Selected Topics in Number Theory: Notes on Xie Junyi's classes

BAO without will of writing papers

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Abstract

These are the notes on Prof. Xie Junyi's classes.

Contents

1 Note on 20250218

REF:

- [1, Berkovich-Koblitz, 1990] Spectral theory and analytic geometry over non-archimedean fields.
- [2, Bosch-Güntzer-Remmert, 1984] Non-Archimedean analysis. A systematic approach to rigid analytic geometry.
- [3, Conrad, 2008] Several approaches to non-archimedean geometry.
- Lecture notes of Mattias Jonsson.

1.1 Berkovich Space

A language to study analytic geometry for non-archimedean field (analogue of complex geometry).

Recall complex manifold: Topological space + structure sheaf.

More generally: complex analytic space.

1.1.1 Valued field

Definition 1.1 (Valued fields). A valued field is a pair $(k, |\cdot|)$, where k is a field, $|\cdot|$ is norm, i.e., $|\cdot|\colon k \to \mathbb{R}_{\geq 0}$ s.t.

- $|a| = 0$ iff $a = 0$;
- $|a - b| \leq |a| + |b|$;
- $|ab| = |a||b|$ (multiplicative).

Definition 1.2. $(k, |\cdot|)$ is complete is k is a complete metric space under $|\cdot|$.

For arbitrary valued space $(k, |\cdot|)$, $(\hat{k}, |\cdot|) :=$ completion of $(k, |\cdot|)$.

Fact: $k = \bar{k} \Rightarrow \hat{k} = \bar{\hat{k}}$.

Example 1.3.
- $(\mathbb{C}, |\cdot|_\infty)$ with the standard Euclidean norm. Subfields: $(\mathbb{R}, |\cdot|_\infty)$, $(\mathbb{Q}, |\cdot|_\infty)$ ... $(\mathbb{R}, |\cdot|_\infty) = (\widehat{\mathbb{Q}}, |\cdot|_\infty)$.
- $k \subseteq \mathbb{C}$ subfield, $\varepsilon \in (0, 1]$. Let $|\cdot| = |\cdot|_\infty^\varepsilon$. Then $(k, |\cdot|)$ is a valued field.
- k any field, $|\cdot|_0 :=$ trivial norm, i.e. $|a| = 0$, $\forall a \in k^*$. $(k, |\cdot|_0)$ complete.
- (p-adic fields) $p =$ prime number. $\forall r \in \mathbb{Q}^*$, write $r = p^n \frac{a}{b}$, $p \nmid a, b$. Fix $c \in (0, 1)$ (e.g. $c = \frac{1}{p}$), $|r|_p := c^n$. The completion of $\mathbb{Q}$ for $|\cdot|_p$ is

$$\mathbb{Q}_p = \{\text{p-adic numbers}\}, \quad \widehat{\overline{\mathbb{Q}_p}} = \mathbb{C}_p.$$

- (Laurent series) $\tilde{k}$ =field,

$$k := \tilde{k}((+)) := \Big\{ f = \sum_{i=-\infty}^{+\infty} a_i T^i \mid a_i \in \tilde{k},\ a_i = 0 \text{ for } i \ll 0 \Big\}.$$

  Fix $c = (0,1)$ (e.g. $c = \mathrm{e}^{-1}$). Define $\mathrm{ord}_0 f := \min\{i \mid a_i \neq 0\}$, $|f| := c^{\mathrm{ord}_0 f}$. $(k, |\cdot|)$ is complete.

Definition 1.4. $(k, |\cdot|)$ is non-archimedean (NA) if $|a+b| \le \max\{|a|, |b|\}$ for every $a, b \in k$. A valued field (norm) that is not NA is called archimedean.

Fact: If $(k, |\cdot|)$ is archimedean and complete, then $k = \mathbb{R}$ or $\mathbb{C}$, and $|\cdot| = |\cdot|_\infty^\varepsilon$ for some $\varepsilon \in (0,1]$.

Proposition 1.5. *$(k, |\cdot|)$ is non-archimedean if $|n| \le 1$, $\forall n \in \mathbb{Z}$.*

Proof. "only if" part: $|n| = |1 + \cdots + 1| \le \ldots \le 1$.
"if" part: Pick $a, b \in k$. Then $\forall n$,

$$\begin{aligned} |a+b| = |(a+b)^n|^{\frac{1}{n}} &= \left| \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \right|^{\frac{1}{n}} \le \left(\sum_{i=0}^{n} \left| \binom{n}{i} \right| |a|^i |b|^{n-i} \right)^{\frac{1}{n}} \\ &\le ((n+1) \cdot \max\{|a|, |b|\}^n)^{\frac{1}{n}} = (n+1)^{\frac{1}{n}} \max\{|a|, |b|\} \\ &\to \max\{|a|, |b|\}\ (n \to +\infty). \end{aligned}$$

■

Focus on the NA case. $(k, |\cdot|)$ NA. Denote

$$k^\circ := \{a \in k \mid |a| \le 1\},\ k^{\circ\circ} := \{a \in k \mid |a| < 1\},$$

where k° is a ring and $k^{\circ\circ}$ is the maximal ideal of k°. $\tilde{k} := k^\circ / k^{\circ\circ}$ residue field. e.g.

$$\widetilde{\tilde{k}((+))} = \tilde{k}, \quad \widetilde{\mathbb{Q}_p} = \mathbb{F}_p, \quad (k, |\cdot|_0) \rightsquigarrow \tilde{k} = k.$$

Fact: If $(k, |\cdot|)$ is NA, then

- $|a| > |b| \Rightarrow |a+b| = |a|$;
- balls in k are both open and closed;
- k is totally disconnected, i.e. only connected subsets are singletons;
- B_1, B_2 balls, then $B_1 \cap B_2 = \varnothing$ of $B_1 \subset B_2$ or $B_2 \subset B_1$.

1.1.2 What is a space?

Naive "definition" of "$k-$manifold" X:

- X covered by $U_\alpha \simeq V_\alpha$ open in k^n;
- transition maps are analytic $V_{\alpha\beta} \to V_{\beta\alpha}$;

- analytic functions are locally convergent power series.

There are some problems:

- balls in k^n are open and closed, $X = \bigsqcup$balls, totally disconnected;
- too many analytic functions. e.g. $k = \mathbb{Q}_p$,

$$\overline{B}(0,1) = \mathbb{Z}_p = \bigsqcup_{i=0}^{p-1} i + p\mathbb{Z}_p,$$

$U_i = i + p\mathbb{Z}_p$, $f \coloneqq i$ on $U_i \Rightarrow$ no meaningful notion of analytic continuation.

General properties for a nice notion of space should have: X should consist of

1. a set X;
2. a topology on X;
3. a structure sheaf $\mathcal{O}_X$ on X, where the sections $\mathcal{O}_X(U)$ on $U \subset X$ open, consist of analytic functions on U.

Various approaches.

1. (J. Tate 1960s) rigid spaces. No topology but a Grothendieck topology.
2. (M. Raynaud 1970s) formal models of rigid spaces.
3. (V. Berkovich 1980s) $k-$analytic space (Berkovich space) (very good topology).
4. (R. Huber 1980s) adic space.

1.1.3 Berkovich affine space

(As a topological space)

$k = (k, |\cdot|)$ complete valued space. $R = k[T_1, \dots, T_n]$. $\operatorname{Max} R = \{\text{maximal ideals of } R\} \simeq k^n$ if $k = \bar{k}$. $\operatorname{Spec} R = \{\text{prime ideals of } R\} = \mathbb{A}_k^n$, algebraic geometry version of affine space.

Definition 1.6 (Berkovich version). $\mathbb{A}_k^{n,\mathbf{an}}$ is defined as a set

$$\mathbb{A}_k^{n,\mathbf{an}} \coloneqq \{\text{multiplicative seminorms on } R \text{ extending the norm on } k\}.$$

A seminorm $|\cdot|_x \colon R \to \mathbb{R}_{\geq 0}$ satisfies the conditions for being a norm except: $|a| = 0 \Rightarrow a = 0$.

Topology on $\mathbb{A}_k^{n,\mathbf{an}}$ is the weakest topology s.t.

$$\begin{aligned} \mathbb{A}_k^{n,\mathbf{an}} &\to \mathbb{R}_{\geq 0} \\ |\cdot|_x &\longmapsto |f|_x \end{aligned}$$

is continuous for every $f \in R$.

Base:

$$\{|f_1|_x < r_1, \dots, |f_m|_x < r_m, |g_1|_x > s_1, \dots, |g_l|_x > s_l\}.$$

Fact: $\mathbb{A}_k^{n,\mathbf{an}}$ is Hausdorff, locally compact and path connected. We have a canonical inclusion

$$\operatorname{Max} R \hookrightarrow \mathbb{A}_k^{n,\mathbf{an}}.$$

Proof. $\mathfrak{m} \in \operatorname{Max} R \Rightarrow R/\mathfrak{m}$ is a finite field extension of k. Thus, $|\cdot|$ in k canonically extends to a norm $|\cdot|'$ on $R/\mathfrak{m}$. Then

$$R \twoheadrightarrow R/\mathfrak{m} \xrightarrow{|\cdot|'} \mathbb{R}_{\geq 0}$$

defines a point $|\cdot|_{\mathfrak{m}} \in \mathbb{A}_k^{n,\mathbf{an}}$.

$$\begin{aligned} \operatorname{Max} R &\longrightarrow \mathbb{A}_k^{n,\mathbf{an}} \\ \mathfrak{m} &\longmapsto |\cdot|_{\mathfrak{m}}, \end{aligned}$$

$\mathfrak{m} = \{a \in R \mid |a|_{\mathfrak{m}} = 0\}$. ■

2 Note on 20250220

2.1 Why seminorm?

Fact: Assume $(k, |\cdot|) = (\mathbb{C}, |\cdot|_\infty)$. Then $\mathbb{A}^{n,\mathbf{an}}_{\mathbb{C}} = \mathbb{C}^n = \operatorname{Max} R$, $R = \mathbb{C}[T_1, \dots, T_n]$.

Proof. **Gelfand-Mazur Theorem**: If A is a complex Banach field, then $A \simeq \mathbb{C}$.
For $x \in \mathbb{A}^{n,\mathbf{an}}_{\mathbb{C}}$,

$$P_x \coloneqq \{f \in R \mid |f|_x = 0\},$$

which is a closed prime ideal of R. $|\cdot|_x$ induces a multiplicative norm on R/P_x, and thus $\widehat{\operatorname{Frac}(R/P_x)}$ is a complex Banach field, which is $\mathbb{C}$. This implies $R/P_x = \mathbb{C}$, $x \in \operatorname{Max} R$. ■

Algebraic geometry: In the definition of seminorm, we used $\times$, $<$ on $\mathbb{R}$ but not $+$.

$$(\mathbb{R}, \times, <) \supset \mathbb{B} = (\{0,1\}, \times, <).$$

Trivial norm: $|\cdot|_0 : k \to \mathbb{B} \subset \mathbb{R}$.

$$\operatorname{Spec} R = \{\text{multiplicative seminorms on } R,\ \text{valuated on } \mathbb{B}\} \hookrightarrow \mathbb{A}^{n,\mathbf{an}}_k,\ (k, |\cdot|_0),$$

which is a closed subset. Then the induced topology on $\operatorname{Spec} R$ is the constructible topology, i.e. topology generated by constructible subset = topology generated by

$$\{|f_1|_x = 0, \dots, |f_m|_x = 0, |g_1|_x = 1, \dots, |g_l|_x = 1\}.$$

AG (with constructible topology) $\sim$ "Berkovich space" for $\mathbb{B}$.

Example 2.1 (Affine line). Assume $k = \bar{k}$, $|\cdot| = |\cdot|_0$. Then $\mathbb{A}^{1,\mathbf{an}}_k =$

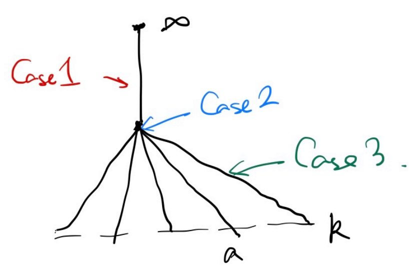

Figure 1: Affine line

Case 2: $|\cdot|_x$ trivial norm on $k[T]$, i.e. $|f|_x = 1$, $\forall f \neq 0$.

Case 1: $|T|_x > 1$, $r = |T|_x \in (1, +\infty)$. Then $|T - a|_x = r$, $\forall a \in k$. $\Rightarrow$ x is uniquely determined by r.

Case 3: $\exists a \in k$ s.t. $|T - a|_x < 1$. Set $r \coloneqq |T - a|_x \in [0, 1)$, then $|T - b|_x = |T - a + (a - b)|_x = 1$, $\forall b \neq a$, x determined by r.

EXAMPLE 2.2. Assume $k = \bar{k}$, $|\cdot| = |\cdot|_0$.

$$\begin{aligned}(\mathbb{R}_{>0}, x) &\hookrightarrow \mathbb{A}_k^{2,\mathbf{an}}\\ (r, |\cdot|_x) &\longmapsto |\cdot|_x^r.\end{aligned}$$

Example of points in $\mathbb{A}_k^{2,\mathbf{an}}$:

- $$\begin{gathered}\mathbb{A}_k^2 \hookrightarrow \mathbb{A}_k^{2,\mathbf{an}}\\ p \to R \to R/p \xrightarrow{|\cdot|_0} \mathbb{R}.\end{gathered}$$

  Indeed,
  $$\mathbb{A}_k^2 = (\mathbb{A}_k^{2,\mathbf{an}})^{\mathbb{R}_{>0}}.$$

- Divisorial valuations:
  $$\mathbb{A}_k^2 \subset \mathbb{P}_k^2 = \mathbb{A}_k^2 \sqcup L_\infty.$$
  Pick $\pi\colon X \xrightarrow{\text{bir.}} \mathbb{P}_k^2$ with X smooth projective. Let $E \subset X$ be a prime divisor, which defines a valuation
  $$\begin{aligned}k[T_1, T_2] = R &\longrightarrow \mathbb{R} \cup \{+\infty\}\\ P &\longmapsto \mathrm{ord}_E \pi^* P.\end{aligned}$$
  Pick $r \in \mathbb{R}_{>0}$. Then
  $$|\cdot| = x_E^r\colon P \longmapsto \mathrm{e}^{-r\mathrm{ord}_E \pi^* P}$$
  is a norm on R which belongs to $\mathbb{A}_k^{2,\mathbf{an}}$.

Fact: Divisorial valuations are dense in $\mathbb{A}_k^{2,\mathbf{an}}$.
Note: E could be contained in $\pi^{-1}(L_\infty)$.
Indeed,
$$\mathbb{A}_k^{2,\mathbf{an}} = \mathbb{D}_k^{2,\mathbf{an}} \sqcup V_\infty,$$
where
$$\mathbb{D}_k^{2,\mathbf{an}} = \{|\cdot|_x : |P|_x \le 1,\ \forall P \in R\} = \{|\cdot|_x : |T_1|_x \le 1,\ |T_2|_x \le 1\}.$$
The decomposition is preserved by the action of $\mathbb{R}_{>0}$.
$$x_E^r \in V_\infty \text{ iff } E \in \pi^{-1}(L_\infty).$$
$\mathbb{D}_k^{2,\mathbf{an}}$ is compact.
How V_∞ looks like?
Define
$$\overline{V_\infty} \coloneqq \{|\cdot|_x \in V_\infty \mid \max\{|T_1|_x, |T_2|_x\} = \mathrm{e}\} = V_\infty/\mathbb{R}_{>0}.$$
$\Rightarrow V_\infty = \overline{V_\infty} \times \mathbb{R}_{>0}$. Let $\mathbb{A}^2 \subset X$ a smooth projective compactification. $\partial X \coloneqq X \setminus \mathbb{A}^2$

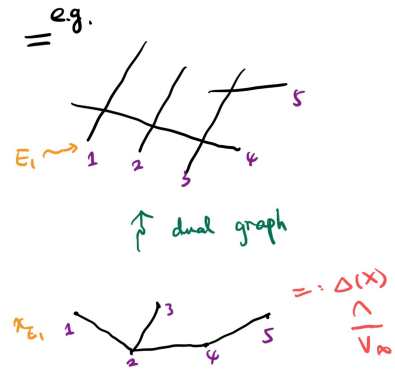

Figure 2: Example of ∂X

For higher model $X' \to X$, we have $\Delta(X) \subset \Delta(X')$, where the inverse contraction is denoted by r_x.

Fact : $\overline{V_\infty} = \varprojlim_X \Delta(X)$, which is a compact tree.

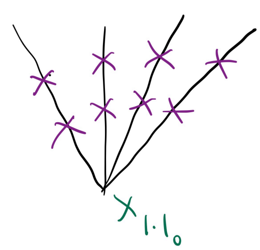

Figure 3: Picture of $\overline{V_\infty}$

There exists natural way to compactify $\mathbb{C}^2$ by $\overline{V_\infty}$.

THEOREM 2.3 (Friedland and Milnor). *$f\colon \mathbb{C}^2 \to \mathbb{C}^2$ polynomial automorphism. Assume $h_{\mathrm{top}}(f) > 0$. Then no $f-$periodic curve.*

A classical example of such automorphism is the Hénon map:

$$(x, y) \longmapsto (y, ax + P(y)),$$

where $\deg P \geq 2$, $a \neq 0$, $h_{\mathrm{top}}(f) = \log \deg P$.

Proof. Let $C \subset \mathbb{C}^2 \subset \mathbb{P}^2$ be an algebraic curve. $s =$ a branch of C at ∞ centered at o.

