by $\mathcal{F}_S$.

Jouanolou's theroem implies two cases:

(a) $A(S) \coloneqq \{$ algebraic leaves$\}$ is finite;

(b) there exists first integration $h_S \colon S \dashrightarrow \mathbb{P}^1$.

If (a) holds, then $f'(\tilde{C}) \subset A(S)$, so only finitely many such C. OK.

Assume (b) holds. Take $\tilde{S}$ sm. proj. with

$$
\begin{tikzcd}
\tilde{S} \arrow[d, "\mathrm{bir}"] \arrow[dr, "\tilde{h}"] & \\
S \arrow[r, dotted, "h"] \arrow[d, "\pi"] & \mathbb{P}^1 \\
X &
\end{tikzcd}
$$

Stein factorization + generic smoothness,

I forgot to take the picture here!

s.t. a general fiber F of ϕ is irreducible and smooth.

X is general type, so X can not be covered by curves of genus 0 or 1 $\Rightarrow g(S) = g(F) \geq 2$.

In case (b), there exists a finite set $A(S)$ of curves $\subset S$ contained in some fiber of h, s.t. any irreducible component F_1 of h outside $A(S)$, we have

(1) $F_1 = \tau(F)$, where F is a smooth fiber of ϕ ($\Rightarrow g(F) = g_S$);

(2) $\tau \colon F \to F_1$ is birational;

(3) $\pi \colon F_1 \to \pi(F_1) \subset X$ is birational.

Then there exists $M_S > 0$ s.t. for such F_1,

$$(F_1 \cdot \pi^* A) = (F_1 \cdot (\pi \circ \tau)^* A) = M_S.$$

As $f'(\tilde{C})$ is invariant by $\mathcal{F}_S$, either $f'(\tilde{C}) \in A(S)$ or $f'(\tilde{C}) =$ such $F_1 \Rightarrow$

$$(C \cdot A) = f'(\tilde{C}) \cdot \pi^* A = (F_1 \cdot \pi^* A) = M_S.$$

Moreover, $g(\tilde{C}) = g(F) \geq 2 \Rightarrow \chi(\tilde{C}) = 2g_S - 2 \geq 1$

$$\Rightarrow \chi(\tilde{C}) \geq \frac{2g_S - 2}{M_S} \cdot (C \cdot A)$$

$\Rightarrow$ If $f'(\tilde{C}) \subset Z$, either

$$f'(\tilde{C}) \in \bigcup_S A(S),$$

or

$$\chi(\tilde{C}) \geq \min_S \left\{ \frac{2g_S - 2}{M_S} \right\} (C \cdot A).$$

■

12 Note on 20241119

Remark 12.1. *Bogomolov, McQuillan's results, not work for hypersurfaces of* $\mathbb{P}^3$.

Let $D \subset \mathbb{P}^3$ smooth hypersurface, $\deg D = d$.

$$K_D = (K_{\mathbb{P}^3}) + \mathcal{O}(d))|_D \Rightarrow D \text{ general type } \Leftrightarrow d \geq 5.$$

Next we compute c_1, c_2 of D.

Exact sequence:

$$0 \to \Omega_{\mathbb{P}^3} \to \bigoplus_{i=1}^{4} \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^3} \to 0. \tag{12.1}$$

Adjunction

$$0 \to \mathcal{O}(-D)|_D \to \Omega_{\mathbb{P}^3}|_D \to \Omega_D \to 0. \tag{12.2}$$

$h = \mathcal{O}(1)$ on $\mathbb{P}^3$. By (12.1),

$$t^3 - c_1 t^2 + c_2 t - c_3 = C_{\Omega_{\mathbb{P}^3}} \times t = (t+h)^4,$$

$h^4 = 0$, $\Rightarrow$

$$C_{\Omega_{\mathbb{P}^3}} = t^3 + 4ht^2 + 6h^2,$$

$h^3|_D = 0$. By (12.2), $= C_D(t) \times (t + dh)$. $\Rightarrow$

$$C_D(t) = t^2 - (d-4)ht + ((d-4)d + 6)h^2,$$

$c_1(D) = (d-4)h|_D$, $c_2(D) = ((d-4)d+6) \times h^2|_D$. Note $h^2|_D = d$. Then

$$c_1(D)^2 > c_2(D) \Leftrightarrow (d-4) \times d \geq ((d-4)d + 6) \times d \Leftrightarrow d \leq \frac{5}{2}.$$

THEOREM 12.2 (Lu-Miyaoka). *Let X smooth projective variety of general type. Then X has only a finite number of non-singular codimension one subvarieties having pseudo-effective anticanonical divisors.*

When $\dim = 2$, only finitely many smooth curves $C \subset X$ with $g(C) \leq 1$.

12.1 Nevanlinna theory

(Intersection theory of entire curves) (Nevanlinna current) (Tautological inequality)

Analogy in algebraic setting: Recall

$$\begin{tikzcd} & \mathbb{P}(T_X) \arrow[d, "\pi"] \\ \tilde{C} \arrow[ur, "f'"] \arrow[r, "f"] & C \subset X \end{tikzcd}$$

then

$$\deg((f')^* \mathcal{O}_{\mathbb{P}(T_X)}(1)) \leq \deg K_C = 2g - 2.$$

12.1.1 Notations and settings

Let X be a smooth projective variety, and $f\colon \mathbb{C} \to X$ a holomorphic map.
Let α be a smooth closed $(1,1)-$form on X. Define the characteristic function

$$T(r,f,\alpha) \coloneqq \int_1^r \frac{dt}{t}\left(\int_{\mathbb{D}(t)} f^*\alpha\right).$$

$T(r,f,\cdot)$ can be think as a positive current of bidegree $(1,1)$.

