L^2 EXTENSION AND EFFECTIVENESS OF STRONG OPENNESS PROPERTY

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Abstract. In this note, we present an L^2 extension approach to the effectiveness result of strong openness property of multiplier ideal sheaves.

1. Introduction

The multiplier ideal sheaf associated to a plurisubharmonic function plays an important role in several complex variables, complex geometry and algebraic geometry (see e.g. [25, 18, 9, 24]). We recall the definition of multiplier ideal sheaf as follows. Let φ be a plurisubharmonic function (see [6]) on a complex manifold X. The multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the sheaf on X whose germs are the holomorphic functions F such that $|F|^2e^{-\varphi}$ is locally integrable.

The following strong openness property was conjectured by Demailly [7, 8] (the so-called strong openness conjecture), and proved by Guan-Zhou[12].

Strong openness property:

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi),$$

where $\mathcal{I}_+(\varphi) := \bigcup_{p>1} \mathcal{I}(p\varphi)$.

When there is also $\mathcal{I}(\varphi) = \mathcal{O}$, the strong openness conjecture is called the openness conjecture, which was posed by Demailly-Kollár [9] and proved by Berndtsson [2].

The effectiveness of openness conjecture was established by Berndtsson [2], which implies the openness conjecture. In the proof of the strong openness conjecture, Ohsawa-Takegoshi L^2 extension was used by Guan-Zhou [12]. After that, Guan-Zhou [13] established the related effectiveness result by solving the $\bar{\partial}$ equations with L^2 estimates. It is natural to ask

Question 1.1. *Can one obtain an L^2 extension approach to the effectiveness result of the strong openness property?*

In the present note, we give an affirmative answer to the above question.

1.1. Optimal L^2 extension and Guan-Zhou Method.

In this section, we illustrate how to use the optimal L^2 extension and Guan-Zhou Method.

First we consider that ξ is an element in the following set

$$\ell_1 := \{\xi = (\xi_\alpha)_{\alpha\in\mathbb{N}^n} : \sum_{\alpha\in\mathbb{N}^n} |\xi_\alpha|\rho^{|\alpha|} < +\infty, \text{ for any } \rho > 0\}.$$

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For any $F(z) \in \mathcal{O}_{z_0}$, $z_0 \in \mathbb{C}^n$, we define the value that ξ acts on $F(z)$ as

$$(\xi \cdot F)(z_0) := \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha \frac{F^{(\alpha)}(z_0)}{\alpha!},$$

for any $\xi \in \ell_1$. Then the Bergman kernel can be defined as follows.

Definition 1.2 (Bergman kernel). *For any bounded domain $D \subseteq \mathbb{C}^n$, $z \in D$, we define the Bergman kernel with respect to ξ as*

$$K_{\xi,D}(z) = \sup_{F \in L^2(D) \cap \mathcal{O}(D)} \frac{|(\xi \cdot F)(z)|^2}{\int_D |F|^2}.$$

The So-called Guan-Zhou Method ([14], see also [21],[22]) shows that an optimal L^2 extension theorem implies the following fibrewised log-plurisubharmonicity property of $K_{\xi,D}(z)$.

Let Ω be a pseudoconvex domain in $\mathbb{C}^{n+1}$ with coordinate (z,t), where $z \in \mathbb{C}^n$, $t \in \mathbb{C}$, and p is the natural projection $p(z,t) = t$ on Ω, $p(\Omega) = D$. For any $t \in D$, $\Omega_t = p^{-1}(t) \subseteq \Omega$. Let $K_{\xi,t}(z) = K_{\xi,\Omega_t}(z)$ be the Bergman kernels of the domains Ω_t with respect to $\xi \in \ell_1$.

Proposition 1.3. *(see* [1]*)* $\log K_{\xi,t}(z)$ *is a plurisubharmonic function with respect to (z,t), for any $\xi \in \ell_1$.*

For $\xi = (1, 0, \cdots, 0, \cdots)$, Berndtsson [1] proved the above log-plurisubharmonicity of the Bergman kernel, which can be seen as a generalization of Maitani and Yamaguchi's result in [17].

1.2. **The effectiveness of the strong openness property.**

In this section, we present the L^2 extension approach to the effectiveness of the strong openness property.