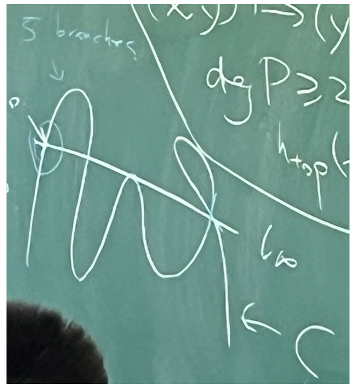

Figure 4: A branch s

Define $x_s = \overline{V_\infty}$ by

$$P \longmapsto \mathrm{e}^{-\mathrm{ord}_o(P|_S)\times r}$$

for some $r > 0$.

Dynamics of f on $\overline{V_\infty}$ with $h_{\mathrm{top}}(f) > 0$,

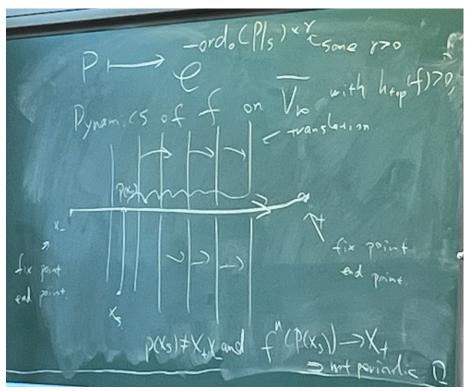

Figure 5: Dynamics of f on $\overline{V_\infty}$

$P(x_s) \neq x_+, x_-$ and $f^n(P(x_s)) \to x_+$, not periodic. ■

2.2 Seminormed groups

Analogy: In AG, local chart $X = \operatorname{Spec} A$ with ring A.

Berkovich: local chart $X = \mathcal{M}(A)$ with A Berkovich ring.

Definition 2.4 (Seminormed groups). Let $\mathcal{M}$ be an abelian group. A seminorm $\|\cdot\|$ on $\mathcal{M}$ is $\|\cdot\|\colon \mathcal{M} \to \mathbb{R}_{\geq 0}$, s.t.

- $\|0\| = 0$;

- $\|f - g\| \leq \|f\| + \|g\|$, and is a non-archimedean (NA) seminorm if
- $\|f - g\| \leq \max\{\|f\|, \|g\|\}$.

Note: $\|\cdot\|$ induces a topology on $\mathcal{M}$. $\|\cdot\|$ is a norm if and only if $(\mathcal{M}, \|\cdot\|)$ is Hausdorff.
Note: $\ker \|\cdot\| := \{m \in \mathcal{M} \mid \|m\| = 0\}$. $\|\cdot\|$ induces a norm on $\mathcal{M}/\ker\|\cdot\|$.

Definition 2.5. $\widehat{\mathcal{M}} :=$ completion of $\mathcal{M}/\ker\|\cdot\|$ for $\|\cdot\|$.

Example 2.6. If $\|\cdot\| = 0$, then $\widehat{\mathcal{M}} = 0$.

Definition 2.7. Two norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if $\exists C > 1$ s.t.

$$C^{-1}\|\cdot\| \leq \|\cdot\|' \leq C\|\cdot\|.$$

If $\mathcal{N} \subset \mathcal{M}$ is a subgroup, then we get a residue seminorm on $\mathcal{M}/\mathcal{N}$:

$$\|f\| := \inf\{\|g\| \colon g \in \pi^{-1}(f)\},$$

where $\pi\colon \mathcal{M} \to \mathcal{M}/\mathcal{N}$ is the quotient map.

Definition 2.8. Let $\mathcal{M}$, $\mathcal{N}$ be seminormed groups, and $\varphi\colon \mathcal{M} \to \mathcal{M}$ be group homomorphism. We say φ is <u>bounded</u>, if $\exists C > 0$ s.t.

$$\|\varphi(f)\| \leq C \cdot \|f\|.$$

We say φ is <u>admissible</u> if $(\mathcal{M}/\ker\varphi,\ \text{residue norm }) \simeq (\operatorname{im}\varphi,\ \text{subspace norm })$, i.e. $\exists C > 1$ s.t.

$$C^{-1} \cdot \inf\{\|f + g\| \colon g \in \ker\varphi\} \leq \|\varphi(f)\| \leq C \cdot \|f\|.$$

Example 2.9.

$$(\mathbb{Z}, \|\cdot\|_0) \xrightarrow{\text{id}} (\mathbb{Z}, \|\cdot\|_p)$$

is bounded, but not admissible.

3 Note on 20250225

3.1 A Lot of Definitions

3.1.1 Multiplicity

$A =$ (commutative ring), $\|\cdot\|$ seminorm on $(A, +)$ with $\|f \cdot g\| \leq \|f\| \cdot \|g\|$, $\forall f, g \in A$, which is submultiplicative.

Remark 3.1. *$\|1\| \geq 1$ or $\|f\| = 0$ for all $f \in A$.*

Definition 3.2. $\|f\|$ is power-multiplicative if $\|f^n\| = \|f\|^n$ for all $f \in A$ and $n \geq 1$, and is multiplicative if $\|fg\| = \|f\| \cdot \|g\|$.

Definition 3.3. A valuation on A is a multiplicative norm.

Definition 3.4. $(A, \|\cdot\|)$ is a Banach ring if the norm $\|\cdot\|$ is complete.

Example 3.5. Examples of Banach rings:

- $A =$ any ring, $\|\cdot\| := |\cdot|_0$ trivial ring.
- $(\mathbb{Z}, \|\cdot\|_\infty)$.
- Zero ring O, with norm $\|\cdot\| = 0$.
- $(\mathbb{R}, |\cdot|_\infty^\varepsilon)$, $\varepsilon \in (0, 1]$.
- $(\mathbb{Q}_p, |\cdot|_p)$, $(k((T)), \|\cdot\|_T)$.
- $(\mathbb{C}, \max\{|\cdot|_\infty, |\cdot|_0\})$.
- $(A, \|\cdot\|)$ Banach ring, $\mathfrak{a} \subset A$ closed ideal, then $(A/\mathfrak{a}, \text{residue norm})$ is a Banach ring.

  Fact: $\mathfrak{m} \subset A$ maximal ideal, then $\mathfrak{m}$ is closed.

  Proof. Assume $1 \neq 0$. Only need to show $1 \notin \overline{\mathfrak{m}}$. Otherwise, there exists $a \in A$ s.t. $1 - a \in \mathfrak{m}$ and $\|a\| < 1$. Thus, $1 + a + a^2 + \cdots$ converges, which implies that $1 - a$ is invertible. Contradiction! ∎

- If $\{A_i, \|\cdot\|)\}_{i \in I}$ Banach rings, then

  $$\prod_{i \in I} A_i := \{(a_i)_I, a_i \in A \mid \exists C > 0 \text{ s.t. } \|a_i\| \leq C,\ \forall i \in I\}$$

  with $\|(a_i)\| := \sup_i \|a_i\|$ is a Banach ring.

- If $(A, \|\cdot\|)$ is a Banach ring and $r > 0$, then

  $$A\langle r^{-1}T\rangle := \left\{ f = \sum_{i=0}^{\infty} a_i T^i \mid a_i \in A,\ \|f\| := \sum_{i=0}^{\infty} \|a_i\| r^i < \infty \right\}$$

  is a Banach ring.

3.1.2 Banach field

Definition 3.6. A Banach ring $(A, \|\cdot\|)$ is a Banach field if A is a field, and is NA if $\|\cdot\|$ is NA.

Note: $(A, \|\cdot\|)$ is a normed ring, $C \geq 1$, then $(A, C\cdot\|\cdot\|)$ is also a normed ring.

Question: Is there a best norm on A equivalent to $\|\cdot\|$?

Let $(A, \|\cdot\|)$ be a Banach ring, $f \in A$,

$$\begin{aligned} \mathbb{Z}_{\geq 0} &\longrightarrow \mathbb{R}_{\geq 0} \\ n &\longmapsto \|f^n\| \end{aligned}$$

is submultiplicative, i.e. $a_{n+m} \leq a_n \cdot a_m$, where $a_n = \|f^n\|$. Then the limit

$$\rho(f) \coloneqq \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}}$$

exists, and $\rho(f) = \inf_n \|f^n\|^{\frac{1}{n}}$, which is called the spectral radius of f.

Proposition 3.7.
- *$\rho(f^n) = \rho(f)^n$;*
- *$\rho(1) = 1$, (assume $\|\cdot\| \neq 0$);*
- *$\rho(fg) \leq \rho(f)\cdot\rho(g)$.*

Lemma 3.8. *$\rho(f-g) \leq \rho(f) + \rho(g)$. If NA, then $\rho(f-g) \leq \max\{\rho(f), \rho(g)\}$.*

Proof. Pick $\alpha > \rho(f)$, $\beta > \rho(g)$. $\exists C > 0$ s.t. $\|f^n\| \leq C\cdot\alpha^n$, $\|g^n\| \leq C\cdot\beta^n$. Then

$$\|(f-g)^n\| \leq \sum_{i=0}^{n}\binom{n}{i}\|f^i\|\|g^{n-i}\| \leq C^2\sum_{i=0}^{n}\binom{n}{i}\alpha^i\beta^{n-i} = C^2(\alpha+\beta)^n.$$

$\Rightarrow \rho(f-g) \leq \alpha + \beta$. ∎

theorem 3.9. *ρ is a seminorm, and if $\|\cdot\|$ is NA then ρ is NA.*

Remark 3.10. *If $\|\cdot\|$ and $\|\cdot\|'$ are equivalent, then ρ for $\|\cdot\|$ and $\|\cdot\|'$ are the same.*

ρ may not be a norm even when $(A, \|\cdot\|)$ is a Banach ring.

Example 3.11. $A = \mathbb{C}[\varepsilon]/(\varepsilon^2)$, then $\rho(\varepsilon) = 0$.

Definition 3.12. $A^u \coloneqq$ separated completion for (A, ρ), $A \to A^u$ is bounded, has the universal property: for any

$$\begin{tikzcd} A \arrow[r] \arrow[d] & (B, \|\cdot\|) \\ A^u \arrow[ur, "\exists!"] & \end{tikzcd}$$

where $\|\cdot\|$ is a power-multiplicative complete norm on B. A^u is called the uniformization of A.

3.1.3 Banach module

Definition 3.13. $A =$ (semi)normed ring, $(M, \|\cdot\|)$ is a (semi)normed $A-$module if

- M is an $A-$module;
- $\|\cdot\|$ is a (semi)norm on M as additive group;
- $\exists C > 0$ s.t. $\|fm\| \le C \cdot \|f\| \cdot \|m\|$ for all $f \in A$, $m \in M$.

Definition 3.14. If $(M, \|\cdot\|)$ is complete, we call it Banach $A-$module.

Definition 3.15 (Complete tensor product). Let A be a normed ring and NA. Let M, N be seminormed NA $A-$modules. Denote $M \otimes_A N$ the usual tensor product. Define a seminorm on $M \otimes_A N$ by

$$\|v\| := \inf\left\{ \max_i \|m_i\| \cdot \|n_i\| \colon v = \sum_i m_i \otimes n_i \right\}.$$

$M \widehat{\otimes}_A N :=$ separated completion of $M \otimes_A N$ w.r.t. the above seminorm. Then $M\widehat{\otimes}_A N$ is a Banach $A-$module ($\widehat{A}-$module). It has the universal property

$$\begin{tikzcd} M \times N \arrow[r] \arrow[dr, "\text{bil. bd.}"'] & M\widehat{\otimes}_A N \arrow[d, "\exists !"] \\ & P \end{tikzcd}$$

where P is any Banach $A-$module.

3.1.4 Berkovich spectrum

Definition 3.16. Let $(A, \|\cdot\|)$ be a Banach ring. Then (Berkovich) spectrum $\mathcal{M}(A)$ is (as a set)

$$\mathcal{M}(A) = \{\text{nontrivial multiplicative seminorm } |\cdot| \colon A \to \mathbb{R}_{\ge 0} \mid |f| \le \|f\|,\ \forall f \in A\}.$$

And the topology on $\mathcal{M}(A)$ is the weakest topology s.t.

$$\begin{aligned} \mathcal{M}(A) &\longrightarrow \mathbb{R}_{\ge 0} \\ |\cdot|_x &\longmapsto |f|_x \end{aligned}$$

is continuous for all $f \in A$.

Example 3.17. If $A = O$, then $\mathcal{M}(A) = \varnothing$.

theorem 3.18 (Berkovich). *If $A \ne O$, then $\mathcal{M}(A)$ is a non-empty, compact, Hausdorff topological space.*

Proof. Consider

$$\begin{aligned} \phi := \mathcal{M}(A) &\longrightarrow \prod_{f \in A} [0, \|f\|] \\ |\cdot|_x &\longmapsto (|f|_x)_{f \in A}. \end{aligned}$$

Easy to check

- ϕ is injective;
- the topology on $\mathcal{M}(A)$ is induced by ϕ, which shows that $\mathcal{M}(A)$ is Hausdorff;
- we have

  $$\phi(\mathcal{M}(A)) = \Big\{(a_f)_{f\in A} \in \prod [0, \|f\|] \colon a_1 = 1, a_0 = 0, a_{f-g} \le a_f + a_g, a_{fg} = a_f \cdot a_g, \forall f, g \in A\Big\},$$

  which is closed.

Now we show $\mathcal{M}(A) \neq \varnothing$.

1. <u>Reduce to A =field</u>.

   By Zorn's lemma, there exists $\mathfrak{m} \subset A$ maximal ideal, thus closed. Replace A by $A/\mathfrak{m}$ with the residue norm, where $A/\mathfrak{m}$ is a field.

2. <u>May assume $\|\cdot\|$ is minimal in nontrivial seminorms: if $\|\cdot\|' \le \|\cdot\|$, then $\|\cdot\|' = \|\cdot\|$</u>.

   If there exists a decreasing chain $\|\cdot\|_i$, $i \in I$ of nontrivial seminorms, then $\|\cdot\|' \coloneqq \inf\{\|\cdot\| \colon i \in I\}$ is a seminorm, and nontrivial as $\|1\|_i \ge 1$ for all i. Zorn's lemma implies there exists minimal nontrivial seminorm $\|\cdot\|$.

3. <u>Replace $(A, \|\cdot\|)$ by $(\widehat{A}, \|\cdot\|')$ for $\|\cdot\|'$ which is minimal and complete</u>.

Since $\rho \le \|\cdot\|$, we have $\rho = \|\cdot\|$, which shows $\|\cdot\|$ is power-multiplicative.
Only need to show $\|\cdot\|$ is multiplicative.
<u>Claim</u>: $\|f^{-1}\| = \|f\|^{-1}$ for all $0 \neq f \in A$.

Proof of Claim. If the claim is not true, the there exists $f \neq 0$ s.t. $\|f^{-1}\| > \|f\|^{-1}$. Set $r \coloneqq \|f^{-1}\|^{-1}$, so $\|f\| > r$. Consider $A' \coloneqq A\langle r^{-1}T\rangle$. Then $f - T = f(1 - f^{-1}T)$ is not invertible, since $\rho(f^{-1}T) = \|f^{-1}\|r = 1$.

Define

$$A'' \coloneqq \Big(A'/\overline{(f-T)}, \|\cdot\|_{\mathrm{res}}\Big).$$

We have

$$A \xrightarrow{\text{isometry embedding}} A' \xrightarrow{\|\cdot\|' \ge \|\cdot\|''} A''.$$

Pull back $\|\cdot\|''$ to A, and we have $\|\cdot\|'' \le \|\cdot\|$. But $\|f\|'' \le \|T\|' = r < \|f\|$. Contradiction! ∎

Using the claim, for any $f \neq 0$, we have

$$\|g\| = \|f^{-1} \cdot f \cdot f\| \le \|f^{-1}\| \cdot \|fg\| = \|f\|^{-1}\|fg\|,$$

which finishes the proof. ∎

4 Note on 20250227

I did not attend this class because of the seminar in AMSS, but there may be not very much new knowledge in this class I wonder.

5 Note on 20250311

5.1 Fibers

Let $\varphi^*\colon A \to B$ be a bounded map of Banach rings (NA). Induce $\varphi\colon \mathcal{M}(B) \to \mathcal{M}(A)$ (write as $Y \to X$). What is the fiber of φ at $x \in X$?

Exercise $\varphi^{-1}(x) \simeq \mathcal{M}(\mathcal{H}(x)\widehat{\otimes}_A B)$.

In particular, if $x \notin \varphi(\mathcal{M}(B))$, i.e. $\varphi^{-1}(x) = \varnothing$, then $\mathcal{H}(x)\widehat{\otimes}_A B = 0$.

Proposition 5.1 (Ground field extension). *k: NA valued field, K: NA valued field extending k, A: (NA) Banach $k-$algebra, $X := \mathcal{M}(A)$, $A_K := A\widehat{\otimes}_k K$ and $X_K := \mathcal{M}(A_K)$.*

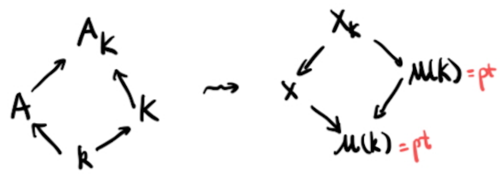

Figure 6: Ground field extension

Then $\forall x \in X$, the fiber of $X_K \to X$ at x is $\mathcal{M}(\mathcal{H}(x)\widehat{\otimes}_k K)$.