Lemma 12.3. *Let α be a smooth exact $(1,1)-$form on X. Then*

$$T(r,f,\alpha) = O(1).$$

Proof. By $\partial\bar{\partial}-$Lemma, $\alpha = dd^c\varphi$ with φ smooth, where $d^c = -\frac{i}{2}(\partial - \bar{\partial})$. Then

$$\begin{aligned} T(r,f,\alpha) &= \int_1^r \frac{dt}{t}\int_{|z|<t} dd^c\varphi \circ f(z) \\ &= \int_1^r \frac{dt}{t}\int_{|z|=t} d^c(\varphi \circ f)(z). \end{aligned}$$

Write $z = \rho e^{i\theta}$, $\beta = \varphi \circ f$. Compute $d^c\beta$ on $|z| = t$ ($\rho = t$ constant):

$$\frac{\partial\beta}{\partial\rho}d\theta = \cdots = \frac{1}{it} \times \frac{3}{-i} \times d^c\beta,$$

so

$$d^c\beta = \frac{1}{2}t\frac{\partial\beta}{\partial\rho}d\theta.$$

Now

$$T(r,f,\alpha) = \frac{1}{2}\int_1^r dt\int_0^{2\pi}\frac{\partial\beta(te^{i\theta})}{\partial t}d\theta = \frac{1}{2}\int_0^{2\pi}(\beta(re^{i\theta}) - \beta(e^{i\theta}))dt = O(1).$$

■

Let L be a line bundle on X. Define

$$T(r,f,L) \coloneqq T(r,f,c_1(L,\|\cdot\|)) + O(1),$$

where $\|\cdot\|$ is a Hermitian metric on L. It is well-defined, as for two Hermitian metrics $\|\cdot\|, \|\cdot\|'$, $c_1(L,\|\cdot\|) - c_1(L,\|\cdot\|')$ is an exact form.

We think $T(r,f,L)$ as the "intersection number" for $f(\mathbb{C})$ and L.

Additivity For L_1, L_2 line bundles on X,

$$T(r,f,L_1 \otimes L_2) = T(r,f,L_1) + T(r,f,L_2).$$

Functoriality Let $p\colon X \to Y$ be a morphism. Then

$$T(r,p\circ f,E) = T(r,f,p^*E).$$

Positivity If L is ample and $f\colon \mathbb{C} \to X$ is non-constant, then

$$T(r, f, L) \to +\infty.$$

Take $\|\cdot\|$ s.t. $c_1(L, \|\cdot\|)$ positive. Then

$$T(r, f, L) = \int_1^r \frac{dt}{t} \int_{\mathbb{D}(t)} f^* c_1(L, \|\cdot\|) \geq \left(\int_{\mathbb{D}(1)} f^* c_1(L, \|\cdot\|) \right) \times \log r \to \infty.$$

The above computation shows

$$\frac{T(r, f, L)}{\log r} \not\to \infty \Rightarrow \operatorname{area}(f(\mathbb{C})) < \infty$$

$\Rightarrow f\colon \mathbb{C} \to X$ is algebraic i.e. f extends to $\mathbb{P}^1 \to X$.

Definition 12.4 (Counting function). $D \subset X$ Cartier divisor on X s.t. $f(\mathbb{C}) \not\subset \operatorname{Supp} D$. Want to count "$f(\mathbb{C}) \cap D$". Counting function:

$$\begin{aligned} N(r, f, D) &\coloneqq \int_1^r \frac{dt}{t} \left(\sum_{z \in \mathbb{D}(t)} \operatorname{ord}_z f^* D \right) \\ &= \int_1^r \frac{dt}{t} \int_{\mathbb{D}(t)} [f^* D] \\ &= \text{“}T(r, f, [D])\text{''}, \end{aligned}$$

where $[f^* D]$ is seen as a current.

Additivity For D_1, D_2 Cartier divisors on X s.t. $f(\mathbb{C}) \not\subset \operatorname{Supp} D_1 \cup \operatorname{Supp} D_2$,

$$N(r, f, D_1 + D_2) = N(r, f, D_1) + N(r, f, D_2).$$

Functoriality Let $p\colon X \to Y$ be a morphism, and D Cartier divisor on Y with $p \circ f(\mathbb{C}) \not\subset \operatorname{Supp} D$. Then

$$N(r, p \circ f, D) = N(r, f, p^* D).$$

Positivity For D effective,

$$N(r, f, D) \geq 0, \ \forall r.$$

D Cartier divisor. Let $\|\cdot\|$ be a smooth Hermitian metric on $\mathcal{O}_X(D)$. s_D: a section of $\mathcal{O}_X(D)$ associated to D. We get the Green function

$$\begin{aligned} g_D\colon X \setminus \operatorname{Supp} D &\longrightarrow \mathbb{R} \\ x &\longmapsto -\log \|s_D(x)\|. \end{aligned}$$

For other g'_D associated to $(D, s'_D, \|\cdot\|')$, we have $g_D - g'_D = O(1)$.

Definition 12.5 (Proximity function). Define

$$m(r, f, D) \coloneqq \int_0^{2\pi} g_D(f(re^{i\theta})) \frac{d\theta}{2\pi} + O(1).$$

"Effect of boundary".