Proposition 1.3 gives the following estimate of the Bergman kernels on the sublevel sets $\{\varphi < -t\}$ when $\xi \in \ell_{\mathcal{I}(\varphi)_o}$, where

$$\ell_I = \{0 \neq \xi \in \ell_1 : (\xi \cdot F)(o) = 0, \ \forall F \in I\},$$

I is an ideal of $\mathcal{O}_o$ such that $I \neq \mathcal{O}_o$.

Proposition 1.4. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ with $o \in D$, and let φ be a negative plurisubharmonic function on D, such that $\varphi(o) = -\infty$. Assume that I is an ideal of $\mathcal{O}_o$ such that $I \neq \mathcal{O}_o$. Let F be a holomorphic function on D. Then for any $p > 1$, and any $\xi \in \ell_{\mathcal{I}(\varphi)_o}$, the inequality*

$$\int_D |F|^2 e^{-\frac{\varphi}{p}} \geqslant \frac{p}{p-1} \cdot \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)}$$

holds.

Denote that

$$B(F, I, D) := \sup_{\xi \in \ell_I} \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)},$$

where $(F, o) \in \mathcal{O}_o$, and ideal $I \subsetneq \mathcal{O}_o$.

The separating theorem in functional analysis theory implies the following property of the above definition.

Proposition 1.5. *Let D be a bounded domain in $\mathbb{C}^n$, with $o \in D$, and let I be an ideal of $\mathcal{O}_o$ such that $I \neq \mathcal{O}_o$. Let $(F, o) \in \mathcal{O}_o$ such that $(F, o) \notin I$. Then*

$$B(F, I, D) > 0.$$

Recall that

$$c_o^F(\varphi) := \sup\{c \geqslant 0 : |F|^2 e^{-2c\varphi} \text{ is integrable near } o\}$$

is the jumping number (see[16]). When $F \equiv 1$, $c_o^F(\varphi)$ will degenerate to $c_o(\varphi)$, which is called the complex singularity exponent (or log canonical threshold) (see [25, 9]).

Using Proposition 1.4 and Proposition 1.5, we complete the L^2 extension approach to the effectiveness of the strong openness property.

Theorem 1.6. *(see [13, 10]) Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $o \in D$, and let φ be a negative plurisubharmonic function on D, such that $\varphi(o) = -\infty$. Let F be a holomorphic function on D. Assume that $\int_D |F|^2 e^{-\varphi} < +\infty$.*

Then for $p > 1$ satisfying

$$\frac{p}{p-1} > \frac{\int_D |F|^2 e^{-\varphi}}{B(F, \mathcal{I}_+(2c_o^F(\varphi)\varphi)_o, D)},$$

we have $|F|^2 e^{-p\varphi}$ is locally integrable near o, where $B(F, \mathcal{I}_+(2c_o^F(\varphi)\varphi)_o, D) > 0$.

The effectiveness result is sharp in some sense, which will be illustrated in the last section of this article.

2. Definition and basic properties of the Bergman kernel

Firstly, we recall a linear space of sequences of complex numbers,

$$\ell_1 := \{\xi = (\xi_\alpha)_{\alpha \in \mathbb{N}^n} : \sum_{\alpha \in \mathbb{N}^n} |\xi_\alpha| \rho^{|\alpha|} < +\infty, \text{ for any } \rho > 0\}.$$

Any element in ℓ_1 can be a linear functional over $\mathcal{O}_{z_0}$ for any $z_0 \in \mathbb{C}^n$ as follows.

For any $F(z) \in \mathcal{O}_{z_0}$, we can write that $F(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha$ near z_0. Then we define the value that ξ acts on $F(z)$ as

$$(\xi \cdot F)(z_0) := \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha \frac{F^{(\alpha)}(z_0)}{\alpha!} = \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha a_\alpha,$$

for any $\xi \in \ell_1$.

We prove that $(\xi \cdot F)(z_0)$ is well defined, which means the summation above is always absolutely convergent for all $F(z) \in \mathcal{O}_{z_0}$. In fact, there exists a polydisc $\Delta^n_{z_0,R}$ centered at z_0 with radius R, such that $F(z)$ is holomorphic on $\Delta^n_{z_0,R}$. Then according to Cauchy's inequality (see [15], Theorem 2.2.7), there exists a constant $M > 0$ such that

$$|a_\alpha| \leqslant \frac{M}{R^{|\alpha|}}, \ \forall \alpha \in \mathbb{N}^n.$$

It follows from $\xi \in \ell_1$ that there exists $M'(\rho) > 0$ such that

$$|\xi_\alpha| \rho^{|\alpha|} < M'(\rho), \ \forall \alpha \in \mathbb{N}^n,$$

for any $\rho > 1/R$. Then

$$\sum_{\alpha \in \mathbb{N}^n} |\xi_\alpha a_\alpha| < \sum_{\alpha \in \mathbb{N}^n} \frac{MM'(\rho)}{(\rho R)^{|\alpha|}} < +\infty,$$

which shows that $(\xi\cdot F)(z_0)$ is well defined.