Note: $\mathcal{H}(x)\widehat{\otimes}_k K$ may not be a field.

theorem 5.2 (Gruson). *If V and W are NA $k-$Banach space, then the canonical map $V \otimes W \to V\widehat{\otimes}_k W$ is injective.*

Corollary 5.3. *$\phi\colon X_K \to X$ is surjective.*

Proof. $\forall x \in X$, $\phi^{-1}(x) = \mathcal{M}(\mathcal{H}(x)\widehat{\otimes}_k K)$. As $\mathcal{H}(x) \otimes_k K \hookrightarrow \mathcal{H}(x)\widehat{\otimes}_k K$, which are both nonempty, which implies $\mathcal{M}(\mathcal{H}(x)\widehat{\otimes}_k K) \neq \varnothing$. ■

Proposition 5.4. *Let A be a Banach $k-$algebra, and K/k be a finite normal extension (valuation on k extends to K).*

- *If K separable over k, then*

$$\mathcal{M}(A \otimes_k K)\big/ \operatorname{Gal}(K/k) \simeq \mathcal{M}(A).$$

- *If K purely inseparable over k, then*

$$\mathcal{M}(A \otimes_k K) \simeq \mathcal{M}(A).$$

Proof. $\phi\colon \mathcal{M}(A \otimes K) \to \mathcal{M}(A)$. $\forall x \in \mathcal{M}(A)$, $\phi^{-1}(x) = \mathcal{M}(\mathcal{H}(x) \otimes_k K) = \operatorname{Max}\{\mathcal{H}(x) \otimes_k K\}$. ■

A valuation on k extends uniquely to $\bar{k}$ hence $\widehat{\bar{k}}$. $\mathrm{Gal}(K/k)$ acts on $\widehat{\bar{k}}$ by isometries.

COROLLARY 5.5.
$$\mathcal{M}\big(A\widehat{\otimes}_k\widehat{\bar{k}}\big)\Big/\mathrm{Gal}(\bar{k}/k)\simeq\mathcal{M}(A).$$

EXAMPLE 5.6. 1. $\mathcal{M}(\mathbb{Z},\|\cdot\|_\infty)$:

$\forall x\in\mathcal{M}(A)$ corresponds to a seminorm $|\cdot|_x$ on $\mathbb{Z}$ and gives a prime ideal $\mathfrak{p}_x=\ker|\cdot|_x\subseteq\mathbb{Z}$.

Case 1: there exists p s.t. $\mathfrak{p}_x=p\mathbb{Z}$. Then $|\cdot|_x$ induces a norm on $\mathbb{Z}/p=\mathbb{F}_p$.
Claim: any non-trivial multiplicative norm on $\mathbb{F}_p$ is the trivial norm.

Proof. $\forall a\neq 0$, $\exists m\geq 1$ s.t. $a^m=1\Rightarrow |a|=1$. ■

$\Rightarrow\forall a\in\mathbb{Z}$,
$$|a|_x=\begin{cases}0 & \text{if} \quad p\mid a,\\ 1 & \text{if} \quad p\nmid a.\end{cases}$$

Case 2: We have $\mathfrak{p}_x=(0)$. In this case $|\cdot|_x$ extends to a valuation on $\mathbb{Q}$ by multiplicity.

THEOREM 5.7 (Ostrowski). *Any valuation on $\mathbb{Q}$ is one of the following:*

(a) trivial norm: $|\cdot|_0$;
(b) the Euclidean norm $|\cdot|_\infty^\varepsilon$, $\varepsilon\in(0,1]$;
(c) then $p-$adic norm $|\cdot|_p^\varepsilon$, $\varepsilon\in(0,+\infty)$.

Proof. Assume $|\cdot|_x$ a nontrivial valuation on $\mathbb{Q}$. Assume $\exists\iota\in\mathbb{Z}_{>1}$, s.t. $|\iota|_x>1$, then we have $|\iota|_x\leq|\iota|\times|1|_x=\iota\Rightarrow\varepsilon\in(0,1]$ s.t. $|\iota|_x=|\iota|_\infty^\varepsilon$.
$\forall m,n\in\mathbb{Z}_{>1}$, write $n=\sum_{i=0}^{r_n}a_im^i$, where $a_i\in\{0,\dots,m-1\}$.
$$r_n=\left[\frac{\log n}{\log m}\right]\leq\frac{\log n}{\log m}.$$
$|n|_x\leq r_n\times(m-1)\times\max\{|m|^{r_n},1\}$, $\Rightarrow$ $(\forall s\geq 1)$
$$|n|_x\leq\left(\frac{\log n}{\log m}\times s\times(m-1)\right)^{1/s}\times\max\{|m|_x^{\frac{\log n}{\log m}},1\}.$$
Let $s\to+\infty$:
$$|n|_x\leq\max\{|m|_x^{\frac{\log n}{\log m}},1\}.$$
Set $n=\iota$. $1<|\iota|x\leq\max\{|m|_x^{\frac{\log n}{\log m}},1\}\Rightarrow|m|_x>1$, $\forall m\in\mathbb{Z}_{>1}$ and $|m|_x\geq|m|_\infty^\varepsilon$.
Set $m=\iota$. $|n|_x\leq|\iota|_x^{\frac{\log n}{\log\iota}}=|\iota|_\infty^{\frac{\log n}{\log\iota}}=|n|_\infty^\varepsilon$.
Thus, $|n|_x=|n|_\infty^\varepsilon$, $\forall n\in\mathbb{Z}_{>1}$. $|-1|_x=1$. OK.
Now assume $|n|_x\leq 1$, $\forall n\in\mathbb{Z}$. Then $|\cdot|_x$ is NA. If nontrivial. $\exists n\in\mathbb{Z}_{>0}$ with $|n|_x<1$ $\Rightarrow$ prime $p\mid n$ with $|p|_x<1$.
$\forall m\in\mathbb{Z}$, $p\nmid m$, $\exists a,b\in\mathbb{Z}$ s.t. $am+bp=1\Rightarrow|am|_x=|1-bp|_x=1\Rightarrow|m|_x=1$.
$\Rightarrow\forall\iota\in\mathbb{Z}_{\geq 0}$, $\iota=m\cdot p^{\mathrm{ord}_p\iota}$, $p\nmid m$. We have $|\iota|=|p|_x^{\mathrm{ord}_p\iota}$. ■

Figure 7: Valuations on $\mathbb{Q}$

Topology

– On each branch:

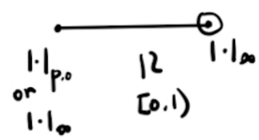

Figure 8: Topology on branches

– A base of open neighborhood of $|\cdot|_0$:

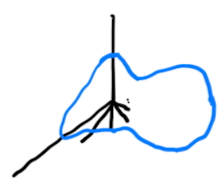

Figure 9: Open neighborhood of $|\cdot|_0$

Contains all but finitely many branches. (Poineau)

Two maps

– Kernel map:

$$\begin{aligned} \ker\colon \mathcal{M}(A) &\longrightarrow \operatorname{Spec} A \\ x &\longmapsto \ker(|\cdot|_x), \end{aligned}$$

which is continuous, and $\ker^{-1}(\{f \neq 0\}) = \{x \colon |f|_x > 0\}$.
e.g. $\ker\colon \mathcal{M}(\mathbb{Z}, \|\cdot\|_\infty) \to \operatorname{Spec}\mathbb{Z}$:

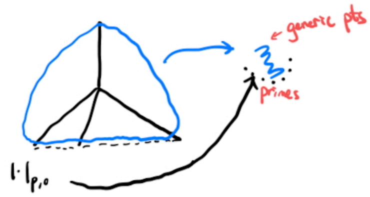

Figure 10: Kernel map of $\mathcal{M}(\mathbb{Z}, \|\cdot\|_\infty) \to \operatorname{Spec}\mathbb{Z}$

If $\|\cdot\|$ on A is the trivial norm, then there exists canonical section

$$\begin{aligned} \operatorname{Spec} A &\longrightarrow \mathcal{M}(A) \\ p &\longmapsto |\cdot|_{p,0}, \end{aligned}$$

where

$$|\cdot|_{p,0} \colon f \mapsto \begin{cases} 0 & f \in p, \\ 1 & f \notin p. \end{cases}$$

– <u>Reduction map (NA)</u>

$$\begin{aligned} \operatorname{red}\colon \mathcal{M}(A) &\longrightarrow \operatorname{Spec} A \\ x &\longmapsto \{|\cdot|_x < 1\}, \end{aligned}$$

where $\{|\cdot|_x < 1\}$ is a prime ideal. If A is noetherian, then red is anticontinuous, i.e. $\operatorname{red}^{-1}(\text{open}) = \text{closed}$, $\operatorname{red}^{-1}(\{f \neq 0\}) = \{|f(x)| = 1\}$.

6 Note on 20250313

6.1 Some examples

Assume $\|\cdot\|$ on A is a trivial norm.

Case 1: A is a field, $\mathcal{M}(A)$ =point.

Case 2: Assume A is a DVR, e.g. $A = \mathbb{Z}_p$ or $k[[T]]$, $\mathfrak{m}$ is the unique maximal ideal, $\mathfrak{m} = (\pi)$, π: uniformizer. Then $\mathcal{M}(A) =$

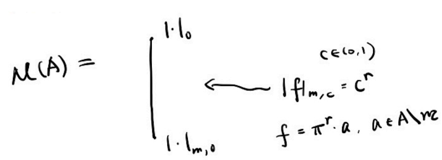

Figure 11: $\mathcal{M}(A)$ in Case 2

$c \in (0,1)$, $|f|_{\mathfrak{m},c} = c^r$, where $f = \pi^r \cdot a$, $a \in A \setminus \mathfrak{m}$.

Proof. – $g \in A$, $|g|_x \leq 1$.

– $\forall g \in A \setminus \mathfrak{m}$, $|g \cdot g^{-1}|_x = 1$, $\Rightarrow |g|_x = 1$.

∎

The ker map:

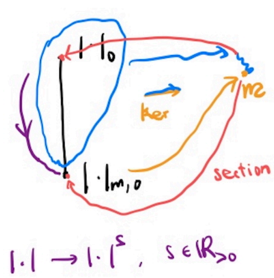

Figure 12: ker map in Case 2

The red map:

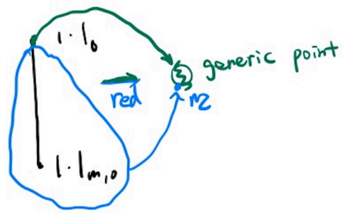

Figure 13: red map in Case 2

Case 3: Global version of Case 2.

A is a Dedekind domain (e.g. $\mathbb{Z}$, $k[T]$), with $\|\cdot\|$ trivial norm.

$\forall x \in \mathcal{M}(A)$, set

$$\mathfrak{p} = \ker(x) = \{f\colon |f|_x = 0\} \quad \mathfrak{q} = \mathrm{red}(x) = \{f\colon |f|_x < 1\},$$

so $0 \subseteq \mathfrak{p} \subset \mathfrak{q} \subset A$. Since A is Dedekind domain, we have the following 3 cases:

Case (a) $\mathfrak{p} = \mathfrak{q} = 0$. Then $|\cdot|_x = |\cdot|_0$ the trivial norm.

Case (b) $\mathfrak{p} = \mathfrak{q}$ =maximal ideal. $|\cdot|_x = |\cdot|_{\mathfrak{p},0}$.

Case (c) $0 = \mathfrak{p} \subsetneqq \mathfrak{q}$ =maximal ideal. Then $\forall f \in A \setminus \mathfrak{q}$, $|f|_x = 1$, $|\cdot|_x$ extends to a norm in $\mathcal{M}(A_{\mathfrak{q}}, |\cdot|_0)$, where $A_{\mathfrak{q}}$ is a DVR. $|\cdot|_x = |\cdot|_{\mathfrak{q},c}$, $c \in (0,1)$:

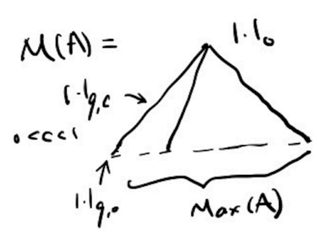

Figure 14: $\mathcal{M}(A)$ in Case 3

Topology:

* On each branch:

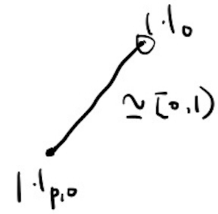

Figure 15: Topology on each branch

* A base of open neighborhood of $|\cdot|_0$:

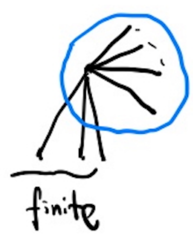

Figure 16: Base of open neighborhood

The ker map:

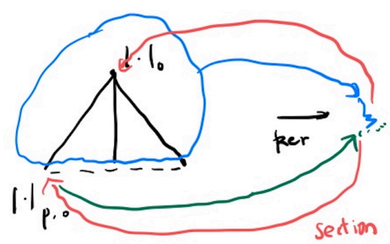

Figure 17: ker map in Case 3

The red map:

Figure 18: red map in Case 3

6.2 Berkovich's closed disc (Preliminaries)

$(k, |\cdot|)$ =complete valued field, $r > 0$ (NA or note NA).

$$\mathcal{A} = k\langle r^{-1}T\rangle \coloneqq \Big\{ f = \sum_{i=0}^{\infty} a_i T^i \colon a_i \in k,\ \|f\| = \sum_{i=0}^{\infty} |a_i| r^i < \infty \Big\}.$$

- Even when k is NA, $\mathcal{A}$ is note NA. e.g., $\|1\| = 1$, $\|T\| = r$, $\|1 + T\| = 1 + r$.
- $\mathcal{A}$ is not uniform.

Assume k NA.

$$k\{r^{-1}T\} \coloneqq \Big\{ \sum_{i=0}^{\infty} a_i T^i \colon a_i \in k,\ \lim_{r\to\infty} |a_i| r^i = 0 \Big\},$$

with norm $\|f\| \coloneqq \max_i |a_i| r^i$.

Lemma 6.1 (Gauss's Lemma). *Then norm $\|\cdot\|$ on $k\{r^{-1}T\}$ is multiplicative.*

Proof. Let $f = \sum a_i T^i$, $g = \sum b_j T^j$, and $fg = \sum c_i T^i$. Only need to show $\|fg\| \geq \|f\|\|g\|$. Pick I minimal s.t. $\|f\| = |a_I| r^I$, and J minimal s.t. $\|g\| = |b_J| r^J$. Then

$$|c_{I+J}| = \left| \sum_{i+j=I+J} a_i b_j \right| = |a_I||b_J|,$$

so $\|fg\| \geq |c_{I+J}| r^{I+J} = \|f\|\|g\|$. ■

Note that

$$(k\{r^{-1}T\}, \|\cdot\|) \simeq A^u \Rightarrow \mathcal{M}(k\{r^{-1}T\}) = \mathcal{M}(k\langle r^{-1}T\rangle),$$

Gauss's Lemma $\Rightarrow \|\cdot\| \in \mathcal{M}(k\{r^{-1}T\})$ a maximal element.

Definition 6.2. Let

$$E(r) \coloneqq \mathcal{M}(k\{r^{-1}T\}),$$

be the Berkovich's closed disc of radius r over k.

Study its set/topology.

A general strategy:

$$(\dim m) \quad X = \mathcal{M}(A) \xrightarrow{\text{finite map}} E(r_1, \ldots, r_m) \quad \text{“polydisc”},$$

where A is a $k-$affinoid algebra, we can see the map as the “Noether normalization”.

See $E_k(r_1, \ldots, r_m)$ as a fibration over $E_k(r_1, \ldots, r_{m-1})$. Total space=Base+Fiber, where Base is $E_k(r_1, \ldots, r_{m-1})$, Fiber: $\forall x \in E_k(r_1, \ldots, r_{m-1})$, $\pi^{-1}(x) = E_{\mathcal{H}(x)}(r_m)$ a disc.

The trivially valued case:

$$k\{r^{-1}T\} = \begin{cases} k[[T]] & r < 1, \\ k[T] & r \geq 1. \end{cases}$$

- If $r = 1$, $E(1)$ =Berkovich's closed unit disc. $\|\cdot\|$ =trivial norm on $k[T]$ (Dedekind domain).

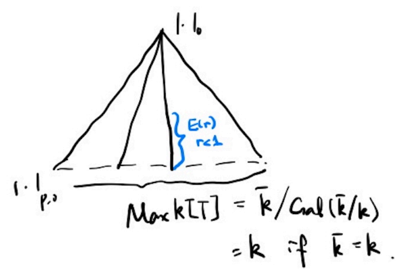

Figure 19: $E(1)$ and $E(r)$ for $0 < r < 1$

- If $0 < r < 1$, then $k\{r^{-1}T\} = k[[T]]$, with

$$\left\| f = \sum_{i=0}^{\infty} a_i T^i \right\| = \max |a_i| r^i = r^{\mathrm{ord}_0(f)},$$

where $\mathrm{ord}_0(f) = \min\{i \colon a_i \neq 0\}$. $\forall x \in \mathcal{M}(A)$, s.t. $t := |T(x)| \in [0, r]$, x is determined uniquely by t as $|f(x)| = t^{\mathrm{ord}_0(f)}$.

Remark 6.3. *$r < s$, we have $k\{s^{-1}T\} \to k\{r^{-1}T\}$, bounded map not admissible. It induces the embedding: $E(r) \hookrightarrow E(s)$ as topological space.*

- If $r > 1$, then $k\{r^{-1}T\} = k[T]$, with norm

$$\left\| f = \sum_{i=0}^{\infty} a_i T^i \right\| = \max |a_i| r^i = r^{\deg f}.$$

Pick $x \in \mathcal{M}(A)$, set $t = |T(x)| \in [0, r]$. If $t \leq 1$, $|\cdot|_x \leq |\cdot|_0 \Rightarrow x \in E(1)$. For $t \in [1, r]$, x is uniquely determined by t as $|f(x)| = t^{\deg f}$. $\Rightarrow E(r) =$

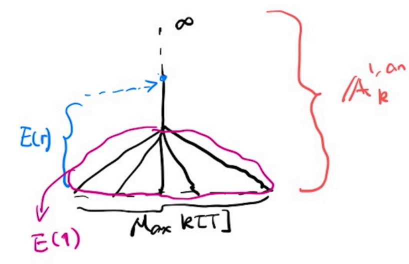

Figure 20: $E(r)$ for $r > 1$.