Additivity For D_1, D_2 Cartier divisors on X s.t. $f(\mathbb{C}) \not\subset \operatorname{Supp} D_1 \cup \operatorname{Supp} D_2$,

$$m(r, f, D_1 + D_2) = m(r, f, D_1) + m(r, f, D_2) + O(1).$$

Functoriality Let $p\colon X \to Y$ be a morphism, and D Cartier divisor on Y with $p \circ f(\mathbb{C}) \not\subset \operatorname{Supp} D$. Then

$$m(r, p \circ f, D) = m(r, f, p^*D) + O(1).$$

Positivity For D effective,

$$m(r, f, D) \geq O(1).$$

THEOREM 12.6 (First Main theorem). *X projective variety, D Cartier divisor on X. Let $f : \mathbb{C} \to X$ holomorphic map s.t. $f(\mathbb{C}) \not\subset \operatorname{Supp} D$. Then*

$$T(r, f, \mathcal{O}_X(D)) = N(r, f, D) = m(r, f, D) + O(1).$$

Proof. Desingularization + functoriality, we can assume X is smooth. Let $\|\cdot\|$ be a smooth Hermitian metric on $\mathcal{O}_X(D)$, s_D a section associated to D, and g_D the Green function.

Poincaré-Lelong formula:

$$f^*c_1(L, \|\cdot\|) = [f^*D] + \frac{1}{\pi} dd^c f^* g_D.$$

Denote $\omega = c_1(L, \|\cdot\|)$, $g = f^*g_D$. Then

$$\begin{aligned} T(r, f, \omega) - N(r, f, D) &= \frac{1}{\pi} \int_1^r \frac{dt}{t} \int_{\mathbb{D}(t)} dd^c g \\ &= \frac{1}{\pi} \int_1^r \frac{dt}{t} \int_{\|z|=t} d^c g(z) \\ &= \frac{1}{2\pi} \int_1^r dt \int_0^{2\pi} \frac{\partial g}{\partial t} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left(g(re^{i\theta}) - g(e^{i\theta}) \right) d\theta \\ &= m(r, f, L) - \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) d\theta. \end{aligned}$$

■

COROLLARY 12.7 (Nevanlinna inequality). *D effective Cartier divisor, $f(\mathbb{C}) \not\subset \operatorname{Supp} D$, we have*

$$0 \leq N(r, f, D) \leq T(r, f, \mathcal{O}_X(D)) + O(1).$$

Especially, if L is an ample line bundle, E big, then there exists $\epsilon > 0$ s.t. $E - \varepsilon L > 0$, which deduces

$$T(r, f, E) > \varepsilon T(r, f, L) + O(1).$$

12.2 Proof of McQuillan's theorem

[Courbes entières et feuiletages holomorphes] Brunella.

13 Note on 20241210

Lemma 13.1. *There exists $C_1 > 0$, s.t.*

$$T(r, f'_H, c_1(L)) \leq T(r, f'_H, c_1(L)^{\text{sing}})$$

Lemma 13.2. *There exists $C_2 > 0$ s.t.*

$$T(r, f, \tilde{\omega}) \leq C_2 T(r, f, c_1(A)).$$

Same argument as the usual case,

$$\begin{aligned}
&T(r, f'_H, c_1(\mathbb{L})) - C_1 \log T(r, f, A) \\
\leq &T(r, f'_H, c_1(L)^{\text{sing}}) + o(1) \\
= &\int_0^{2\pi} \log(1 + F(re^{i\theta})) \frac{d\theta}{2\pi} \\
\leq &\log T(r, f, \tilde{\omega}) + o(T(r, f, \tilde{\omega})) \| \\
\leq &T(r, f, A) + o(T(r, f, A)) \|.
\end{aligned}$$

Corollary 13.3. *Let X be smooth projective variety, H a SNC divisor on X, and $f\colon \mathbb{C} \to X$ an entire curve. Assume $f(\mathbb{C}) \cap H = \varnothing$. Let $f'_H\colon \mathbb{C} \to \mathbb{P}(T_x(\log(H)))$ the canonical lift of f.*

$$\begin{tikzcd}
\mathbb{P}(T_x(\log(H))) \arrow[d, "h_H"] \\
X
\end{tikzcd}$$

Then we can choose Nevanlinna currents $T_f, T_{f'}$ associated to $f.f'$ s.t.

$$(h_H)_*(T_{f'_H}) = T_f,$$

and

$$[T_{f'_H}] \wedge \left[\mathcal{O}_{\mathbb{P}_{T_x(\log(H))}}(1)\right] \leq 0.$$

13.1 Foliation on surfaces

Definition 13.4. Let X be a smooth surface. A foliation $\mathcal{F}$ on X is a locally free rank 1 subsheaf $N^\vee_{\mathcal{F}}$ of Ω_X s.t. $\Omega / N^\vee_{\mathcal{F}}$ is torsion free.

- $N_{\mathcal{F}}$: normal bundle.
- $N^\vee_{\mathcal{F}}$: conormal bundle.

- $K_{\mathcal{F}} := K_X \otimes N_{\mathcal{F}}^{\vee}$ conanical sheaf of $\mathcal{F}$.

Exact sequence:

$$0 \to N_{\mathcal{F}}^{\vee} \to \Omega_X \to \mathcal{I}_Z \otimes K_{\mathcal{F}} \to 0, \tag{13.3}$$

where $\mathcal{I}_Z$ is the ideal sheaf of Z, $\dim Z = 0$, and Z is the singular locus of $\mathcal{F}$.

$X \setminus Z$: regular points.