Now the Bergman kernel can be defined as follows. For any $z\in D$, $\xi\in\ \ell_1$,

$$K_{\xi,D}(z):=\sup_{F\in A^2(D)}\frac{|(\xi\cdot F)(z)|^2}{\int_D|F|^2},$$

where D is a bounded domain in $\mathbb{C}^n$, $A^2(D)=L^2(D)\cap\mathcal{O}(D)$.

If $\xi\neq(0,0,\cdots,0,\cdots)$, ($\xi\neq 0$ for short), we can see that $K_{\xi,D}(z)>0$.

Secondly, we list some properties of the Bergman kernel.

Lemma 2.1. *If $z\in D_1\subseteq D_2$, then $K_{\xi,D_1}(z)\geqslant K_{\xi,D_2}(z)$, where D_1 and D_2 are bounded domains in $\mathbb{C}^n$, $\xi\in\ell_1$.*

Proof. For any $F\in A^2(D_2)$, it implies that $F\in A^2(D_1)$, then

$$\begin{aligned}K_{\xi,D_2}(z)&=\sup_{F\in A^2(D_2)}\frac{|(\xi\cdot F)(z)|^2}{\int_{D_2}|F|^2}\\&\leqslant\sup_{F\in A^2(D_2)}\frac{|(\xi\cdot F)(z)|^2}{\int_{D_1}|F|^2}\\&\leqslant\sup_{F\in A^2(D_1)}\frac{|(\xi\cdot F)(z)|^2}{\int_{D_1}|F|^2}=K_{\xi,D_1}(z).\end{aligned}$$

□

The following lemma shows that the functionals preserve the functions being holomorphic.

Lemma 2.2. *Let D be a bounded domain in $\mathbb{C}^n$, and let F be a holomorphic function on D. Then $(\xi\cdot F)(z)$ is also a holomorphic function on D, where $\xi\in\ell_1$.*

Proof. It suffices to prove that the summation

$$(\xi\cdot F)(z)=\sum_{\alpha\in\mathbb{N}^n}\xi_\alpha\frac{F^{(\alpha)}(z)}{\alpha!}$$

is absolutely and uniformly convergent on any compact subset of D.

Let K be a compact subset of D, then $|F(z)|\leqslant M(K)$ holds for some positive constant $M(K)$ and any $z\in\bigcup_{w\in K}\Delta^n_{w,d}$. Here $d=\mathrm{dist}(K,D^c)/2\sqrt{n}$. Recall the Cauchy's inequality ([15]), then

$$\left|\frac{F^{(\alpha)}(z)}{\alpha!}\right|\leqslant\frac{M(K)}{d^{|\alpha|}}$$

holds for any $\alpha\in\mathbb{N}^n$, $z\in K$. By the definition of $\xi\in\ell_1$, the above summation is absolutely and uniformly convergent on K, and so for any compact subset of D.

Note that $F^{(\alpha)}(z)$ is holomorphic for any $\alpha\in\mathbb{N}^n$, then $(\xi\cdot F)(z)$ is holomorphic on D by the theorem of Weierstrass (see [20], Theorem 1.6). □

The following lemma shows that the Bergman kernel is finite.

Lemma 2.3. *Let D be a bounded domain in $\mathbb{C}^n$. Then $K_{\xi,D}(z_0)<+\infty$ for any $z_0\in D$, $\xi\in\ell_1$.*

In fact, we can prove the following stronger result.

Lemma 2.4. *Let D be a bounded domain in $\mathbb{C}^n$, and let $\xi \in \ell_1$. Then for any compact subset $K \subseteq D$, there is a finite constant $C > 0$ such that*

$$|(\xi \cdot F)(z)|^2 \leqslant C \int_D |F|^2,$$

for any L^2 integrable holomorphic function F on D, and any $z \in K$.