7 Note on 20250318

7.1 Points in Berkovich's closed disc

In the following, we consider the non-trivial valued, algebraically closed case.
k =complete NA field, $k = \bar{k}$, $|\cdot| \neq |\cdot|_0$.

DEFINITION 7.1. The Berkovich's disc is defined by

$$E(r) := \mathcal{M}\big(k\{r^{-1}T\}\big), \quad r > 0,$$

and the rigid disc is defined by

$$\overline{\mathbb{D}}(r) := \big\{a \in k\colon |a| \leq k\big\}.$$

REMARK 7.2. *As $k = \bar{k}$, $x \in E(r)$ is uniquely determined by $|(T-a)(x)|$, $\forall a \in k$.*
Reason: $k[T] \subseteq k\{r^{-1}T\}$ dense + for any $p \in k[T]$, $p = c(T-a_1)\cdots(T-a_n)$.

There are 4 types of points in $E(r)$.

Type 1 points Given $a \in \overline{\mathbb{D}}(r) \subseteq k$, define $|\cdot|_a \in E(r)$ by $|f|_a := |f(a)|$. $|\cdot|_a$ is a seminorm with $\ker|\cdot|_a = (T-a)$.

$$\begin{aligned}\overline{\mathbb{D}}(r) &\hookrightarrow E(r)\\ a &\mapsto |\cdot|_a\end{aligned}$$

is a homeomorphism on to its image (not clear, indeed dense).
$\forall x \in E(r)$, if $\ker|\cdot|_x \neq 0$, then x is Type 1.
Indeed, all prime ideals of $k\{r^{-1}T\}$ take the form $T-a$, $a \in \overline{\mathbb{D}}(r)$. For $f = \sum_{i\geq 0} a_i T^i \neq 0$, there exists I maximal with $a_I r^I = \|f\|$. Hensel's Lemma implies $f = p \times U$, where $p \in k[T]$ with $\deg p = I$, and U is invertible.
If $\ker|\cdot|_x \neq 0$, then $\ker|\cdot|_x = (T-a)$, and thus $|\cdot|_x = |\cdot|_a$.

The remaining points in $E(r)$ correspond to multiplicative norms on $k\{r^{-1}T\}$.
We study them using discs in $\overline{\mathbb{D}}(r), E(r)$.

DEFINITION 7.3. Given $a \in \overline{\mathbb{D}}(r)$, $0 < \rho \leq r$, define

$$\overline{\mathbb{D}}(a,\rho) := \big\{b \in k\colon |b-a| \leq \rho\big\} \subseteq \overline{\mathbb{D}}(r);$$

$$E(a,\rho) := \big\{x \in E(r) := |(T-a)(x)| \leq \rho\big\} \subseteq E(r).$$

Facts:

(1) $E(r) \cap \overline{\mathbb{D}}(a,\rho) = E(a,\rho) \cap \overline{\mathbb{D}}(r)$;

(2) $\overline{\mathbb{D}}(a,\rho) = \overline{\mathbb{D}}(b,\rho)$ ($E(a,\rho) = E(b,\rho)$ respectively) iff $|a-b| \leq \rho$;

(3) The radius $\rho(\overline{\mathbb{D}}(\cdot))$ is well-defined, i.e., $\overline{\mathbb{D}}(a,\rho) = \overline{\mathbb{D}}(b,\rho')$, then $\rho = \rho'$ (Here need k non-trivial, algebraically closed);

(4) Given $\overline{\mathbb{D}}_1, \overline{\mathbb{D}}_2$, either $\overline{\mathbb{D}}_1 \cap \overline{\mathbb{D}}_2 = \varnothing$ or $\overline{\mathbb{D}}_1 \subseteq \overline{\mathbb{D}}_2$ or $\overline{\mathbb{D}}_2 \subseteq \overline{\mathbb{D}}_1$ (same for E_1, E_2);

(5) If $a \in \overline{\mathbb{D}}(0,r)$, then $T \to T - a$ is a bounded map of $k\{r^{-1}T\}$, inducing a homeomorphism of $E(r)$ sending $E(b,\rho)$ to $E(a+b,\rho)$.

Definition 7.4. The valued group of k is

$$|k^* := \{|a| \colon a \in k^*\},$$

which is a subgroup of $(\mathbb{R}_{>0}, \times)$.

k non-trivially valued $\Leftrightarrow |k^*| \neq \{1\}$; $k = \bar{k} \Rightarrow |k^*|$ divisible. Thus, $|k^*|$ is dense in $\mathbb{R}_{>0}$.

Remark 7.5. *If $E_1, E_2 \subseteq E(r)$, then there exists affine map $T \to aT + b$ mapping E_1 onto E_2 iff $\rho(E_1)/\rho(E_1) \in |k^*|$ (same for $\overline{\mathbb{D}}_1, \overline{\mathbb{D}}_2$).*

Type 2 / Type 3 points Given a closed disc $E = E(a,\rho) \subseteq E(r)$, define a norm $|\cdot|_E$ on $k\{r^{-1}T\}$ by

$$|f|_E := \max_i |c_i|\rho^i,$$

where $f = \sum_{i=0}^{\infty} c_i (T-a)^i$. We may check that $\|\cdot\|_E$ is a multiplicative norm $\leq \|\cdot\|$. $|\cdot|_E$ only depends on E not on a.

$p(E) := \|\cdot\|_E$.

Set $\rho(E) := \rho$ =radius of E.

If $\rho = 0$, $|\cdot|_E = |\cdot|_a$ Type 1.

Definition 7.6. $x \in E(r)$ is called

- Type 2 if $x = p(E)$ with $\rho \in |k^*|$.

Type 3 if $x = p(E)$ with $\rho \notin |k^*|$.

Facts:

- $p(E) \in E$ and it is the maximal point in E, i.e., $|f(x)| \leq |f|_E$ for all $x \in E$ and $f \in k\{r^{-1}T\}$.
- $E \subseteq E'$ iff $|\cdot|_E \leq |\cdot|_{E'}$ on $k\{r^{-1}T\}$.
- Given $f \in k\{r^{-1}T\}$,
$$|f|_E = \sup\left\{|f(b)| \colon b \in E \cap \overline{\mathbb{D}}(r)\right\}.$$
Moreover, if $p(E)$ is Type 2, the supremum is attained.
- Given $a \in \overline{\mathbb{D}}(r)$, the map
$$\begin{aligned} [0,r] &\longrightarrow E(r) \\ \rho &\longmapsto p(E(a,\rho)) \end{aligned}$$
is a homeomorphism onto its image.
- On each sagement, Type 2 points are dense as $|k^*| \subseteq \mathbb{R}_{>0}$ is dense.
- At each Type 2 point, there exists infinite branches ($\simeq \mathbb{P}^1(\tilde{k})$).
- At Type 3 point, no branch.

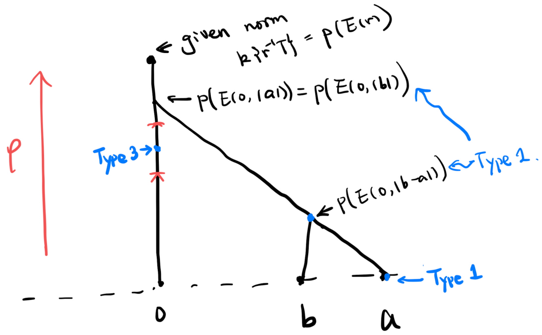

Figure 21: Type 2 and Type 3 points

Type 4 points

Definition 7.7. A collection $\mathcal{E}$ of discs in $E(r)$ is <u>nested</u> if

(i) $E, E' \in \mathcal{E}$, $E \subset E'$ iff $\rho(E) \leq \rho(E')$;

(ii) $E \in \mathcal{E}$ and $E \subset E' \subseteq E(r)$, then $E' \in \mathcal{E}$.

Then $\rho\colon \mathcal{E} \longrightarrow \mathbb{R}_{>0}$ is a bijection to $[s,r]$ or $(s,r]$, $0 \leq s \leq r$. Define

$$|f|_{\mathcal{E}} \coloneqq \inf_{E \in \mathcal{E}} |f|_E,$$

which is a multiplicative seminorm. Indeed, $|f|_E$ is decreasing when $\rho \downarrow s$.

$p(\mathcal{E})$:=the corresponding point in $E(r)$. Set

$$\rho(\mathcal{E}) \coloneqq \inf \rho(E) \in [0,r],$$

and

$$\sigma(\mathcal{E}) \coloneqq \bigcap_{E \in \mathcal{E}} E \cap \overline{\mathbb{D}}(r) \subseteq \overline{\mathbb{D}}(r).$$

We have 3 cases:

<u>Case 1</u> : $\rho(\mathcal{E}) = 0$. $k = \bar{k} \Rightarrow$ is a point in $\overline{\mathbb{D}}(r) \Rightarrow p(\mathcal{E})$ Type 1.

<u>Case 2</u> : $\rho(\mathcal{E}) > 0$ and $\sigma(\mathcal{E}) \neq \varnothing \Rightarrow \sigma(\mathcal{E})$ is a closed disc in $\overline{\mathbb{D}}(r)$. Thus, $p(\mathcal{E})$ is Type 2 or Type 3 (depending on $\rho(\mathcal{E}) \in |k^*|$ or $\notin |k^*|$).

<u>Case 3</u> : $\rho(\mathcal{E}) > 0$, $\sigma(\mathcal{E}) = \varnothing$. This corresponds to new type of point.

Definition 7.8. $p(\mathcal{E})$ with $\rho(\mathcal{E}) > 0$ and $\sigma(\mathcal{E}) = \varnothing$ is called a <u>Type 4</u> point.

8 Note on 20250320

I did not attend this class.

9 Note on 20250325

I did not attend this class.

9.1 Affinoid algebra

10 Note on 20250401

10.1 Examples of affinoid algebras

Example 10.1. $X = \mathcal{M}(A)$.

- **Polydisc**. Let $A = k\{r^{-1}T\}$. Then $X = E(r)$ is the Berkovich polydisc of (poly)radius r.
- **Annuli**. Fix $0 < r_1 \le r_2 < \infty$. Let
$$A = \frac{k\{r_1T_1, r_2^{-1}T_2\}}{(T_1T_2 - 1)}$$
with the residue norm. Think:
$$A = \{\text{functions converge on } r_1 \le |T| \le T_2\}.$$
$X =$Berkovich annulus of radius r_1, r_2.
Think it as $T = T_2$ and $T_1 = T^{-1}$. The condition on convergence means $r_1 \le |T| \le r_2$.
- **Circles**. Let $r \in \mathbb{R}_{>0}$, and let
$$A = \frac{k\{rT_1, r^{-1}T_2\}}{(T_1T_2 - 1)}.$$
Then X is the Berkovich circle of radius r.
Let $r \notin \sqrt{|k^*|}$. Consider
$$K_r \coloneqq \left\{ f = \sum_{i=-\infty}^{+\infty} a_iT^i \colon a_i \in k,\ \lim_{|i|\to\infty} |a_i|r^i = 0 \right\},$$
with norm $\|f\| \coloneqq \max |a_i|r^i$. Recall that $K_r = \mathcal{H}(x)$ where $x \in E(r)$ is a Type 3 point with $\rho(x) = r$. K_r is a NA field extension of k.
Indeed, if $f \neq 0$, then $f = a_iT^i(1+h)$, where a_iT^i is the unique term maximizing $|a_i|r^i$. Then $\|h\| < 1 \Rightarrow 1 + h$ invertible $\Rightarrow f$ invertible.
Think $\mathcal{M}(A) =$"circle with one point".

Higher-dimensional analogue.

Definition 10.2. We say an $n-$tuple $r = (r_1, \dots, r_n) \in (\mathbb{R}_{>0})^n$ is free if $\log r_i$ are $\mathbb{Q}-$linearly independent in $\mathbb{R}/\log\sqrt{|k^*|}$.

If r is $k-$free, then we define
$$K_r \coloneqq \left\{ f = \sum_{v\in\mathbb{Z}^n} a_vT^v \colon a_v \in k,\ \lim_{|v|\to\infty} |a_v|r^v = 0 \right\},$$
which is a NA field extension of k, and $|K_r^*|$ is generated by $|k^*|$ and $r_1, \dots, r_n$.

Look at algebra/metric properties of

- Tate algebra;
- strict $k-$affinoid algebra;
- general $k-$affinoid algebra.

10.2 Properties of Tate algebra

Recall Tate algebra

$$\mathcal{T}_n \coloneqq k\{T_1, \ldots, T_n\} = \left\{ f = \sum_{v \in \mathbb{Z}_+^n} a_v T^v \colon a_v \in k,\ \lim_{|v| \to \infty} |a_v| r^v = 0 \right\},$$

where $\|f\| = \max |a_v|$.

Valuation ring:

$$\mathcal{T}_n^\circ \coloneqq k^\circ\{T_1, \ldots, T_n\} = \{f \in \mathcal{T}_n \colon \|f\| \leq 1\} = \{f \in \mathcal{T}_n \colon a_v \in k^\circ\}.$$

Ideal:

$$\mathcal{T}_n^{\circ\circ} \coloneqq \{f \colon \|f\| < 1\} = \{f \colon a_v \in k^{\circ\circ}\},$$

which is a prime ideal, since $\|\cdot\|$ is multiplicative. Then

$$\widetilde{\mathcal{T}_n} = \mathcal{T}_n^\circ / \mathcal{T}_n^{\circ\circ} \simeq \widetilde{k}[T_1, \ldots, T_n].$$

Facts :

(1) $f \in \mathcal{T}_n$ is invertible iff

$$\|f - f(0)\| < \|f\| = |f(0)|).$$

(2) $\forall f \in \mathcal{T}_n$, $\exists a \in k$ s.t.

$$|a| = \|f\| \quad \& \quad f + a \text{ is not invertible.}$$

(3)

$$(0) = \bigcap_{\mathfrak{m} \text{maximal ideal}} \mathfrak{m}.$$

(4) Any $k-$alg. homomorphism $\varphi \colon \mathcal{T}_n \to \mathcal{T}_m$ is a contraction:

$$\|\varphi(f)\| \leq \|f\|, \quad \forall f \in \mathcal{T}_n,$$

"Schwarz lemma".

Proof. 1. $\|f - f(0)\| < |f(0)|$, denote $h = f - f(0)$, $a = f(0) \in k^*$. Then $f = a + h = a(1 + a^{-1}h)$, where $\|a^{-1}h\| < 1$. Thus, f is invertible.

Assume f invertible. There exists a s.t. $\|a\| = \|f\| \neq 0$. Replacing f by f/a, assume $\|f\| = 1$. There exists $g \in \mathcal{T}_n$ s.t. $fg = 1 \Rightarrow \|g\| = 1 \Rightarrow \widetilde{f}\widetilde{g} = 1$, $\widetilde{f}, \widetilde{g} \in \widetilde{k}[T_1, \ldots, T_n] \Rightarrow \widetilde{f} \in \widetilde{k} \Rightarrow \widetilde{f} = \widetilde{f(0)} \Rightarrow f - f(0) \in \mathcal{T}_n^{\circ\circ} \Rightarrow \|f - f(0)\| < 1$.

2. Left to the readers.

3. Assume $f \in R(T_n) \setminus \{0\}$. $\exists a \in k$ with $|a| = \|f\| > 0$ s.t. $f + a$ is not invertible. $\exists \mathfrak{m}$ maximal ideal s.t. $f + a \in \mathfrak{m}$ and $f \in \mathfrak{m} \Rightarrow a \in \mathfrak{m}$, contradiction.

4. Assume $\|\varphi(f)\| > \|f\|$. By (2), $\exists a \in k$ s.t. $|a| = \|\varphi(f)\|$ and $\varphi(f+a) = \varphi(f) + a$ is not invertible. Thus, $f + a$ is not invertible. But $\|f\| < \|\varphi(f)\| = |a| \Rightarrow f + a = a(1 + f/a)$ (with $\|f/a\| < 1$) is invertible. Contradiction.

■

Then

$$\varphi(\mathcal{T}_n^\circ) \subseteq \mathcal{T}_m^\circ, \quad \varphi(\mathcal{T}_n^{\circ\circ}) \subseteq \mathcal{T}_m^{\circ\circ}.$$

So we get reduction

$$\widetilde{\varphi} := \widetilde{\mathcal{T}_n} \to \widetilde{\mathcal{T}_m}.$$

(5) If $\varphi\colon \mathcal{T}_n \to \mathcal{T}_m$ is a $k-$algebra isomorphism, then $m = n$ and φ is an isometric isomorphism, i.e., $\|\varphi(f)\| = \|f\|$, $\forall f$.

Proof. $\widetilde{\varphi}\colon \widetilde{k}[T_1, \dots, T_n] \to \widetilde{k}[T_1, \dots, T_m]$ iso. $\Rightarrow m = n$.
$\|f\| = 1 \Leftrightarrow \|\varphi(f)\| = 1$, so φ is an isometric isomorphism. ■

(6) A $k-$algebra homomorphism $\varphi\colon \mathcal{T}_n \to \mathcal{T}_n$ is bijective iff $\widetilde{\varphi}$ is bijective.