- Let M be a Riemann surface (not necessarily compact nor algebraic). A morphism $\iota\colon M \to X$ is said to be a piece of leaf of $\mathcal{F}$ if
  1. there exists a discrete set of points $P \subset M$ s.t. $\iota|_{M\setminus P}\colon M \setminus P \to X$ is a local embedding;
  2. the natural map
  $$\iota^*(N_{\mathcal{F}}) \to \iota^*(\Omega_X) \to \Omega_M$$
  is the zero map.

Facts:

1. If $x \in X$ is a regular point of $\mathcal{F}$, there exists a unique maximal piece of leaf passing through x (any other piece of leaf passing through x is contained in its image).
2. For $x \in X$ regular point of $\mathcal{F}$, there exists an analytic neighborhood $x \in U \subset X$, $U \simeq \mathbb{D} \times \mathbb{D}$, with coordinates (z_1, z_2), s.t. the restriction of exact sequence eq. (12.1) on U becomes
$$0 \to \mathcal{O}_U dz_1 \to \mathcal{O}_u dz_1 \otimes \mathcal{O}_u dz_2 \to \mathcal{O}_u dz_2 \to 0$$
local leaves in U are given by $\{z_1 = c\}$, $c \in \mathbb{D}$.

- A leaf M of $\mathcal{F}$ is called algebraic if the Zariski closure of M in X is a curve.

Definition 13.5. Let Y be a smooth surface, $\pi\colon Y \to X$ be a dominant morphism. We have an inclusion

$$\pi^*(N_{\mathcal{F}}^{\vee}) \to \pi^*(\Omega_X) \to \Omega_Y.$$

The saturation $N_{\mathcal{F}_Y^{\vee}}$ of $\pi^*(N_{\mathcal{F}}^{\vee})$ in Ω_Y defines a foliation $\mathcal{F}_Y$ on Y. We call $\mathcal{F}_Y$ the pull-back of $\mathcal{F}$ to Y via π and write $\pi^*\mathcal{F}\colon \mathcal{F}_Y$.

Note: In general, $K_{\pi^*(\mathcal{F})} \neq \pi^* K_{\mathcal{F}}$.

Fibration: Let S be a smooth curve, and $f\colon X \to S$ a dominant morphism. We have exact sequence

$$0 \to f^*(\Omega_S) \to \Omega_X \to \Omega_{X/S} \to 0,$$

which induces a foliation $\mathcal{F}_f$ on X, by $N_{\mathcal{F}_f}^{\vee} :=$ saturation of $f^*(\Omega_S)$ in Ω_X.

We say $\mathcal{F}$ comes from a fibration if there exists

$$\pi\colon \quad \begin{array}{ccc} Y & \xrightarrow{\text{bir}} & X \\ \Big\downarrow f \text{ dominant} & & \\ S & & \end{array}$$

with S smooth curve, s.t. $\pi^*\mathcal{F} = \mathcal{F}_f$.

13.1.1 Singularity

Locally $\mathcal{F}$ is defined by a vector field $\partial = a_1(z,w)\partial_z + a_2(z,w)\partial_w$ s.t. $\mathrm{Sing}(\partial) = \{a_1 = a_2 = 0\}$ zero dimensional.

Assume $\mathrm{Sing}(\partial) = \{0\}$, $\mathfrak{m}$ maximal ideal for o. $\partial\colon \mathfrak{n} \to \mathfrak{m}$ verifying the Liebnitz rule, which defines a linear map

$$L_{\mathcal{F}}\colon \mathfrak{m}/\mathfrak{m}^2 \to \mathfrak{m}/\mathfrak{m}^2$$

$L_{\mathcal{F}} =$ "linear part of $\mathcal{F}$ at o". λ_1, λ_2 the two eigenvalues of $L_{\mathcal{F}}$.

Definition 13.6. The singularity o of $\mathcal{F}$ is reduced, if

1. one of $\lambda_i \neq 0$ (assume $\lambda_2 \neq 0$)
2. $\lambda_1/\lambda_2 \neq \mathbb{Q}_{>0}$.

Divided to two cases

- Simple: $\lambda_1, \lambda_2 \neq 0$;
- Saddle-node: $\lambda_1 = 0$.

theorem 13.7 (Seidenberg). *X smooth surface, $\mathcal{F}$ foliation on X. Then there exists $\pi\colon \tilde{X} \to X$ proper birational, s.t. the foliation $\pi^*(\mathcal{F})$ has only reduced singularities.*

Moreover, π can be obtained by successive blowups of singular points for corresponding surfaces.

Properties for reduced singularities:

1. For $p \in \mathrm{Sing}(\mathcal{F})$, there exists exactly 1 or 2 leaves passing through p. They are smooth. If have 2, they intersect transversally.

   Indeed,

   - if p simple, in a suitable coordinate, $\mathcal{F}$ is

     $$\partial = z\partial_z + \lambda w(1 + \text{h.o.t.})\partial_w,\ \lambda = \lambda_1/\lambda_2,$$

     $\{z = 0\}, \{w = 0\}$ are the 2 leaves passing through o. ("Separatrix")
   - If p saddle-node, $\mathcal{F}$ is locally defined by

     $$\partial = [z(1 + vw^{d-1}) + wF(z,w)]\partial_z + w^d\partial_z$$

     (normal form of Dulac), $d \geq 2$, $v \in \mathbb{C}$, F vanishes at o to order $d-1$. $\{w = 0\}$ is $\mathcal{F}-$invariant ("strong separatrix").

     It is possible to have another leave passing through o tangent to $\{z = 0\}$ (If exists, "weak separatrix").