Proof. It is trivial when $\xi = 0$. Now we assume $\xi \neq 0$. For the compact subset K, we are able to find some $R > 0$ such that the polydisc $\Delta^n_{z,R} \subseteq D$ for any $z \in K$. Then for any nonzero holomorphic function $F(z)$ on D, if we write that $F(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha$ in $\Delta^n_{z_0,R}$ for any $z_0 \in K$, then

$$\int_D |F|^2 \geqslant \int_{\Delta^n_{z_0,R}} |F|^2 = \sum_{\alpha \in \mathbb{N}^n} \frac{\pi^n |a_\alpha|^2}{(\alpha_1 + 1) \cdots (\alpha_n + 1)} R^{2(|\alpha| + n)}.$$

By Cauchy-Schwarz's inequality,

$$|(\xi \cdot F)(z_0)|^2 \leqslant \Big(\sum_{\alpha \in \mathbb{N}^n} \frac{(\alpha_1 + 1) \cdots (\alpha_n + 1)}{\pi^n R^{2(|\alpha| + n)}} |\xi_\alpha|^2\Big)\Big(\sum_{\alpha \in \mathbb{N}^n} \frac{\pi^n |a_\alpha|^2}{(\alpha_1 + 1) \cdots (\alpha_n + 1)} R^{2(|\alpha| + n)}\Big),$$

and it implies that

$$\frac{|(\xi \cdot F)(z_0)|^2}{\int_D |F|^2} \leqslant \sum_{\alpha \in \mathbb{N}^n} \frac{(\alpha_1 + 1) \cdots (\alpha_n + 1)}{\pi^n R^{2(|\alpha| + n)}} |\xi_\alpha|^2.$$

Since $\xi \in \ell_1$, we can choose some ρ with $\rho > 1/R$ such that

$$|\xi_\alpha| \rho^{|\alpha|} < M, \ \forall \alpha \in \mathbb{N}^n,$$

for some positive constant M. Hence

$$\begin{aligned}
\frac{|(\xi \cdot F)(z_0)|^2}{\int_D |F|^2} &\leqslant \sum_{\alpha \in \mathbb{N}^n} \frac{(\alpha_1 + 1) \cdots (\alpha_n + 1)}{\pi^n R^{2(|\alpha| + n)}} |\xi_\alpha|^2 \\
&\leqslant \sum_{\alpha \in \mathbb{N}^n} \frac{(\alpha_1 + 1) \cdots (\alpha_n + 1)}{\pi^n R^{2(|\alpha| + n)}} \cdot \frac{M^2}{\rho^{2|\alpha|}} \\
&= \frac{M^2}{\pi^n R^{2n}} \sum_{\alpha \in \mathbb{N}^n} (\alpha_1 + 1) \cdots (\alpha_n + 1) \frac{1}{(R\rho)^{2|\alpha|}} < +\infty.
\end{aligned}$$

Now we choose

$$C = \frac{M^2}{\pi^n R^{2n}} \sum_{\alpha \in \mathbb{N}^n} (\alpha_1 + 1) \cdots (\alpha_n + 1) \frac{1}{(R\rho)^{2|\alpha|}},$$

which is independent of the choice of $z_0 \in K$, and get the result. □

Now let $K = \{z_0\}$, then Lemma 2.3 can be induced by Lemma 2.4.
Lemma 2.4 will also be used to prove the following lemma.

Lemma 2.5. *Let D be a bounded domain in $\mathbb{C}^n$, and let $\{F_j\}$ be a sequence of holomorphic functions on D uniformly converging to F on every compact subset of D. Then for any $z \in D$, and $\xi \in \ell_1$, $\{(\xi \cdot F_j)(z)\}$ converges to $(\xi \cdot F)(z)$ uniformly on every compact subset of D.*

Proof. For any compact set $K \subseteq D$, we can find some open set D' such that $K \subseteq D' \subset\subset D$. Then Lemma 2.4 shows that there exists a positive constant C such that $|(\xi \cdot f)(z)|^2 \leqslant C \int_{D'} |f|^2$ for any holomorphic function f on D and $z \in K$. Then we have

$$|(\xi \cdot F_j)(z) - (\xi \cdot F)(z)|^2 = |(\xi \cdot (F_j - F))(z)|^2 \leqslant C \int_{D'} |F_j - F|^2 \to 0, \ j \to +\infty,$$

for any $z \in K$. This means $\{(\xi \cdot F_j)(z)\}$ converges to $(\xi \cdot F)(z)$ uniformly on every compact subset of D. $\square$

Lemma 2.5 can be used to prove the following lemma.