Proof. "Only if" part is OK. Now assume $\widetilde{\varphi}$ bijective $\Rightarrow \varphi$ is isometric. Only need to show φ surjective. Pick $\psi\colon \mathcal{T}_n \to \mathcal{T}_m$ s.t. $\widetilde{\psi}\colon \mathcal{T}_n \to \mathcal{T}_n$ s.t. $\widetilde{\psi} = \widetilde{\varphi}^{-1}$. Only need to show $F := \varphi \circ \psi$ is surjective. Note $\widetilde{F} = \widetilde{\varphi} \circ \widetilde{\psi} = \mathrm{Id}$,

$$\|FT_i - T_i\| < 1, \quad \forall i,$$

$\epsilon := \max\{\|FT_i - T_i\|\} < 1$. Write $\Delta := F - \mathrm{Id} \in \hom_k(\mathcal{T}_n, \mathcal{T}_n) \Rightarrow \|\Delta T_i\| \le \epsilon < 1$. For any monomial

$$\begin{aligned} F(T_{i_1} \cdots T_{i_m}) &= (\mathrm{Id} + \Delta)(T_{i_1}) F(T_{i_1} \cdots T_{i_m}) \\ &= T_{i_1} F(T_{i_1} \cdots T_{i_m}) + \Delta(T_{i_1}) F(T_{i_1} \cdots T_{i_m}) \\ &= T_{i_1} \cdots T_{i_m} + \cdots, \end{aligned}$$

$\Rightarrow$ the operator norm $\|\Delta\| \le \epsilon$. Let

$$G = \sum_{i=0} (-1)^i \Delta^i,$$

which converges.

$$F \circ G = (\mathrm{Id} + \Delta)(\mathrm{Id} - \Delta + \Delta^2 - \Delta^3 + \cdots) = \mathrm{Id}.$$

Thus, F surjective. ■

10.3 Maximum modulus principle

Given a NA field K/k. Set

$$E^n(K) \coloneqq \overline{\mathbb{D}_k^n(1)} = \{x \in K^n \colon |x_i| \leq 1, \forall i\}.$$

Then $\forall f = \sum_{v \in \mathbb{Z}_+^n} a_v T^v \in \mathcal{T}_n$ defines a continuous function $E^n(K) \to K$. NA triangle inequality implies $|f(x)| \leq \|f\|$, $\forall x \in E^n(K)$.

THEOREM 10.3 (Maximum modulus principle). *If $\widetilde{K}$ is infinite, then $\exists x \in E^n(K)$ s.t. $|f(x)| = \|f\|$ i.e. $\|f\| = \max_{x \in E^n(K)} |f(x)|$.*

Proof. We may assume $\|f\| = 1$. $\widetilde{K}$ infinite $\Rightarrow \widetilde{x} \in \widetilde{K}^n$ s.t. $\widetilde{f}(\widetilde{x}) \neq 0$. Pick $x \in E^n(K)$ with $\widetilde{x} = x \mod K^{\circ\circ}$. Then $|f(x)| = 1 = \|f\|$. ∎

11 Note on 20250403

THEOREM 11.1 (Thm A). *$\mathcal{T}_n$ is Noetherian, UFD, and Jacobson.*

"Weierstrass-Rückert theory".

11.1 Weierstrass' theorems

Let A be a NA Banach ring with norm $\|\cdot\|$. Assume $\|\cdot\|$ is multiplicative.

$$A\{r^{-1}T\} \coloneqq \left\{ f = \sum_{n\geq 0} a_n T^n : a_n \in A,\ \|a_n\| r^n \to 0 \right\},$$

$\|f\| \coloneqq \max\{\|a_n\|r^n\}$. "Gauss's Lemma" shows $\|\cdot\|$ is multiplicative on $A\{r^{-1}T\}$. Thus, for any $f \in A\{r^{-1}T\}$, the principle ideal (f) is closed. Indeed, any fg_n Cauchy sequence $\Rightarrow$ g_n Cauchy sequence.

DEFINITION 11.2. Order $\mathrm{ord}(f)$ of a non-zero $f = \sum_{n=0}^{\infty} a_n T^n$ is the maximal n with $\|f\| = \|a_n\|r^n$. Say that f is distinguished if $a_{\mathrm{ord}(f)}$ is invertible in A.

Easy Facts: $\forall f, g \in A\{r^{-1}T\} \setminus \{0\}$, we have

1. $\mathrm{ord}(fg) = \mathrm{ord}(f) + \mathrm{ord}(g)$;
2. if $\|f\| < \|g\|$ OR $\|f\| = \|g\|$ and $\mathrm{ord}(f) < \mathrm{ord}(g)$, then $\|f+g\| = \|g\|$.

Let $A\{r^{-1}T\}_{<n} \coloneqq$ the free Banach $A-$submodule of degree $< n$.

PROPOSITION 11.3 (Weierstrass division thm). *Let $f \in A\{r^{-1}T\} \setminus \{0\}$ of order n. Then*

(i) the following homomorphism of Banach $A-$modules is an isometric monomorphism:

$$\begin{aligned} \Phi \colon A\{r^{-1}T\}f \otimes A\{r^{-1}T\}_{<n} &\longrightarrow A\{r^{-1}T\} \\ (Q\cdot f, R) &\longmapsto g = Qf + R; \end{aligned}$$

(ii) Φ is an isomorphism if f is distinguished.

Proof. (i) Easy Fact (1) $\Rightarrow$ $\mathrm{ord}(Qf) \geq \mathrm{ord}(f) = n$. Since $\mathrm{ord}(R) < n$, Easy Fact (2) $\Rightarrow$ $\|Qf + R\| = \max\{\|Qf\|, \|R\|\}$, so Φ is an isometry, and automatically monomorphism.

(ii) First assume Φ is isomorphism. Then $T^n = Qf + R$ for $Q \in A\{r^{-1}T\}$, $R \in A\{r^{-1}T\}_{<n}$, so $\mathrm{ord}(Q) = 0$. Thus, $Q = \sum_{i\geq 0} c_i T^i$ with $\|c_0\| > \|c_i\|r^i$, $\forall i \geq 1$. Write $f = \sum_{i\geq 0} a_i T^i$. Consider the $n-$th coefficients: $1 = c_0 a_n + \cdots + c_n a_0$, and $\|c_i a_{n-i}\| < \|c_0 a_n\|$, $\forall 0 \leq i \leq n$. $\Rightarrow c_0 a_n = 1 - \sum_{i=1}^{n} c_i a_{n-i}$, with $\|\sum_{i=1}^{n} c_i a_{n-i}\| < 1$, which shows a_n is invertible.

Conversely, we assume a_n is invertible.

Observation: (i)$\Rightarrow$ $\mathrm{im}(\Phi)$ is closed in $A\{r^{-1}T\}$.

Pick $\varepsilon \in \left(\max_{i\geq n+1}\left\{\frac{\|a_i\|r^i}{\|f\|}\right\}, 1\right)$. Set $\widetilde{f} = \sum_{i=0}^{n} a_i T^i$. For any $g = \sum_{i=0}^{\infty} c_i T^i \in A\{r^{-1}T\}$, take $\widetilde{g} = \sum_{i=0}^{m} c_i T^i$, where m is maximal with $\|c_m\|r^m > \varepsilon\|g\|$. Then

$$\|f - \widetilde{f}\| \leq \varepsilon\|f\|, \quad \|g - \widetilde{g}\| \leq \varepsilon\|g\|.$$

By Euclid's division algorithm, $\exists \widetilde{Q}, \widetilde{R} \in A[T]$ with $\deg(\widetilde{R}) \leq n-1$ s.t. $\widetilde{g} = \widetilde{Q}\widetilde{f} + \widetilde{R}$. By (i), $\|\widetilde{g}\| = \max\{\|\widetilde{Q}\| \cdot \|\widetilde{f}\|, \|\widetilde{R}\|\}$, where $\|\widetilde{f}\| = \|f\|$. $\Rightarrow \|\widetilde{Q}\| \cdot \|f\| \leq \|\widetilde{g}\| = \|g\| \Rightarrow$

$$\|g - (\widetilde{Q}f + \widetilde{R})\| = \|g - \widetilde{g} + \widetilde{g} - (\widetilde{Q}\widetilde{f} + \widetilde{R}) - \widetilde{Q}(f - \widetilde{f})\| \leq \max\{\|g - \widetilde{g}\|, \|\widetilde{Q}\| \cdot \|f - \widetilde{f}\|\} \leq \varepsilon \|g\|.$$

In other words, there exists a map

$$\begin{aligned} P\colon A\{r^{-1}T\} &\longrightarrow \mathrm{im}(\Phi) \\ g &\longmapsto P(g) = \widetilde{Q}f + \widetilde{R}, \end{aligned}$$

s.t. $\|g - P(g)\| \leq \varepsilon \|g\|$. Write $\Delta(g) \coloneqq g - P(g)$, i.e., $\Delta = \mathrm{Id} - P$. Then

$$\mathrm{Id} = P + \mathrm{Id} \cdot \Delta = P + P\Delta + \Delta^2 = P + P\Delta + P\Delta^2 + \cdots + \Delta^{n+1},$$

where $P + P\Delta + P\Delta^2 + \cdots \in \mathrm{im}(\Phi)$, and $\|\Delta^{n+1}\| \leq \varepsilon^{n+1}$. Since $\mathrm{im}(\Phi)$ is closed, we have $g \in \mathrm{im}(\Phi)$.

■

THEOREM 11.4 (Weierstrass preparation thm). *Let $f \in A\{r^{-1}T\}$ be a distinguished element of order n. Then there exists a unique decomposition $f = e \cdot w$, where $w \in A[T]$ is a monic polynomial of degree and order n and e is an invertible element of $A\{r^{-1}T\}$.*

Proof. Weierstrass division thm $\Rightarrow$ $\exists$ unique $Q = A\{r^{-1}T\}$, $R \in A\{r^{-1}T\}_{<n}$ s.t. $T^n = Qf + R$, and $\max\{\|Qf\|, \|R\|\} = \|T^n\| = r^n$. Define $w \coloneqq T^n - R$, which is a monic polynomial of degree n. Since $\|R\| \leq r^n$ and $\mathrm{ord}(R) < n$, $\mathrm{ord}(w) = n$ and $\|w\| = r^n$. We have $w = Qf \Rightarrow \|Q\| = \|a_n\|^{-1}$ and $\mathrm{ord}(Q) = 0$.

Write $Q = q_0 + \sum_{i \geq 1} q_i T^i$, where $\|a_n\|^{-1} = \|q_0\| > \|q_i\| r^i$. w is monic $\Rightarrow 1 = a_n q_0 + \sum_{i=1}^{n} q_i a_{n-i}$, where the last term satisfies $\|\cdot\| < 1$. Thus, Q is invertible in $A\{r^{-1}T\}$. Set $e \coloneqq Q^{-1}$. We get $f = e \cdot w$.

For the uniqueness, if $f = e \cdot w$ with w and e satisfy the desired assumptions. Then $T^n = e^{-1}f + (T^n - w)$, where $\deg(T^n - w) < n$. Now, the uniqueness of w and e follows from the Uniqueness part in Weierstrass division thm. ■

DEFINITION 11.5. A Weierstrass polynomial is a monic polynomial $w \in A[T] \subseteq A\{r^{-1}T\}$ whose order equals to its degree.

Facts:

- For $w_1, w_2 \in A[T]$ monic polynomials, $w_1 \cdot w_2$ is a Weierstrass polynomial iff both w_1 and w_2 are Weierstrass polynomials.
- For any $w \in A[T]$ a Weierstrass polynomial, we have

$$A\{r^{-1}T\} \big/ w \cdot A\{r^{-1}T\} \simeq A[T] \big/ w \cdot A[T].$$

Write (and let $A = \mathcal{T}_{n-1}$)

$$\mathcal{T}_n = \mathcal{T}_{n-1}\{T_n\}.$$

Proposition 11.6. *For any $f \in \mathcal{T}_n \setminus \{0\}$, there exists a $k-$automorphism $\sigma\colon \mathcal{T}_n \xrightarrow{\sim} T_n$ s.t. $\sigma(f)$ is distinguished in A.*

Proof. We may assume $\|f\| = 1$. Write $f = \sum_{v \in \mathbb{Z}_+^n} a_v T^v$. Set

$$S \coloneqq \{x \colon |a_v| = 1\},$$

which is a finite set.

Let $\mu = (\mu_1, \ldots, \mu_n)$ be the maximal among them w.r.t. the lexicographical ordering.

Pick $d \in \mathbb{Z}_+$ s.t.

$$d > \max_{v \in S} |v|.$$

Define a k endomorphism of $\mathcal{T}_n$ by

$$\begin{cases} \sigma(T_1) = T_1 + T_n^{d^{n-1}} \\ \sigma(T_2) = T_2 + T_n^{d^{n-2}} \\ \quad \cdots \\ \sigma(T_{n-1}) = T_{n-1} + T_n^{d} \\ \sigma(T_n) = T_n. \end{cases}$$

Then $\widetilde{\sigma}$ is an automorphism on $\widetilde{k}[T_1, \ldots, T_n] \Rightarrow \sigma$ is an automorphism on $\mathcal{T}_n$. Write

$$f = \sum_{v \in S} a_v T_1^{v_1} \cdots T_n^{v_n} + \mathcal{E},$$

where $\|a_v\| = 1$ and $\|\mathcal{E}\| < 1$. Then

$$\sigma(f) = \sum_{v \in S} a_v \left(T_n^{v_1 d^{n-1} + v_2 d^{n-2} + \cdots + v_n} + \mathcal{L} \right) + \mathcal{E},$$

where $\mathcal{L}$ is a lower degree term in T_n. Now,

$$\sigma(f) = a_\mu T_n^{\mu_1 d^{n-1} + \mu_2 d^{n-2} + \cdots + \mu_n} + \mathcal{L} + \mathcal{E},$$

$\Rightarrow \sigma(f)$ is distinguished of order $m = \mu_1 d^{n-1} + \cdots + \mu_n$. ■

11.2 Rückert's theory

Let A be a commutative ring with unity, and B be a commutative ring containing $A[T]$.

Definition 11.7. We call that B is <u>Rückert</u> over A if there exists a set of monic polynomials $W \subseteq A[T]$ with the following properties:

(1) For monic $w_1, w_2 \in A[T]$, if $w_1, w_2 \in W$, then both $w_1, w_2 \in W$;

(2) $\forall w \in W$, $A[T] \big/ wA[T] \simeq B \big/ wB$;

(3) $\forall \in B \setminus \{0\}$, there exists an automorphism σ of B s.t. $\sigma(f) = e \cdot w$, where e is invertible and $w \in W$.

Fact : $\mathcal{T}_n$ is Rückert over $\mathcal{T}_{n-1}$.

PROPOSITION 11.8. *Assume B is Rückert over A. Then*

(i) A Noetherian $\Rightarrow$ B Noetherian;

(ii) If A is Jacobson, then $\mathrm{Rad}(B/\mathfrak{b}) = \mathrm{Jac}(B/\mathfrak{b}b)$ for any non-zero ideal $\mathfrak{b} \subseteq B$;

(iii) A UFD $\Rightarrow$ B UFD.

Proof of Thm A. Noetherian and UFD are OK. For Jacobson, (ii) + Fact + $\mathrm{Jac}(\mathcal{T}_n) = \{0\} \Rightarrow$ $\mathcal{T}_n$ Jacobson by induction. ■

REMARK 11.9. *It is not true in general that in (ii) B is Jacobson.*

EXAMPLE 11.10. $A = k$ field and $B = k[[T]]$. Then (0) is a prime ideal but only have one maximal ideal $\mathfrak{m} = (T)$, and $(0) \neq \mathfrak{m}$.

12 Note on 20250408

12.1 Proof of Proposition 11.8

Proof of Proposition 11.8. (i) Let $\mathfrak{b}$ be a non-zero ideal of B. By (3), we may assume $\exists w \in \mathfrak{b} \cap W$. By (2), $B/wB \simeq A[T]/wA[T]$ which is Noetherian $\Rightarrow$ $\mathfrak{b}/wB$ is finitely generated $\Rightarrow$ $\mathfrak{b}$ is finitely generated $\Rightarrow$ B is Noetherian.

(ii) We may assume that $\mathfrak{b}$ is a prime ideal. Only need to show $\mathrm{Jac}(B/\mathfrak{b}) = 0$. Let $\mathfrak{a} := A \cap \mathfrak{b}$. We may assume $\exists w \in \mathfrak{b} \cap W$. Then $B/\mathfrak{b} = (B/wB)\Big/(\mathfrak{b}/wB)$, where $B/wB \simeq A[T]/wA[T]$ is finite over $A/\mathfrak{a}$. $\forall g \in \mathrm{Jac}(B/b)$, we have equation

$$g^n + a_1 g^{n-1} + \cdots + a_n = 0 \quad \text{over} \quad A/\mathfrak{a}$$

of minimal degree $\Rightarrow$ $a_n = -(g^n + a_1 g^{n-1} + \cdots + a_{n-1} g) \in \mathrm{Jac}(B/\mathfrak{b}) \cap A/\mathfrak{a} \subseteq \mathrm{Jac}(A/\mathfrak{a}) = 0$ $\Rightarrow g = 0$.