     The d, v are important invariants. Define

     $$d_p := \begin{cases} d^{\geq 2} & \text{saddle node,} \\ 1 & \text{simple.} \end{cases}$$

14 Note on 20241217

15 Note on 20241219

Step 2:

THEOREM 15.1 (McQuillan).

$$N_{\mathcal{F}} \cdot [\Phi] \geq 0.$$

Fact : Φ is $\mathcal{F}-$invariant, i.e., $\Phi(\eta) = 0$ for every $2-$form η which vanishes on $\mathcal{F}$.

Indeed, f tangent to $\mathcal{F} \Rightarrow f^*\eta = 0 \Rightarrow \Phi(\eta) = 0$.

For $x \in X \setminus \mathrm{Sing}(\mathcal{F})$, pick neighborhood $x \in U \simeq \mathbb{D} \times \mathbb{D}$ with coordinate (z_1, z_2) s.t. $\mathcal{F}$ defined by dz_1. Consider $\pi_1 \colon \mathbb{D} \times \mathbb{D} \to \mathbb{D}$, $(z_1, z_2) \mapsto z_1$.

LEMMA 15.2. *There exists a measure μ_Φ on $\mathbb{D}$ s.t. $\Phi = \pi_1^* \mu_\Phi$.*

$\Phi(\alpha) = \int \Phi \wedge \alpha = \int \int_{\pi^{-1}(z)} \alpha \mu_\Phi$.

Proof. Write

$$\Phi = \sum_{i,j=1,2} A_{ij} dz_i \wedge d\bar{z}_j,$$

A_{ij} are measures.

Φ is $\mathcal{F}-$invariant $\Rightarrow A_{ij} = 0$ if $2 \in \{i, j\}$. Thus, $\Phi = A_{1,1} dz_1 \wedge d\bar{z}_1$. Φ is closed $\Rightarrow d\Phi = 0 \Rightarrow \frac{\partial A_{1,1}}{\partial z_1} = \frac{\partial A_{1,1}}{\partial \bar{z}_1} = 0$.

$\Rightarrow A_{1,1}$ does not depend on $z_1, \bar{z}_1$. $\Rightarrow \Phi = \pi_1^* \mu_\Phi$ for some measure μ_Φ on $\mathbb{D}$. ∎

μ_Φ is called the invariant transverse measure.

PROPOSITION 15.3. $K \coloneqq \mathrm{Supp}\, \Phi$. *$\mathcal{F}|_{X \setminus \mathrm{Sing}(\mathcal{F})}$ has an invariant transverse measure whose support coincide with $K \setminus \mathrm{Sing}(\mathcal{F})$.*

THEOREM 15.4 (Siu's decomposition of currents).

$$\Phi = \Phi_{\mathrm{alg}} + \Phi_{\mathrm{diff}},$$

where

$$\Phi_{\mathrm{alg}} = \sum_{i=1}^{\infty} \lambda_i \delta_{C_i},\ \lambda_i \geq 0,\ \sum \lambda_i < +\infty,$$

and Φ_{diff} has Lelong number 0 outside a finite set.

$\Phi_{\mathrm{alg}}, \Phi_{\mathrm{diff}}$ are $\mathcal{F}-$invariant $\Rightarrow C_i$ are $\mathcal{F}-$invariant. By Jouanolou's Thm $\Rightarrow C_i$ are finite $\Rightarrow \Phi_{\mathrm{alg}} = \sum_{i=1}^{N} \lambda_i \delta_{C_i}$.

In our setting, the condition for Φ_{diff} means the invariant transverse measure μ_{diff} does not have atom (i.e. there is no single point p s.t. $\mu_{\mathrm{diff}}(\{p\}) > 0$).

First treat the algebraic part:

As C_i are $\mathcal{F}-$invariant, $\bigcup_{i=1}^N C_i$ is smooth outside $\mathrm{Sing}(\mathcal{F})$. If $\bigcup_{i=1}^N C_i$ is not smooth at $x \in \mathrm{Sing}(\mathcal{F})$, then locally looks like [cross] $\Rightarrow \bigcup_{i=1}^N C_i$ SNC, except some C_i (self-intersect). Blowup some singular pts, we may assume $\bigcup_{i=1}^N C_i$ SNC.

$$C := \bigcup_{i=1}^{N} C_i.$$

Lemma 15.5.

$$N_{\mathcal{F}} \cdot C_i \geq C_i \cdot C.$$

Proof. Assume $i = 1$. Only need to show

$$\deg\left(\mathcal{F}(-C_1)|_{C_1}\right) \geq \left(\sum_{j \neq 1} C_j\right) \cdot C.$$

Pick U_i cover of X s.t.

1. $\alpha_i \in N^{\vee}_{\mathcal{F}}(U_i)$ defines $\mathcal{F}$;
2. If $U_i \cap U_1 \neq \varnothing$, then we have coordinates (z_i, w_i) s.t. $C_1|_{U_i} = \{w_i = 0\}$. As C_1 is a $\mathcal{F}-$invariant, we may write $\alpha_i = h_i dw_i + w_i \eta$.

For every $p \in C_1$, set

$$Z(\mathcal{F}, C_1, p) := \mathrm{ord}_p(h_i|_{C_1}),$$

where $p \in U_i$.

- $Z(\mathcal{F}, C_1, p)$ does not depend on the choice of U_i, α_i;
- If $p \notin \mathrm{Sing}(\mathcal{F})$, then $Z(\mathcal{F}, C_i, p) = 0$;
- If $p \in \mathrm{Sing}(\mathcal{F})$, then $Z(\mathcal{F}, C_1, p) \geq 1$.