Lemma 2.6. *Let D be a bounded domain in $\mathbb{C}^n$, and let $z \in D$. Then for any $\xi \in \ell_1$, there exists a holomorphic function F_0 on D such that*

$$K_{\xi,D}(z) = \frac{|(\xi \cdot F_0)(z)|^2}{\int_D |F_0|^2}.$$

Proof. It is trivial when $\xi = 0$. Now we assume $\xi \neq 0$.

By the definition of Bergman kernel, there exists a sequence of holomorphic functions $\{F_j\}$ on D such that $\int_D |F_j|^2 = 1$, and $\lim\limits_{j\to+\infty} |(\xi \cdot F_j)(z)|^2 = K_{\xi,D}(z)$. Then by Montel's theorem (see [20], Theorem 1.5), there is a subsequence of $\{F_j\}$ which is uniformly convergent on every compact subset of D. Denote the limit of the subsequence by F_0. Then Fatou's lemma and Lemma 2.5 imply that $\int_D |F_0|^2 \leqslant 1$, and $|(\xi \cdot F_0)(z)|^2 = K_{\xi,D}(z)$. It means that $\frac{|(\xi \cdot F_0)(z)|^2}{\int_D |F_0|^2} \geqslant K_{\xi,D}(z)$. Then by the definition of $K_{\xi,D}(z)$, we get $\frac{|(\xi \cdot F_0)(z)|^2}{\int_D |F_0|^2} = K_{\xi,D}(z)$. $\square$

Combining Lemma 2.2 with Lemma 2.5, we get the following result.

Lemma 2.7. *Let D be a bounded domain in $\mathbb{C}^n$, and let $\{z_j\}$ be a sequence of points in D such that $\lim\limits_{j\to+\infty} z_j = z \in D$. Let $\{F_j\}$ be a sequence of holomorphic functions on D uniformly converging to F on every compact subset of D. Then for any $\xi \in \ell_1$, $\lim\limits_{j\to+\infty} (\xi \cdot F_j)(z_j) = \xi \cdot F(z)$.*

Proof. For any $\varepsilon > 0$, using Lemma 2.2, we can find a positive integer N_1, such that for any $j > N_1$,

$$|(\xi \cdot F)(z_j) - (\xi \cdot F)(z)| < \frac{\varepsilon}{2}.$$

Note that the set $\{z_j\}_{j=1}^{+\infty} \cup \{z\}$ is a compact subset of D, then using Lemma 2.5, we can also find a positive integer N_2, such that for any $j > N_2$,

$$|(\xi \cdot F_j)(z_j) - (\xi \cdot F)(z_j)| < \frac{\varepsilon}{2}.$$

Let $N = \max\{N_1, N_2\}$, then for any $j > N$,

$$|(\xi \cdot F_j)(z_j) - (\xi \cdot F)(z)| < \varepsilon.$$

The proof is done. $\square$

The following lemma shows that the Bergman kernel is continuous.

Lemma 2.8. *Let D be a bounded domain in $\mathbb{C}^n$. Then for any $\xi \in \ell_1$, $K_{\xi,D}(z)$ is a continuous function on D.*

Proof. It is trivial when $\xi = 0$. Now we assume $\xi \neq 0$. By the definition of $K_{\xi,D}(z)$,

$$K_{\xi,D} = \sup\{|(\xi \cdot F)(z)| : \int_D |F|^2 = 1, F \in \mathcal{O}(D)\}.$$

Combining with Lemma 2.2, we know that $K_{\xi,D}(z)$ is lower semicontinuous.