(iii) Let $f \in B \setminus \{0\}$. By (3), we may assume $f \in W$. Set $K := \operatorname{Frac} A$. $K[T]$ is a UFD. $f = p_1 \cdots p_m$ in $K[T]$ into monic irreducible polynomials in $K[T]$. A is a UFD $\Rightarrow \exists a_1, \ldots, a_m \in A$, s.t. $a_1 p_1, \ldots, a_m p_m \in A[T]$ and primitive (i.e., there does not exist prime element of A divides all coefficients of p). Then

$$\left(\prod_{i=1}^n a_i\right) f = \prod_{i=1}^n (a_i p_i)$$

is a primitive polynomial ("Gauss Lemma"). Thus, $\prod_{i=1}^n a_i$ is invertible $\Rightarrow p_i \in A[T]$, $\forall i$. By (1), $p_i \in W$, $\forall i$. By (2), $B/p_iB = A[T]/p_iA[T]$. "Gauss Lemma" $\Rightarrow A[T] \cap p_i K[T] = p_i A[T]$. $\Rightarrow$

$$A[T]\Big/p_iA[T] \longrightarrow K[T]\Big/p_ik[T]$$

is injective $\Rightarrow$ (p_i) are prime ideals $\Rightarrow$ is UFD. ∎

12.2 Metric properties

Assume k non-trivially valued field.

Fact: Let V, W be $k-$normed spaces. $T\colon V \to W$ linear map. Then T is bounded iff T is continuous.

Proof. "Bounded $\Rightarrow$ Continuous" is easy.

"Continuous $\Rightarrow$ Bounded": $\exists w \in k^*$ with $|w| < 1$. $\exists r > 0$ s.t. $\forall |v| < r$, $\|T(v)\| < 1$. $\forall v \in W \setminus \{0\}$. $\exists$ minimal n s.t. $\|w^n v\| < r \Rightarrow$

$$\frac{r}{\|w\|^{n-1}} \leq \|v\| < \frac{r}{\|w\|^n}, \quad \|T(w^n v)\| \leq 1,$$

$\Rightarrow \|T(v)\| < \frac{1}{\|w\|^n} \Rightarrow$

$$\frac{\|T(v)\|}{\|v\|} \leq \frac{1}{\|w\|^n}\Big/\frac{r}{\|w\|^{n-1}} = \frac{1}{r\|w\|}.$$

∎

THEOREM 12.1 (Open Mapping Thm). *V, W are $k-$Banach spaces. $T\colon V \to W$ surjective linear map. Then T is open.*

Proof. Let
$$B(r) \coloneqq \{v \in V \colon \|v\| < r\}.$$
Only need to show $T(B(r))$ is open for all $r > 0$. As k non-trivially valued, there exists $w \in k^*$ with $|w| < r$. Then
$$B(r) = \bigcup_{v \in B(r)} \Big(v + wB(1)\Big).$$
Only need to show that $T(B(1))$ is open.

Claim $T(B(1))$ contains a non-empty open subset.

Proof. Otherwise, $\forall r \in k^*$, $T(B(r))$ does not contain any non-empty open subset. Pick $a \in k$ with $|a| > 1$. Then $W = T(V) = \bigcup_{n \geq 0} T(B(|a|^n))$. Contradiction by Baire Category Thm. ■

Let $U \subseteq T(B(1))$ be a non-empty open subset. Then $U + (-U) \subseteq T(B(1))$. We may assume $0 \in U$. Pick $w \in k^*$ with $|w| < 1$. $\forall v \in B(1)$, we have
$$T(v) + wU \subseteq T(v) + wT(B(1)) = T(v + wB(1)) \subseteq T(B(1))$$
$\Rightarrow T(v) \in T(B(1))^\circ \Rightarrow T(B(1))$ open. ■

THEOREM 12.2 (Closed Graph Thm). *Let V, W be $k-$Banach spaces. $T\colon V \to W$ linear map. Then T is bounded iff the graph*
$$\Gamma(T) \coloneqq \{(x, T(x)) \colon x \in V\} \subseteq V \times W$$
is closed.

Proof. "Only if" part is easy.

Assume $\Gamma(T)$ is closed. The projections
$$\begin{aligned} \pi_1\colon \Gamma(T) &\longrightarrow V \\ (x, y) &\longmapsto x \end{aligned}$$
$$\begin{aligned} \pi_2\colon \Gamma(T) &\longrightarrow W \\ (x, y) &\longmapsto y \end{aligned}$$
are bounded. π_1 is bijective (hence surjective) $\Rightarrow \pi_1$ is open $\Rightarrow \pi_1^{-1}$ is bounded $\Rightarrow T = \pi_2 \circ \pi_1^{-1}$ is bounded. ■

COROLLARY 12.3. *Let $T\colon A \to B$ be a continuous surjective $k-$linear map between $k-$Banach spaces. Then T is admissible.*

Proof. T continuous $\Rightarrow T$ bounded. $\ker(T)$ closed. Replace A by $(A/\ker(T), \|\cdot\|_{\mathrm{res}})$. We may assume that T is injective, hence bijective. Open Mapping Thm + Fact $\Rightarrow T$ is an isomorphism. ■

REMARK 12.4. *The above results are not true when k trivially valued. Reason: Can not scale the balls.*

Example 12.5. Let $R > r \geq 1$. Consider the map

$$\phi\colon k\{R^{-1}T\} \longrightarrow k\{r^{-1}T\}.$$

For $(k, |\cdot|_0)$, it is actually $\phi\colon k[T] \to k[T]$. Denote the norms by $\|\cdot\|_1$ and $\|\cdot\|_2$. ϕ is isomorphism as $k-$algebra. ϕ is bounded. $\psi := \phi^{-1}$ is not bounded: $\dfrac{\|\psi(T^n)\|_1}{\|T^n\|_2} = \dfrac{R^n}{r^n} \to +\infty$. Thus, ϕ is not admissible (Cor not true).

$\forall f \in k\{r^{-1}T\}$, $\|f\|_2 = 0$ or $\|f\|_2 \geq r \Rightarrow$ the topology induced by $\|\cdot\|_2$ is discrete $\Rightarrow \psi$ is continuous (Fact not true, Closed Graph Thm not true). Considering the similar example for $1 > R > r$, one can show that Open Mapping Thm not true.

12.3 Properties of Noetherian Banach $k-$algebras

Let A be a Noetherian Banach $k-$algebra over k, where k is non-trivially valued (e.g., $A = \mathcal{T}_n$).

Proposition 12.6. *Let M be a normed $A-$module s.t. $\widehat{M}$ is a finite $A-$module. Then $M = \widehat{M}$ i.e., M is complete.*

Corollary 12.7. *Any ideal in A is closed.*

Proposition 12.6 $\Rightarrow$ Corollary 12.7. Apply Proposition 12.6 to any ideal of A. ■

Proof of Proposition 12.6. $\exists x_1, \cdots, x_n \in \widetilde{M}$, s.t.

$$\begin{aligned} \pi\colon A^n &\longrightarrow \widehat{M} \\ (a_1, \ldots, a_n) &\longmapsto \sum_{i=1}^n a_i x_i \end{aligned}$$

is surjective. Hence admissible. Open Mapping Thm implies π is open. Thus, $\sum_{i=1}^n B(1)x_i$ contains a neighborhood of $0 \in \widehat{M}$, where $B(1) \subseteq A$ is the unit ball. Since M is dense in $\widehat{M}$,

$$\widehat{M} = M + \sum_{i=1}^n B(1)x_i.$$

Write

$$x_i = y_i + \sum_{i=1}^n f_{ij} x_j,$$

where $y_i \in M$, $\|f_{ij}\| < 1$. Then

$$\mathbf{y} = (\mathrm{Id} - \mathbf{F})\mathbf{x}$$

where $\mathbf{y} = (y_1, \ldots, y_n)^\dagger$, $\mathbf{x} = (x_1, \ldots, x_n)^\dagger$, and $\mathbf{F} = (f_{ij})$. Note that $\mathrm{Id} - \mathbf{F}$ is invertible in $M^{n\times n}(A)$. Let $\mathbf{G} := (\mathrm{Id} - \mathbf{F})^{-1}$. Then $\mathbf{x} = \mathbf{G}\mathbf{y} \Rightarrow x_i \in M$, $\forall i \Rightarrow M = \widehat{M}$. ■

Corollary 12.8. *The forgetful functor*

$$\{\textit{finite Banach } A\textit{-modules}\} \longrightarrow \{\textit{finite } A-\textit{modules}\}$$

is an equivalence of categories.

Proof. "Faithful" is trivial.

"Full": M, N finite Banach $A-$modules. There exists surjective morphism $\varphi\colon A^n \to M$ (admissible) $\Rightarrow M \simeq (A^n/\ker(\varphi), \|\cdot\|_{\mathrm{res}})$. May assume $M = A^n = \oplus Ae_i$. Any $\phi\colon A^n \to N$ where $\phi(\sum_i a_i e^i) = \sum_i a_i \phi(e_i)$ is bounded.

"Essentially surjective": M finite $A-$module. There exists surjection $\phi\colon A^n \to M$ for some $n \geq 1$. Then $M \simeq A^n/\ker(\phi)$. A Noetherian $\Rightarrow \ker(\phi)$ is closed. We may give M the residue norm. ■

Corollary 12.9. *An $A-$module homomorphism of finite Banach $A-$modules is in fact admissible.*

13 Note on 20250415

13.1 Tensor products

Let M be a Banach $A-$module. Then for any $n \geq 1$, $A^n \otimes_A M \simeq M^n$ (with tensor product semi-norm). Thus, $A^n \otimes_A M$ is complete. We have

$$A^n \otimes_A M \simeq A^n \widehat{\otimes}_A M \simeq M^n.$$

Lemma 13.1. *If $\varphi\colon M \to M'$, $\psi\colon N \to N'$ are admissible epimorphisms of Banach $A-$modules, then the induced homomorphisms $M \otimes_A N \to M' \otimes_A N'$ and $M \widehat{\otimes}_A N \to M' \widehat{\otimes}_A N'$ are admissible epimorphisms.*

Proof. Only need to prove the first case.

Bounded is OK.

$\exists C > 0$ s.t. $\forall m' \in M'$, $n' \in N'$, $\exists m \in \varphi^{-1}(m')$, $n \in \psi^{-1}(n')$ s.t.

$$\|m\| \leq C\|m'\|, \quad \|n\| \leq C\|n'\|.$$

Given $x' \in M' \otimes_A N'$ and $\varepsilon > 0$, there exists a finite sum $x' = \sum m_i' \otimes n_i'$ with

$$\max \|\|m_i'\| \cdot \|n_i'\|\} \leq \|x'\| + \varepsilon.$$

Take $m_i \in \varphi^{-1}(m_i'), n_i \in \psi^{-1}(n_i')$ s.t.

$$\|m_i\| \leq C\|m_i'\|, \quad \|n_i\| \leq C\|n_i'\|.$$

Then for $x = \sum m_i \otimes n_i$, we have

$$\|x'\|_{\text{res}} \leq \|x\| \leq \max\{\|m_i\| \cdot \|n_i\|\} \leq C^2 \max\{\|m_i'\| \cdot \|n_i'\|\} \leq C^2(\|x'\| + \varepsilon).$$

Thus, $\|x'\|_{\text{res}} \leq C^2\|x'\| \Rightarrow$ admissible. ■

Lemma 13.2. *Let A be a Noetherian Banach $k-$algebra, and k is non-trivially valued. Let M, N be finite Banach $A-$modules. Then*

$$M \otimes_A N \simeq M \widehat{\otimes}_A N.$$

Proof. Let $\varphi\colon A^m \twoheadrightarrow M$, $\psi\colon A^n \twoheadrightarrow N$ be admissible epimorphisms. Then Lemma 13.1 shows $\varphi \otimes \psi\colon A^{mn} \simeq A^m \widehat{\otimes} A^n \twoheadrightarrow M \widehat{\otimes}_A N$ is an admissible epimorphism. Thus $M \widehat{\otimes}_A N$ is a finite $A-$module, which implies the desired result. ■

Corollary 13.3. *The forgetful functor $\{$finite Banach $A-$algebras$\} \to \{$finite $A-$algebras$\}$ is an equivalence of categories.*

Proof. Any B a finite $A-$algebra, B has a canonical norm $\|\cdot\|$ up to equivalence makes it to be a finite Banach $A-$module, so

$$B \widehat{\otimes} B \simeq B \otimes B \xrightarrow{\text{multiplication}} B$$

is bounded. $\exists C > 0$ s.t. $\forall x, y \in B$, $\|xy\| \leq C\|x\| \cdot \|y\|$. Replace $\|\cdot\|$ by $\|\cdot\| \times C$. We have $\|xy\| \leq \|x\|\|y\| \Rightarrow (B, \|\cdot\|)$ is a Banach $A-$algebra. ■

Corollary 13.4. *Let M, N be finite Banach $A-$modules, and B a finite Banach $A-$algebra. Then the maps*

$$M \otimes_A N \to M\widehat{\otimes}_A N, \quad M \otimes_A B \to M\widehat{\otimes}_A B$$

are bijective. Hence, $M\widehat{\otimes}_A N$ is a finite Banach $A-$algebra and $M\widehat{\otimes}_A B$ is a finite Banach $B-$module.

13.2 Properties of strictly $k-$affinoid algebras

Definition 13.5. A Banach $k-$algebra A is strictly $k-$affinoid if there exists an admissible epimorphism $\pi\colon \mathcal{T}_n \twoheadrightarrow A$ of Banach $k-$algebras for some $n \geq 0$.

In this case $A = \mathcal{T}_n / \ker(\pi)$, where $\mathcal{T}_n$ is Noetherian and Jacobson.

Corollary 13.6. *If A is strictly $k-$affinoid, then A is Noetherian and Jacobson.*

Corollary 13.7. *Any ideal of A is closed.*

theorem 13.8 (Noether Normalization Thm). *Any strictly $k-$affinoid algebra A, $\exists d \geq 0$, and a finite bounded monomorphism $\mathcal{T}_d \hookrightarrow A$.*

Definition 13.9. A chart of $\mathcal{T}_n$ is a system $(f_1, \ldots, f_n)$ of elements in $\mathcal{T}_n^\circ$ s.t. the homomorphism

$$\begin{aligned} k\{S_1, \ldots, S_n\} &\longrightarrow \mathcal{T}_n \\ S_i &\longmapsto f_i \end{aligned}$$

is an isomorphism.

Proposition 13.10. *Let A be a strictly $k-$affinoid algebra. Then for any bounded finite homomorphism $\varphi\colon \mathcal{T}_n \to A$, there exists a chart $(s_1, \ldots, s_n)$ s.t. the induced homomorphism $\mathcal{T}_d = k\{s_1, \ldots, s_d\} \to A$ is finite and injective for some d.*

Proof. When $n = 0$, trivial.

Assume $n \geq 1$. If $\ker(\varphi) = 0$, then nothing to prove. Now assume $\ker(\varphi) \neq 0$.

We can find a chart $(s_1, \ldots, s_n)$ and a Weierstrass polynomial $w \in \mathcal{T}_{n-1}[s_n] \cap \ker(\varphi)$. Then

$$\mathcal{T}_n \Big/ w\mathcal{T}_{n-1} \simeq \mathcal{T}_{n-1}[s_n] \Big/ w\mathcal{T}_{n-1}[s_n],$$

where the latter is finite over $\mathcal{T}_{n-1}$. So

$$\mathcal{T}_{n-1} \longrightarrow \mathcal{T}_n / w\mathcal{T}_{n-1} \longrightarrow A$$

is finite. We conclude the proof by induction on n. ∎

Proof of Theorem 13.8. A is strictly $k-$affinoid $\Rightarrow \exists$ admissible epimorphism $\mathcal{T}_n \twoheadrightarrow A$ (which is finite). We concule the proof by Proposition 13.10. ∎

Corollary 13.11. *Let $\mathfrak{a}$ be an ideal of a strictly $k-$affinoid algebra A s.t. its radical $\operatorname{Rad}(\mathfrak{a})$ is maximal ideal. Then $A/\mathfrak{a}$ is of finite dimension over k.*

Proof. By Proposition 13.10, there exist an admissible finite monomorphism $\varphi\colon \mathcal{T}_n \to A/\mathfrak{a}$ for some $n \geq 0$. Since $\mathcal{T}_n$ is reduced, the induced homomorphism $\mathcal{T}_n \to A/\operatorname{Rad}(\mathfrak{a})$ is finite and injective. As $A/\operatorname{Rad}(\mathfrak{a})$ is a field, $\mathcal{T}_n$ must be a field. Thus, $n = 0$, i.e. $\mathcal{T}_n = k$. So $\dim_k A/\mathfrak{a} < \infty$. ∎

Corollary 13.12. *Any homomorphism between strictly $k-$affinoid algebras $\varphi\colon A \to B$ is bounded.*

Proof. If A is trivially valued, then A, B are finitely generated $k-$algebras with trivial norm. So φ is bounded.