$\Rightarrow$

$$\sum_{p \in C_1} Z(\mathcal{F}, C_1, p) \geq \mathrm{Card}(\mathrm{Sing}(\mathcal{F}) \cap C_1) \geq \mathrm{Card}\left((\sum_{j \neq 1} C_j) \cap C_1 \right) = \mathrm{Card}\left((\sum_{j \neq 1} C_j) \cdot C_1 \right).$$

Lemma 15.6.

$$\deg\left(N_{\mathcal{F}}(-C_1)|_{C_1}\right) = \sum_{p \in C_1} Z(\mathcal{F}, C_1, p).$$

Proof. Construct a section.

Assume $U_i \cap U_j \cap C_1 \neq \varnothing$. There exists $f_{j,i} \in \mathcal{O}^*(U_i \cap U_j)$ s.t. $\alpha_j = f_{j,i}\alpha_i$. Write $w_j = g_{j,i} w_i$, $g_{j,i} \in \mathcal{O}^*(U_i \cap U_j)$.

$$\alpha_j = h_j dw_j + w_j \eta_j = h_j g_{j,i} dw_i + w_i dg_{j,i} + w_i g_{j,i} \eta_j,$$

and

$$\alpha_j = f_{j,i}\alpha_i = f_{j,i} h_i dw_i + f_{j,i} w_i \eta_i.$$

$\Rightarrow$

$$h_j g_{j,i}|_{C_1} = f_{j,i} h_i|_{C_1} \Rightarrow h_j|_{C_1} = \left(\frac{f_{j,i}}{g_{j,i}}\right) h_i|_{C_1}.$$

Pick $\alpha_1^\vee \in N_{\mathcal{F}}(U_i)$ dual to $\alpha_i \in N_{\mathcal{F}}^\vee(U_i)$. $w_i \in \mathcal{O}(-C_1)(U_i)$. Consider

$$h_i \cdot \alpha_i^\vee \otimes w_i|_{C_1} \in N_{\mathcal{F}}(-C_1)|_{C_1}(U_i \cap C_1).$$

They define a global section.

Indeed, on $U_i \cap U_j$,

$$h_i \cdot \alpha_i^\vee \otimes w_i|_{C_1} = \left(\frac{f_{j,i}}{g_{j,i}}\right)\bigg|_{C_1} \times h_j \times (f_{j,i})^{-1} \times (g_{j,i})\alpha_j^\vee \otimes w_j = h_i \alpha_j^\vee \otimes w_i|_{C_1}.$$

Thus,

$$\deg(N_{\mathcal{F}}(-C_1)|_{C_1}) = \sum_{p \in C-1} Z(\mathcal{F}, C_1, p).$$

■

■

Then

$$N_{\mathcal{F}} \cdot [\Phi_{\text{alg}}] = \sum_{i=1}^{N} \lambda_i N_{\mathcal{F}} \cdot [C_i] \geq \sum_{i=1}^{N} \lambda_i [C] \cdot [C_i] = [C] \cdot [\Phi_{\text{alg}}].$$

As $[\Phi]$ is nef, $[C] \cdot [\Phi] \geq 0$, and thus

$$[C] \cdot [\Phi_{\text{alg}}] \geq -[C] \cdot [\Phi_{\text{diff}}].$$

We have

$$N_{\mathcal{F}} \cdot [\Phi] = N_{\mathcal{F}} \cdot [\Phi_{\text{alg}}] + N_{\mathcal{F}} \cdot [\Phi_{\text{diff}}] \geq -[C] \cdot [\Phi_{\text{diff}}] + N_{\mathcal{F}} \cdot [\Phi_{\text{diff}}] = N_{\mathcal{F}} \cdot [\Phi_{\text{diff}}].$$

To prove the theorem, only need to show

Lemma 15.7.

$$N_{\mathcal{F}} \cdot [\Phi_{\text{diff}}] \geq 0.$$

If this lemma holds, then $N_{\mathcal{F}} \cdot [\Phi] \geq 0$, then $N_{\mathcal{F}}^\vee \cdot [\Phi] \leq 0$. By the Theorem showed in the last section which I do not know, $K_{\mathcal{F}} \cdot [\Phi] \leq 0 \Rightarrow (K_{\mathcal{F}} + N_{\mathcal{F}}^\vee) \cdot [\Phi] \leq 0$ ($K_X = K_{\mathcal{F}} + N_{\mathcal{F}}^\vee$). As $[\Phi]$ is nef, K_X can not be big $\Rightarrow$ McQuillan's thm.

Now prove Lemma 15.7.

To do so, we construct a closed $2-$form Θ which represents $c_1(N_{\mathcal{F}} \otimes \mathcal{O}(-C))$.

We choose an open cover $\{U_i\}$ of X and logarithmic $1-$forms $\Omega_k \in \Omega(\log C)(U_k)$ and $(1,0)-$forms $\beta_k \in A^{1,0}(U_k)$ s.t.

1. In U_k, $\mathcal{F}$ is defined by $f_k \Omega_k$ where $C|_{U_k} = \{f_k = 0\}$;
2. $\text{Card}(U_k \cap \text{Sing}(\mathcal{F})) \leq 1$;
3. $d\Omega_k = \beta_k \wedge \Omega_k$ on $U_k \setminus V_k$, where $V_k \Subset U_k$ and $V_k \cap U_i = \varnothing$ when $i \neq k$.

Indeed, we will choose V_k in the following way, if $U_k \cap \text{Sing}(\mathcal{F}) = \varnothing$, then $V_k = \varnothing$; if $U_k \cap \text{Sing}(\mathcal{F}) = \{x\}$, then V_k is a small ball centered at x.

We construct (U_k, Ω_k, β_k) as follows:

(1) At a regular point, local coordinate (z, w), $\mathcal{F}$ defined by dz:

(1.1) if $U_k \cap C = \varnothing$, $\Omega_k = dz$;

(1,2) if $U_k \cap C \neq \varnothing$, $C = \{z = 0\}$, $\Omega_k = \frac{dz}{z}$. Pick $\beta_k = 0$;

(2) At a singular point, $(0,0) \in U_k$ singular, write $\mathcal{F} = adw - bdz$ with $a = b = 0$ at $(0,0)$:

(2.1) if $U_k \cap C = \varnothing$, pick $\Omega_k = adw - bdz$,

$$\beta_k = F \cdot \frac{a_z + b_w}{|a|^2 + |b|^2}(\bar{a}dz + \bar{b}dw),$$

where F is C^∞, real, s.t. $F \equiv 0$ near $(0,0)$, $F \equiv 1$ on $U_k \setminus V_k$;

(2.2) if $U_k \cap C = \{w = 0\}$, then $\dfrac{b}{w}$ is holomorphic. Take $\Omega_k = a\frac{dw}{w} - \frac{b}{w}dz$,

$$\beta_k = F \cdot \frac{a_z + b_w - \frac{b}{w}}{|a|^2 + |b|^2}(\bar{a}dz + \bar{b}dw);$$

(2,3) if $U_k \cap C = \{zw = 0\}$, both $\frac{a}{z}$ and $\frac{b}{w}$ are regular. $\Omega_k = \frac{a}{z} \cdot \frac{dw}{w} - \frac{b}{w} \cdot \frac{dz}{z}$,

$$\beta_k = F \cdot \frac{a_z - \frac{a}{z} + b_w - \frac{b}{w}}{|a|^2 + |b|^2}(\bar{a}dz + \bar{b}dw).$$

16 Note on 20241224

I woke up late this day, so I missed the class.

17 Note on 20241226

Case 1: $\lambda \in \mathbb{R}_+ \setminus \mathbb{Q}_+$. Case 2: $\lambda \in \mathbb{R}_-$.

17.1 Case 1

Poincaré: In case 1, $\mathcal{F}$ linearisable. $\mathcal{F}$ defined by $zdw - \lambda wdz$.

LEMMA 17.1. *In Case 1,*

$$\Phi_{\text{diff}}\left(\frac{d\beta_k}{2\pi i}\right) \geq 0.$$

Proof. If $C = \{zw = 0\}$, then $\Omega_k = \frac{dw}{w} - \lambda\frac{dz}{z}$, $\beta \wedge \Omega_k = d\Omega_k$ in $U_k \setminus V_k$, $\beta_k = 0$ is OK.
If $p \notin C$, then $\Omega_k = zdw - \lambda wdz$,

(a)
$$\beta_k = F_k \frac{1 + \lambda}{|z|^2 + |\lambda w|^2}(\bar{z}dz + \lambda\bar{w}dw);$$

If $C = \{w = 0\}$, then $\Omega_k = dw - \lambda w\frac{dz}{z}$.

(b)

$$\beta_k = F_k \frac{1}{|z|^2 + |\lambda w|^2}(\bar{z}dz + \lambda \bar{w}dw).$$

Pick $U_k = \{|z| < \epsilon,\ |w| < \epsilon\}$. $S_k = \partial U_k$. $\mathcal{L} = \mathcal{F} \cap S_k$ is a real foliation with dimension 1 on S_k (with the induced orientation).

μ_{diff} induces an invariant transverse measure $\mu_{\mathcal{L}}$ for $\mathcal{L}$. By Stokes + Laminar structure of Φ_{diff}, calculating $\Phi_{\text{diff}}(d\beta/2\pi i)$ is to compute the integration of $\beta/2\pi i$ along $\mathcal{L}$, then integrate by $\mu_{\mathcal{L}}$.

$S = \{|z| = \epsilon,\ |w| \le \epsilon\} \cup \{|w| = \epsilon,\ |z| \le \epsilon\} = S_1 \cup S_2$. On S_1, write $z = \epsilon e^{i\theta}$, the leaves takes the form $w = Ce^{\lambda i\theta}$, $|C| \le \epsilon$. On this leaf, Case (a):

$$\begin{aligned}\beta =& F \frac{1+\lambda}{|z|^2 + |\lambda w|^2}(\bar{z}dz + \lambda \bar{w}dw) \\ =& i \frac{1+\lambda}{\epsilon^2 + \lambda^2 C^2}(1 + \lambda |C|^2) d\theta.\end{aligned}$$

$\Rightarrow \frac{\beta}{2\pi i} = \boxdot d\theta$, where $\boxdot > 0$. Case (b) is the same. On S_2, we have similar computation. Thus, the integration along any leaf of $\mathcal{L}$ is ≥ 0, i.e. $\Phi_{\text{diff}}(d\beta_k/2\pi i) \ge 0$. ■

17.2 Case 2

Case 2: $\lambda \in \mathbb{R}_-$.

Lemma 17.2. *In Case 2,*

$$\Re \Phi_{\text{diff}} \left(\frac{d\beta_k}{2\pi i} \right) = 0.$$

Proof. Consider the case $p \notin C$.

$$\Omega_k = zdw - \lambda w(1+h)dz = adw - bdz,\ h \in (z, w).$$

If $\lambda = -1$, then we may pick β holomorphic. Set

$$\beta = Adz + Bdw.$$

To satisfy $\beta \wedge \Omega_k = d\Omega_k$, only need $Aa + Bb = a_z + b_w = 1 - (1+h) - wh_w = -h - wh_w$. Write $h = zf(z) + wg$. Pick $A = -f(z)$, $B = \frac{-g + h_w}{1+h}$, then OK.