Now assume k is not trivially valued. By Closed Graph Thm, only need to show that $\Gamma(\varphi) \subseteq A \times B$ is closed. By contraction, $\exists\ f_n \in A,\ f_n \to 0$ but $\varphi(f_n) \to g \neq 0$. Given a maximal ideal $\mathfrak{m} \subseteq B$ and $l \geq 1$, consider the induced injection:

$$A/\varphi^{-1}(\mathfrak{m}^l) \longrightarrow B/\mathfrak{m}^l.$$

Both are finite dimensional over k, so the induced homeomorphism is bounded $\Rightarrow \varphi(f_j) \to 0$ in $B/\mathfrak{m}^l \Rightarrow g \in \mathfrak{m}^l$ for every $\mathfrak{m}$ maximal ideal and $l \geq 1 \Rightarrow$

$$g \in \bigcap_{\mathfrak{m} \text{ maximal ideal}} \bigcap_{l \geq 1} \mathfrak{m}^l.$$

Only need to show the right term is 0. Pick h in it. Observe that

$$\mathfrak{m} \cdot \bigcap_{l \geq 1} \mathfrak{m}^l = \bigcap_{l \geq 2} \mathfrak{m}^l = \bigcap_{l \geq 1} \mathfrak{m}^l,$$

where $\bigcap_l \mathfrak{m}^l$ is a finitely generated $A-$module $\Rightarrow \exists r_m \in \mathfrak{m}$ s.t.

$$(1 - r_m) \cdot \bigcap_{l \geq 1} \mathfrak{m}^l = 0.$$

Thus, $(1 - r_m) \cdot h = 0$. Let

$$\mathfrak{b} = \big(1 - r_m, \mathfrak{m} \text{ maximal ideal}\big).$$

Then $\mathfrak{b} \cdot h = 0$. For every $\mathfrak{m}$ maximal ideal, $1 - r_m \notin \mathfrak{m} \Rightarrow \mathfrak{b} \not\subset \mathfrak{m}$ for every $\mathfrak{m}$ maximal ideal $\Rightarrow 1 \in \mathfrak{b} \Rightarrow h = 1 \cdot h = 0$. ■

Corollary 13.13. *For any maximal ideal $\mathfrak{m}$, the valuation on k extends uniquely to a valuation on $A/\mathfrak{m}$, hence defines a point $x \in \mathcal{M}(A)$:*

$$|f(x)| \coloneqq \textit{norm of the image of } f \textit{ on } A/\mathfrak{m}$$

gives an injective map $\mathrm{Max}(A) \to \mathcal{M}(A)$.

Remark 13.14. $\mathrm{Max}(A)$ *is the space associated to A in rigid geometry.*

Proposition 13.15. *If k is not trivially valued, then* $\mathrm{Max}(A)$ *is dense in* $\mathcal{M}(A)$.

Proof. Pick $x_0 \in \mathcal{M}(A)$ and $U \ni x_0$ open in $\mathcal{M}(A)$. Want to show $\mathrm{Max}(A) \cap U \neq \varnothing$. We may assume that

$$U = \{x \in \mathcal{M}(A) \colon |f_i(x)| < a_i, 1 \leq i \leq m, \quad |g_j(x)| > b_j, 1 \leq j \leq n\},$$

where $f_i, g_j \in A$, $a_i, b_j \in \mathbb{R}_{\geq 0}$. Pick $p_i, q_j \in \sqrt{|k^*|}$ s.t. $|f_i(x_0)| < p_i < a_i$ and $|g_j(x_0)| > q_j > b_j$. Replacing f_i, g_j by their powers, we may assume $p_i, q_j \in |k^*| \Rightarrow p_i = |c_i|, q_j = |d_j|$, where $c_i, d_j \in k^*$. Replacing f_i, g_j by $f_i/c_i, g_j/d_j$, we may assume $p_i, q_j = 1$, $\forall i, j$.

Define

$$B := \frac{A\{S_1, \dots, S_m, T_1, \dots, T_n\}}{(f_i - S_i, g_j T_j - 1)}.$$

Then B is a strictly $k-$affinoid algebra. The morphism $A \to B$ induces $\phi\colon \mathcal{M}(B) \to \mathcal{M}(A)$ and $\mathrm{Max}(B) \to \mathrm{Max}(A)$.

Claim:

$$\phi(\mathcal{M}(B)) = \{x \in \mathcal{M}(A)\colon |f_i(x)| \le 1, |g_j(x)| \ge 1, \forall i, j\} \subseteq U.$$

Claim $\Rightarrow x_0 \in \phi(\mathcal{M}(B)) \Rightarrow B \neq 0 \Rightarrow \mathrm{Max}(B) \neq \varnothing \Rightarrow \phi(\mathrm{Max}(B)) \subset \mathrm{Max}(A) \cap U \neq \varnothing$.

Proof of Claim. Let $x \in B$. Then

$$|(f_i - S_i)(x)| = 0, \quad |(g_j T_j - 1)(x)| = 0,$$

$\Rightarrow |f_i(x)| = |S_i(x)| \le \|S_i\| \le 1$, $1 = |g_j T_j(x)| = |g_j(x)||T_j(x)| \le |g_j(x)| \Rightarrow$ "$\subseteq$" holds.

Let $x \in \mathcal{M}(A)$ with $|f_i(x)| \le 1$, $|g_j(x)| \ge 1$, $\forall i, j$. Define

$$\begin{aligned} A\{S_1, \dots, S_m, T_1, \dots, T_n\} &\longrightarrow \mathcal{H}(x) \\ S_i &\longmapsto f_i \\ T_j &\longmapsto g_j^{-1}. \end{aligned}$$

As $|f_i(x)|, |g_j^{-1}(x)| \le 1$, the above map is well-defined and bounded. Its kernel contains $S_i - f_i$, $g_j T_j - 1$, $\forall i, j$. It induces a bounded morphism

$$B \longrightarrow \mathcal{H}(x).$$

The composition

$$A \longrightarrow B \longrightarrow \mathcal{H}(x)$$

is the natural morphism $A \to \mathcal{H}(x)$, which induces

$$\mathcal{M}(\mathcal{H}(x)) \longrightarrow \mathcal{M}(B) \longrightarrow \mathcal{M}(A),$$

and the image of the composition map is $\{x\} \Rightarrow x \in \phi(\mathcal{M}(B))$. ∎

Q.E.D. ∎

Remark 13.16. *For $(k, |\cdot|_0)$, not true.*

For example, $A = k\{T\} = k[T]$. Assume $k = \bar{k}$. The open subset

$$U = \{x\colon 0.1 < |T(x)| < 0.2\} \neq \varnothing$$

of $\mathcal{M}(A)$ (not point in $\mathcal{M}(A)$) satisfies $U \cap \mathrm{Max}(A) = \varnothing$.

14 Note on 20250417

14.1 Constructions in strictly $k-$affinoid algebras

PROPOSITION 14.1 (Grounded field extension). *If A is strictly $k-$affinoid algebra, k'/k is a NA field extension, then $A\widehat{\otimes}_k k'$ is a strictly $k-$affinoid algebra.*

Proof. There exists $\mathcal{T}_n \twoheadrightarrow A$ admissible epimorphism. $\mathrm{Id}\colon k' \to k'$. Then

$$\mathcal{T}_{n,k'} = \mathcal{T}_n\widehat{\otimes}_k k' \longrightarrow A\widehat{\otimes}_k k'$$

is also an admissible epimorphism, which implies the desired result. ■

PROPOSITION 14.2 (Fiber product). *Let A, B, C be strictly $k-$affinoid algebras with bounded morphisms*

$$\begin{tikzcd} k \arrow[r] & A \arrow[r] \arrow[d] & B \\ & C & \end{tikzcd}$$

Then $B\widehat{\otimes}_A C$ is strictly $k-$affinoid algebra.

Proof. The morphism

$$B\widehat{\otimes}_k C \twoheadrightarrow B\widehat{\otimes}_A C$$

is an admissible epimorphism by the definition of tensor product seminorm.

Only need to show $B\widehat{\otimes}_k C$ is strictly $k-$affinoid. There exists admissible epimorphisms

$$\mathcal{T}_n \twoheadrightarrow B, \quad \mathcal{T}_m \twoheadrightarrow C.$$

Then $\mathcal{T}_{n+m} = \mathcal{T}_n\widehat{\otimes}\mathcal{T}_m \twoheadrightarrow B\widehat{\otimes}_k C$ is an admissible epimorphism. ■

14.2 The spectral radius I

Recall that for any A nonzero Banach ring, $f \in A$, the spectral radius

$$\rho(f) \coloneqq \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} = \max_{x\in\mathcal{M}(A)} |f(x)|.$$

THEOREM 14.3 (Maximum Modulus Principle). *Assume A is strictly $k-$affinoid. Then*

$$\rho(f) = \max_{x\in\mathrm{Max}(A)} |f(x)|.$$

COROLLARY 14.4. $\rho(f) \subseteq \sqrt{|k^*|} \cup \{0\}$.

Let $P(T) = T^n + a_1T^{n-1} + \cdots + a_n$ be a monic polynomial in $k[T]$. The quotient $K = K[T]\Big/P$ is a finite $k-$algebra. Then it has the structure of a strictly $k-$affinoid algebra with $\mathcal{M}(K) = \mathrm{Max}(K)$ a finite set. Thus, the Maximum Modulus Principle holds.

LEMMA 14.5. *In the above case, let f be the image of T in K. Then*

$$\rho(f) = \max_{1\leq i\leq n} |a_i|^{\frac{1}{i}}.$$

Proof. Observe

$$K \simeq \bigoplus_{i=0}^{n-1} kT^i,$$

with $\| \sum_i b_i T^i \| = \max_i |b_i|$. Multiply by T defines a linear map

$$\times T\colon K \longmapsto K,$$

with matrix

$$M := \begin{pmatrix} 0 & \cdots & 0 & -a_n \\ 1 & \cdots & 0 & -a_{n-1} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & -a_1 \end{pmatrix}$$

$T_m = T^m \times 1$, $m \geq 0$ generates K. Thus,

$$\lim_{m\to\infty} \|T^m\|^{\frac{1}{m}} = |\text{maximal eigenvalue of } M| = \max\{|\alpha|\colon \alpha \text{ root of } P\}.$$

Let $\alpha_1, \ldots, \alpha_n$ be roots of P with $|\alpha_1| = \cdots = |\alpha_s| > |\alpha_{s+1}| > \cdots > |\alpha_n|$. We have

$$\max |a_i|^{\frac{1}{i}} = |\alpha_1 \cdots \alpha_s|^{\frac{1}{s}} = \rho(f).$$

■

Proof of Theorem 14.3. If k is trivially valued, then OK.
Now assume k is not trivially valued and $f \neq 0$.

Case 1 If $A = \mathcal{T}_n$, we already proved.

Case 2 Assume A is an integral domain. Noether normalization thm implies: there exists a finite morphism $\varphi\colon \mathcal{T}_n \to A$. Let

$$P(T) = T^d + g_1 T^{d-1} + \cdots + g_d \in \mathrm{Frac}(\mathcal{T}_n)[T]$$

be the minimal polynomial of f over $\mathrm{Frac}(T_n)$. $\mathcal{T}_n$ UFD $\Rightarrow$ $\mathcal{T}_n$ integrally closed $\Rightarrow$ $g_i \in \mathcal{T}_n$, $\forall i = 1, \ldots, d$. We have

$$\mathcal{T}_n \hookrightarrow B := \mathcal{T}_n[T] \big/ P(T) \hookrightarrow A$$
$$T \mapsto f,$$

and $\rho_A(f) \leq \rho_B(T)$. Since $\mathrm{Max}(A) \to \mathrm{Max}(B)$ is surjective,

$$\max_{y\in \mathrm{Max}(B)} |T(y)| = \max_{x\in\mathrm{Max}(A)} |f(x)|.$$

We may assume $A = B = \mathcal{T}_n[T]/P$. Then

$$\sup_{y\in \mathrm{Max}(A)} = \sup_{x\in \mathrm{Max}(\mathcal{T}_n)} \max_{y\to x} |f(y)| = \sup_{x\in \mathrm{Max}(\mathcal{T}_n)} \max_{1\le i\le d} |g_i(x)|^{\frac{1}{i}},$$

by the above lemma. Case 1 implies there exists $x \in \mathrm{Max}(\mathcal{T}_n)$ s.t.

$$\prod_{g_i\neq 0} |g_i(x)| = \left|\Big(\prod_{g_i\neq 0} g_i\Big)(x)\right| = \rho_{\mathcal{T}_n}\Big(\prod_{g_i\neq 0} g_i\Big) = \prod_{g_i\neq 0} \rho_{\mathcal{T}_n}(g_i),$$

$\Rightarrow |g_i(x)| = \rho(g_i)|, \forall i$. Therefore, there exists $x \in \mathcal{M}(A)$ taking the maximal value.

Case 3 A is arbitrary. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the minimal prime ideals of A and f_i be the image of f in the quotient ring $A_i = A/\mathfrak{p}_i$. Then

$$\rho(f) = \max_{1\le i\le n} \rho(f_i).$$

We conclude the proof by Case 2.

Q.E.D. ■

Proposition 14.6. *Let $\varphi\colon A \to B$ be finite homomorphism of strictly $k-$affinoid algebras. Then $\forall g \in B$, there exists a monic polynomial $P(T) = T^n + f_1T^{n-1} + \cdots + f_n \in A[T]$ s.t. $P(g) = 0$ and $\rho(g) = \sigma(P) \coloneqq \max_{1\le i\le n} \rho(f_i)^{1/i}$.*

Proof. Prove by the following cases.

Case 1 B is an integral domain. Noether normalization thm implies: there exists a homomorphism $\psi\colon \mathcal{T}_n \to A$ s.t. the composition with φ

$$\mathcal{T}_d \xrightarrow{\psi} A \xrightarrow{\varphi} B$$

is a finite monomorphism.

There exists $Q(T) = T^n + h_1T^{n-1} + \cdots + h_n \in \mathcal{T}_d[T]$ such that $Q(g) = 0$ and $\rho(g) = \max_{1\le i\le n} \rho(h_i)^{1/i}$.

Set

$$P(T) = T^n + \psi(h_1)T^{n-1} + \cdots + \psi(h_n) \in A[T],$$

wee have $P(g) = 0$. Set $f_i = \psi(h_i)$. Then $\rho(f_i) \le \rho(h_i) \Rightarrow$

$$\rho(g) = \max_{1\le i\le n} \rho(h_i)^{\frac{1}{i}} \ge \max_{1\le i\le n} \rho(f_i)^{\frac{1}{i}}.$$

Claim:

$$\rho(g) \le \max_{1\le i\le n} \rho(f_i)^{\frac{1}{i}}.$$

Proof of Claim. If $\rho(g) > \max_{1\le i\le n} \rho(f_i)^{1/i}$, then

$$\rho(g^n) > \max_{1\le i\le n} \rho(f_i)\rho(g)^{n-i}.$$

$$\Rightarrow \quad \rho(g^n) > \rho\left(\sum_{i=1}^{n}(-f_i)g^{n-i}\right),$$

$$\Rightarrow \quad g^n + \sum_{i=1}^{n} f_i g^{n-i} \neq 0,$$

contradiction. ■

<u>Case 2</u> B is arbitrary. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$ be the minimal prime ideals of B. Let g_i denote the image of g in $B_i \coloneqq B/\mathfrak{p}_i$. <u>Case 1</u> $\Rightarrow$ there exists monic polynomials $P_i(T) \in A[T]$ with $P_i(g_i) = 0$ and $\rho(g_i) = \sigma(P_i)$. Set

$$Q(T) \coloneqq \prod_{i=1}^{m} P_i(T).$$

Then $Q(g) \in \bigcap_{i=1}^{m} P_i$, i.e., $\exists e \ge 1$, s.t. $Q(g)^e = 0$. Set $P \coloneqq Q(T)^e$.

$$\sigma(P) = \sigma(Q(T)^e) = \sigma(Q(T)) \le \max_{1\le i\le m} \sigma(P_i) = \max_{1\le i\le m} \rho(g_i) = \rho(g).$$

As in **Proof of Claim**, we have $\rho(g) \le \sigma(P)$. Then we get $\rho(g) = \sigma(P)$.

■

15 Note on 20250429

15.1 The spectral radius II

Proposition 15.1. *Let A be a strictly $k-$affinoid algebra, $f \in A$. Then the followings are equivalent:*

(1) f is power bounded,

$$\sup_n \|f^n\| < \infty;$$

(2) $\rho(f) \leq 1$.

Proof. (1) $\Rightarrow$ (2) trivial. Now prove (2) $\Rightarrow$ (1). Assume $\rho(f) \leq 1$. Take a finite homomorphism $\varphi\colon \mathcal{T}_n \to A$. Proposition 14.6 $\Rightarrow \exists P(T) = T^m + g_1 T^{m-1} + \cdots + g_m \in A[T]$ with $P(f) = 0$ and

$$1 \geq \rho(f) = \max_{1 \leq i \leq m} \rho(g_i)^{\frac{1}{i}}.$$

Then $\rho(g_i) \leq 1$, $\forall i = 1, \ldots, m$. Claim:

$$f^n \in \sum_{i=0}^{m-1} \mathcal{T}_n^\circ \cdot f^i.$$

Induction:

$$\begin{aligned} f^{n+1} &\in \sum_{i=0}^{m-1} \mathcal{T}_n^\circ \cdot f^{i+1} \\ &= \sum_{i=1}^{m-1} \mathcal{T}_n^\circ f^i + \mathcal{T}_n^\circ f^m \\ &\subseteq \sum_{i=1}^{m-1} \mathcal{T}_n^\circ f^i + \mathcal{T}_n^\circ \sum_{i=0}^{m-1} \mathcal{T}_n^\circ f^i \\ &\subseteq \sum_{i=0}^{m-1} \mathcal{T}_n^\circ \cdot f^i. \end{aligned}$$

Thus, $\sup_n \|f^n\| < \infty$. ∎

Proposition 15.2. *If A is strictly $k-$affinoid algebra and $f \in A$. Then*

1. *$\rho(f) = 0$ (i.e. f quasi-nilpotent) iff f is nilpotent.*
2. *If f is not nilpotent, then $\exists C > 0$ s.t. $\|f^n\| \leq C \cdot \rho(f^n)$, $\forall n \geq 1$.*

Proof. For (1), we have

$$\rho(f) = 0 \Leftrightarrow f \in \bigcap_{\mathfrak{m} \in \mathrm{Max}(A)} \mathfrak{m},$$

by maximal modulus principle. Since A is Jacobson, we have

$$f \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}\, A} \mathfrak{p} \Leftrightarrow f \text{ nilpotent}.$$

For (2), Corollary 14.4 implies $\rho(f) \in \sqrt{|k^*|} \cup \{0\}$. By (1), $\rho(f) \neq 0$, so $\exists a \in k^*$ and $m \geq 1$ s.t. $\rho(f)^m = |a| \Rightarrow \rho(a^{-1}f^m) = 1$. By Proposition 15.1, $\exists C_1 > 0$ s.t.

$$(\|a^{-1}f^m)^n\| \leq C_1, \quad \forall n \geq 0 \quad \Rightarrow \quad \|f^{mn}\| \leq C_1|a|^n,$$

$$\Rightarrow \quad \|f^n\| \leq \|f^{[n/m]m}\|\|f^{n-[n/m]m}\| \leq C_1\rho(f)^{m[n/m]} \times C_2 \leq C\rho(f)^n.$$

■

Corollary 15.3. *The $k-$Banach algebra*

$$\mathcal{T}_{n,r} = k\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$$

is strictly $k-$affinoid iff $r_i \in \sqrt{|k^*|}$, $\forall\, i$.