If β holomorphic, $d\beta \in A^{2,0} \Rightarrow d\beta \cdot [\Phi_{\text{diff}}] = 0$ ((2,0) paired to (1,1)).

Assume $\lambda \ne -1$. Define $\mathcal{R}$ to be a 1−dim real foliation tangent to $\mathcal{F}$ on $\overline{U_k} \setminus \{p\}$, generated by the kernel of $\eta|_{\mathcal{F}}$, where

$$\eta = \Im \left[\frac{a_z + b_w}{|a|^2 + |b|^2}(\bar{a}dz + \bar{b}dw) \right].$$

Denote $\tilde{\beta} = \frac{a_z + b_w}{|a|^2 + |b|^2}$.

On $\{w = 0\}$ and $\{z = 0\}$, $\mathcal{R}$ are radical. So the picture of $\mathcal{R}$ is

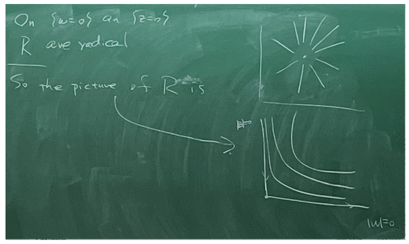

Figure 7: Fig.7

Now we can choose a bidisk U_k s,t,

1. $\partial U_k = T_1 \cup T_2$, $T_1 \simeq \mathbb{S}^1 \times \overline{\mathbb{D}}$, $T_2 \simeq \overline{\mathbb{D}} \times \mathbb{S}^1$;
2. $\partial T_1 = \partial T_2 = \{$ the points in ∂U_k where $\mathcal{R}$ external tangent to $U_k\}$;
3. The leaves of $\mathcal{R}$ in $\overline{U_k}$ except these in $\{w = 0\}$, $\{z = 0\}$ induces a diffeomorphism ϕ between T_1 and T_2.

On T_1, T_2, $\mathcal{F}$ induces a foliation $\mathcal{L}_1, \mathcal{L}_2$ with invariant transverse measures μ_1, μ_2. The diffeomorphism ϕ maps $\mathcal{L}_1$ to $\mathcal{L}_2$ with reverse orientation, and $\phi_* \mu_1 = \mu_2$.

Observe $\tilde{\beta} \wedge \Omega_k = d\Omega_k \Rightarrow d\tilde{\beta} \wedge \Omega_k = 0$, so $d\tilde{\beta}$ closed on $\mathcal{F}$. Thus, $\eta = \Im \tilde{\beta}$ closed on $\mathcal{F}$.

Locally write (x, y, t), $\mathcal{L}(x, y) =$ const. η vanish on $L \Rightarrow$

$$\eta = \alpha_1(x, y, t)dx + \alpha_2(x, y, t)dy.$$

$d\eta = 0 \Rightarrow \frac{\partial \alpha_1}{\partial t} = \frac{\partial \alpha_2}{\partial t} = 0$. Then

$$\eta = \alpha_1(x, y)dx + \alpha_2(x, y)dy,$$

$\Rightarrow \phi^*(\eta|_{\mathcal{L}_2}) = \eta|_{\mathcal{L}_1}$.

Near ∂U_k, $\tilde{\beta_k} = \beta_k \Rightarrow \Im\tilde{\beta} = \Im\beta$ on ∂U_k. By Stokes,

$$\begin{aligned}
\Phi_{\text{diff}}(\Im d\beta) &= \int \left(\int \Im\beta|_{\mathcal{L}_1} \right) \mu_1 + \int \left(\int \Im\beta|_{\mathcal{L}_2} \right) \mu_2 \\
&= \int \left(\int \eta|_{\mathcal{L}_1} \right) \mu_1 + \int \left(\int \eta|_{\mathcal{L}_2} \right) \mu_2 \\
&= \int \left(\int \eta|_{\mathcal{L}_1} \right) \mu_1 - \int \left(\int \phi_*\eta|_{\phi_*\mathcal{L}_2} \right) \phi_*\mu_1 \\
&= 0.
\end{aligned}$$

Thus, $\Re\Phi_{\text{diff}}(d\beta/2\pi i) = 0$. ■

If $C = \{zw = 0\}$, $\Omega_k = \frac{dz}{z} + \lambda(1 + \text{h.o.t.})\frac{dw}{w}$ close to a closed form. We can pick β holomorphic. Then OK as the $\lambda = -1$ case. If $C = \{w = 0\}$, the proof similar to the case $p \notin C$.

Then we get

$$N_{\mathcal{F}}(-C) \cdot [\Phi_{\text{diff}}] = \sum_{\text{Case 1}} \Phi_{\text{diff}} \left(\frac{d\beta_k}{2\pi i} \right) + \sum_{\text{Case 2}} \Phi_{\text{diff}} \left(\frac{d\beta_k}{2\pi i} \right),$$

where the LHS is real, the first term in the RHS is nonnegative, and the real part of the second term in the RHS is nonnegative. We conclude that the LHS is nonnegative.

18 Final Exam

"Kobayashi Hyperbolic and Higher-dimensional Nevanlinna Theory", Yamanoi.

THEOREM 18.1 (Bloch-Ochai Theorem). *Let A be an abelian variety, and $f\colon \mathbb{C} \to A$ an entire curve. Then $\overline{f(\mathbb{C})}^{\text{Zar}}$ is a translate of an abelian subvariety of A.*

Using Jet Space.