Proof. "$\Rightarrow$": $r_i = \|T_i\| = \rho(T_i) \in \sqrt{|k^*|}$.

"$\Leftarrow$": May assume $|k^*| \neq 1$. $\exists\; m_i > 1$, s.t. $r_i^{m_i} = |a_i|$, where $a_i \in k^*$, $\forall 1 \leq i \leq n$. $\exists b_i \in k^*$ s.t. $|b_i| \leq r_i^{-1}$. Consider the morphism

$$\begin{aligned} \phi\colon k\{x_1, y_1, \ldots, x_n, y_n\} &\longrightarrow \mathcal{T}_{n,r} \\ x_i &\longmapsto b_iT_i \\ y_i &\longmapsto a_i^{-1}T_i^{m_i}. \end{aligned}$$

Easy to check ϕ is well-defined bounded.

$\forall f \in \mathcal{T}_{n,r}$,

$$f = \sum_{1\leq j_i\leq m_i-1, \forall\, j} G(T_1^{m_1}, \ldots, T_n^{m_n})T_1^{j_1}\cdots T_n^{j_n},$$

where $G \in \phi(k\{y_1, \ldots, y_n\})$. $\Rightarrow f \in \mathrm{im}(\phi) \Rightarrow \phi$ surjective, thus admissible. ■

15.2 Properties of $k-$affinoid algebras I

Definition 15.4. Let A be a Banach $k-$algebra. A is <u>$k-$affinoid</u> if there exists an admissible epimorphism

$$k\{r^{-1}T\} = \mathcal{T}_{n,r} \twoheadrightarrow A$$

for some $r \in \mathbb{R}^n_{>0}$ and $n \geq 1$.

Geometrically, $\mathcal{M}(A) \hookrightarrow E(r)$ closed embedding.

Recall the field K_r. Assume $r \in \mathbb{R}^n_{>0}$ which is <u>$k-$free</u>, i.e. $\forall v \in \mathbb{Z}^n$, $r^v \in |k^*|$ iff $v = 0$. Define

$$K_r \coloneqq \left\{ f = \sum_{v\in\mathbb{Z}^n} a_vT^v \colon a_v \in k, \lim_{|v|\to\infty} |a_v|r^v = 0 \right\},$$

with the norm $\|f\| \coloneqq \max_v |a_v|r^v$. Then

$$K_r \simeq \frac{k\{r_1S_1, \ldots, r_nS_n, r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}}{(S_iT_i - 1)_{1\leq i\leq n}},$$

which is "polycircle of radius r". Moreover, K_r is a NA field extension of k s.t. $|K_r^*|$ is generated by $|k^*|, r_1, \ldots, r_n$.

Lemma 15.5. *Let V be a Banach $k-$space. Then*

1. *the canonical map*
$$V \longrightarrow V \widehat{\otimes}_k K_r$$
is an isometric embedding.

2. *A sequence $V' \to V \to V''$ of bounded homomorphisms of $k-$Banach spaces is exact and admissible iff*
$$V' \widehat{\otimes}_k K_r \to V \widehat{\otimes}_k K_r \to V'' \widehat{\otimes}_k K_r$$
is exact and admissible.

Proof. Check that

$$V \widehat{\otimes}_k K_r \simeq \left\{ \sum_{v \in \mathbb{Z}^n} V_v T^v : V_v \in V, \lim_{|v| \to \infty} \|V_v\| r^v = 0 \right\}$$

with norm $\| \cdot \| = \max_v |V_v| r^v$, and the "$\simeq$" is isometry. ■

Corollary 15.6. *Let A be a $k-$affinoid algebra. Then there exists $r \in \mathbb{R}^n_{>0}$, $n \geq 0$ free over k, s.t. $A \widehat{\otimes} K_r$ is strictly K_r-affinoid algebra.*

Proof. There exists an admissible epimorphism

$$k\{s_1^{-1}T_1, \ldots, s_m^{-1}T_m\} \twoheadrightarrow A.$$

Induction on

$$l := \dim_{\mathbb{Q}} \frac{\langle \log \sqrt{|k^*|}, \log |s_1|, \ldots, \log |s_m| \rangle}{\log \sqrt{|k^*|}} \leq m.$$

If $l = 0$, A is strict.
If $l > 0$, may assume $s_1 \notin \sqrt{|k^*|}$. Then

$$K_r \langle s_1^{-1}T_1, \ldots, s_n^{-1}T_n \rangle \twoheadrightarrow A_{K_{s_1}}$$

admissible epimorphism, $l \to l - 1$. OK by induction. ■

Proposition 15.7. *Let A be a $k-$affinoid algebra and $f \in A$. Then*

1. *A is Noetherian and all of its ideals are closed;*

2. *$\rho(f) = 0$ iff f nilpotent;*

3. *if f is not nilpotent, then $\exists C > 0$ s.t.*
$$\|f^n\| \leq C \rho(f)^n, \quad \forall n \geq 1.$$

Proof. (1) Only need to show that if $A\widehat{\otimes}K_r$ with $r \notin \sqrt{|k^*|}$ is Noetherian and all ideals are closed, then A is Noetherian and all ideals are closed.

Let $\mathfrak{a} \subset A$ be an ideal of A. Then the ideal $\mathfrak{a} \cdot (A\widehat{\otimes}K_r)$ is generated by elements $f_1, \ldots, f_n \in \mathfrak{a}$. $\forall f \in \mathfrak{a}$, we can write f in $\mathfrak{a} \cdot (A\widehat{\otimes}K_r)$ as

$$f = \sum_{i=1}^{n} f_i g_i, \quad \text{where} \quad g_i = \sum_{j=-\infty}^{+\infty} g_{ij} T^j,$$

$$\Rightarrow \quad f = \sum_{j=-\infty}^{+\infty} f_j g_{i0}.$$

Hence, $f_1, \ldots, f_n$ generate $\mathfrak{a}$. The above argument shows $\mathfrak{a} \subseteq A \cap \mathfrak{a}(A\widehat{\otimes}K_r) \subseteq \mathfrak{a}$, which must be an equality. $\mathfrak{a} \cdot (A\widehat{\otimes}K_r)$ closed $\Rightarrow$ $\mathfrak{a}$ is closed.

When A is strictly $k-$affinoid, (2)(3) hold.

Assume (2)(3) hold for $A\widehat{\otimes}K_r$. As $A \hookrightarrow A\widehat{\otimes}K_r$ is an isometry, (2)(3) hold for A. ■

16 Note on 20250508

16.1 Properties of $k-$affinoid algebras II

Corollary 16.1. *Let $\varphi\colon A \to B$ be a bounded homomorphism between $k-$affinoid algebras. Let $f_1, \dots, f_n \in B$ and let $r_1, \dots, r_n \in \mathbb{R}_{>0}$ with $r_i \geq \rho(f_i)$. Then there exists a unique bounded homomorphism $\Phi\colon A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to B$ extending φ and sending T_i to f_i.*

Proof. If Φ exists, we should have

$$\Phi\left(\sum_{v\in\mathbb{Z}^n_{\geq 0}} a_v T^v\right) = \sum_v \varphi(a_v) f^v.$$

Only need to show the latter term converges. Proposition 15.7 implies

$$\|f^v\| \leq C\rho(f_1)^{v_1}\cdots\rho(f_n)^{v_n} \leq Cr^v.$$

Then the result follows from $\|\varphi(a_v)\| \leq C' \cdot \|a_v\|$. ■

Corollary 16.2. *If A is $k-$affinoid, then A is strictly $k-$affinoid iff $\rho \in \sqrt{|k^*|}$ for every $f \in A$.*

Proof. "$\Rightarrow$" holds by maximum modulus principle.

"$\Leftarrow$": There exists an admissible epimorphism

$$\varphi\colon k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \twoheadrightarrow A.$$

Set $f_i = \varphi(T_i) \in A$, $s_i := \rho(f_i) \leq \rho(T_i) = r_i$. By Corollary 16.1, there exists an extension $\psi\colon k\{s_1^{-1}T_1, \dots, s_n^{-1}T_n\} \to A$ s.t. $\psi(T_i) = f_i$.

$$\begin{tikzcd}
k\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \arrow[r, two heads, "\varphi"] \arrow[d, "T_i \to T_i"] & A \\
k\{s_1^{-1}T_1, \dots, s_n^{-1}T_n\} \arrow[ur, "\psi"] &
\end{tikzcd}$$

φ admissible epimorphism $\Rightarrow$ ψ admissible epimorphism. ■

16.2 Finite modules and algebra over $k-$affinoid algebras

Definition 16.3. Let A be a $k-$affinoid algebra. A Banach $A-$module M is finite if there exists an admissible epimorphism $A^n \twoheadrightarrow M$ for some $n \geq 0$.

$\mathrm{Mod}_b^h(A)$ =Category of finite Banach $A-$module with bounded morphism.
$\mathrm{Mod}^h(A)$ =finite $A-$module.
The forget functor:

$$\theta\colon \mathrm{Mod}_b^h(A) \to \mathrm{Mod}^h(A).$$

Proposition 16.4. *θ is an equivalence of categories.*

Proof. Already proved when $|k^*| \neq 1$.

Assume k trivially valued. Pick $r \in \mathbb{R}_{>0} \setminus \{1\}$.

(1) Let $M \in \mathrm{Mod}^h(A)$. $\exists \pi \colon A^n \twoheadrightarrow M$ surjective.

Claim: $\ker \pi$ is closed.

Observe $\pi_{K_r} \colon A^n \widehat{\otimes}_k K_r \twoheadrightarrow M \widehat{\otimes}_k K_r$. $|K_r^*| \neq 1 \Rightarrow$ admissible $\Rightarrow \ker \pi_{K_r}$ closed. $\ker \pi = \ker \pi_{K_r} \cap A^n$ closed. Then $M \simeq A^n / \ker \pi$ which is a Banach module with $\| \cdot \|_{\mathrm{res}}$.

(2) For $M, N \in \mathrm{Mod}_b^h(A)$, $\varphi \colon M \to N$ morphism of $A-$modules. We show φ bounded.

$\exists \pi \colon A^m = \oplus_{i=1}^m Ae_i \twoheadrightarrow M$ admissible epimorphism. Replace φ by $\varphi \circ \pi$. Assume $M = A^m$. Then

$$\left\| \varphi(\sum_{i=1}^m a_i e_i) \right\| \leq \max_{i=1}^m |a_i| \|\varphi(e_i)\| \leq \max_{i=1}^m |a_i| \times \max_{i=1}^m \|\varphi(e_i)\| \leq C \| \sum a_i e_i \|.$$

■

Remark 16.5. *M =Banach $A-$module. M finite $A-$module can not imply M finite Banach $A-$module.*

Example 16.6. $0 < r < R < 1$, $|k^*| = 1$.

$$\varphi \colon A = k\{R^{-1}T\} \simeq k[[T]] \longrightarrow B = k\{r^{-1}T\} \simeq k[[T]],$$

isomorphism as $k-$algebra. B is not a finite Banach $A-$module since φ^{-1} is not bounded.

Definition 16.7. An affinoid $A-$algebra is an $A \widehat{\otimes}_k K-$algebra for some NA field K over k.

(e.g. $K = \mathcal{H}(x)$)

Proposition 16.8. *Let $M, N \in \mathrm{Mod}_b^h(A)$. Then*

1. *Any $A-$linear map $M \to N$ is admissible.*
2. *$M \otimes_A N \simeq M \widehat{\otimes}_A N$.*
3. *For any affinoid $A-$algebra B, we have $M \otimes_A B \simeq M \widehat{\otimes}_A B$.*

Proof. Just $\widehat{\otimes} K_r$ for some $r \in \mathbb{R}_{>0}^n$ for some n to reduce to the strict case. ■

Corollary 16.9. *{ finite Banach $A-$algebras } $\simeq$ { finite $A-$algebras }.*

Proof. Almost same as the strict case. ■

Proposition 16.10. *Let B be a finite Banach algebra over a $k-$affinoid algebra A and assume the canonical morphism $A \to B$ is injective. Then the map $\mathcal{M}(B) \to \mathcal{M}(A)$ is surjective and has finite fibers.*

The blue part implies B is $k-$affinoid automatically.

Proof. For any $x \in \mathcal{M}(A)$, the fiber at $x \simeq \mathcal{M}(\mathcal{H}(x) \widehat{\otimes}_A B) = \mathcal{M}(\mathcal{H}(x) \otimes_A B)$, which is finite.

For non-emptiness, consider $f^{-1}(x) = \mathcal{M}(\mathcal{H}(x) \widehat{\otimes}_A B)$. Pick $r \in \mathbb{R}_{>0}^r$ which is $|k^*|-$free s.t. $A \widehat{\otimes}_k K_r$, $B \widehat{\otimes}_k K_r$ are strictly K_r-affinoid and $|K_r^*| \neq 1$.

Then we have

$$\begin{array}{ccc} \mathcal{M}(B\widehat{\otimes}_k K_r) & \longrightarrow & \mathcal{M}(A\widehat{\otimes}_k K_r) \\ \big\downarrow & & \big\downarrow \\ \mathcal{M}(B) & \longrightarrow & \mathcal{M}(A) \end{array}$$

(Gruson's thm shows the two down arrows are surjective).

May assume A, B strictly $k-$affinoid. $A \to B$ finite injective $\Rightarrow$ $\mathrm{Max}(B) \twoheadrightarrow \mathrm{Max}(A)$. Due to denseness of Max() in $\mathcal{M}()$ and $\mathcal{M}(B)$ is compact, we have $\mathcal{M}(B) \twoheadrightarrow \mathcal{M}(A)$ is surjective. ■

16.3 Weierstrass, Laurent and rational domains

Motivation Let $X = \mathcal{M}(A)$, where A is $k-$affinoid algebra. Want: Define a structure sheaf $\mathcal{O}_X$ on X, so that for suitable $V \subset X$, $A_V \coloneqq \Gamma(V, \mathcal{O}_X)$ is the ring of analytic functions on V.

In our case, "V" are closed.

Idea Use $V \subset X$ closed defined by (non-strict) inequalities, so that A_V is $k-$affinoid algebra. Such V are called "domain" as $V \cap \mathrm{Max}(A)$ is open in $\mathrm{Max}(A)$.

Definition 16.11 (Weierstrass domains). Given $f_1, \dots, f_n \in A$, $p_1, \dots, p_n \in \mathbb{R}_{>0}$, define

$$V \coloneqq X(p^{-1}f) = \{x \in X \colon |f_i(x)| \le p_i\},$$

which is a closed subset of X, analogous to a polydisc.

The analytic functions on V are

$$A_V \coloneqq \frac{A\{p_1^{-1}T_1, \dots, p_n^{-1}T_n\}}{(T_i - f_i)} = \left\{ g = \sum_{v \in \mathbb{Z}^n_{\ge 0}} a_v f^v \colon a_v \in A,\ \lim_{|v| \to \infty} \|a_v\| p^v = 0 \right\},$$

with residue norm.

Definition 16.12 (Laurent domains). Given $f_1, \dots, f_n, g_1, \dots, g_n \in A$, $p_1, \dots, p_n, q_1, \dots, q_n \in \mathbb{R}_{>0}$, define

$$V \coloneqq X(p^{-1}f, qg^{-1}) = \{x \in X \colon |f_i(x)| \le p_i,\ |g_j(x)| \ge q_j\},$$

which is a subset of X analogous to an annulus.

The analytic functions on V are

$$\begin{aligned} A_V &\coloneqq \frac{A\{p_1^{-1}T_1, \dots, p_n^{-1}T_n, q_1S_1, \dots, q_nS_n\}}{(T_i - f_i, g_jS_j - 1)} \\ &= \left\{ h = \sum_{(\mu,\nu) \in \mathbb{Z}^{n+n}_{\ge 0}} a_{\mu\nu} f^\mu g^{-\nu} \colon a_{\mu\nu} \in A,\ \lim_{|\mu|+|\nu| \to \infty} \|a_{\mu\nu}\| p^\mu q^{-\nu} = 0 \right\}, \end{aligned}$$

(with residue norm).

Remark 16.13. *Given $x \in X$, the family of all Laurent domains containing x form a basis of closed neighborhoods of x.*

Indeed,

$$U = \{x \in X \colon |f_i(y)| < t_i,\ |g_j(y)| > s_j\}$$

with $|f_i(x)| < t_i$, $|g_j(x)| > s_j$ form an open neighborhood basis. Pick $|f_i(x)| < p_i < t_i$, $|g_j(x) > q_j > s_j$. Then $V = X(p^{-1}f, qg^{-1}) \subset U$, $x \in V$.

17 Notes of the rest days in this month

I will be in Chongqing.

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