Next we prove that $K_{\xi,D}(z)$ is also upper continuous. Let $\{z_j\}$ be a sequence of points in D such that $\lim_{j\to+\infty} z_j = z_0 \in D$. And we may assume that $\{z_{k_j}\}$ is the subsequence of $\{z_j\}$ such that

$$\lim_{j\to+\infty} K_{\xi,D}(z_{k_j}) = \limsup_{j\to+\infty} K_{\xi,D}(z_j).$$

Using Lemma 2.6, we can get a sequence of holomorphic function $\{F_j\}$ on D, such that $\int_D |F_j|^2 = 1$, and $|(\xi \cdot F_j)(z_j)|^2 = K_{\xi,D}(z_j)$, for any $j \geqslant 1$. Then using Montel's theorem and the diagonal method, we can select a subsequence of $\{F_{k_j}\}$ which is uniformly convergent on every compact subset of D. We may denote the subsequence by $\{F_{k_j}\}$ itself, and denote the limit function by F_0. It follows from Fatou's lemma and Lemma 2.7 that $\int_D |F_0|^2 \leqslant 1$, $\lim_{j\to+\infty} (\xi \cdot F_{k_j})(z_{k_j}) = (\xi \cdot F_0)(z_0)$. Then

$$\begin{aligned}
K_{\xi,D}(z_0) &\geqslant \frac{|(\xi \cdot F_0)(z_0)|^2}{\int_D |F_0|^2} \\
&\geqslant |(\xi \cdot F_0)(z_0)|^2 \\
&= \lim_{j\to+\infty} |(\xi \cdot F_{k_j})(z_{k_j})|^2 \\
&= \lim_{j\to+\infty} K_{\xi,D}(z_{k_j}) \\
&= \limsup_{j\to+\infty} K_{\xi,D}(z_j).
\end{aligned}$$

We get that $K_{\xi,D}(z)$ is upper semicontinuous.

It is known that $K_{\xi,D}(z)$ is lower semicontinuous, which implies that $K_{\xi,D}(z)$ is continuous. □

The Bergman kernel is also able to be approximated by which of the exhausted domains.

Lemma 2.9. *Let D_j and D be bounded domains in $\mathbb{C}^n$, such that $D_j \subseteq D$ for all $j \geqslant 1$. Assume that for any compact subset of D denoted by K, there exists $j_K \geqslant 1$, such that $K \subseteq D_j$ for any $j \geqslant j_K$. Let $\{z_j\}$ be a sequence of points in D such that $z_j \in D_j$, $\lim_{j\to+\infty} z_j = z \in D$. Then for any $\xi \in \ell_1$,*

$$\lim_{j\to+\infty} K_{\xi,D_j}(z_j) = K_{\xi,D}(z).$$

Proof. On the one hand, it is clear that $K_{\xi,D_j}(z_j) \geqslant K_{\xi,D}(z_j)$ by Lemma 2.1, and $\lim_{j\to+\infty} K_{\xi,D}(z_j) = K_{\xi,D}(z)$ by Lemma 2.8. Then we have

$$\liminf_{j\to+\infty} K_{\xi,D_j}(z_j) \geqslant \liminf_{j\to+\infty} K_{\xi,D}(z_j) = K_{\xi,D}(z).$$

On the other hand, we may assume that $\{z_{k_j}\}$ is the subsequence of $\{z_j\}$ such that

$$\lim_{j\to+\infty} K_{\xi,D_{k_j}}(z_{k_j}) = \limsup_{j\to+\infty} K_{\xi,D_j}(z_j).$$

And we may assume $z \in D_{j_0}$. Then using Lemma 2.6, we can get a sequence of holomorphic function $\{F_j\}$ on D_j, such that $\int_{D_j} |F_j|^2 = 1$, and $|(\xi \cdot F_j)(z_j)|^2 = K_{\xi,D_j}(z_j)$, for any $j \geqslant j_0$. Since for any compact subset K of D, there is $j_K \geqslant 1$, such that $K \subseteq D_j$ for any $j \geqslant j_K$, then we can use Montel's theorem and the diagonal method to get a subsequence of $\{F_{k_j}\}$ which is uniformly convergent on every compact subset of D. We may denote the subsequence by $\{F_{k_j}\}$ itself, and denote the limit function by F_0. Fatou's lemma and Lemma 2.7 imply that $\int_D |F_0|^2 \leqslant 1$, and $\lim_{j\to+\infty}(\xi \cdot F_{k_j})(z_{k_j}) = (\xi \cdot F_0)(z)$. It follows that

$$\begin{aligned} K_{\xi,D}(z) &\geqslant \frac{|(\xi \cdot F_0)(z)|^2}{\int_D |F_0|^2} \\ &\geqslant |(\xi \cdot F_0)(z)|^2 \\ &= \lim_{j\to+\infty} |(\xi \cdot F_{k_j})(z_{k_j})|^2 \\ &= \lim_{j\to+\infty} K_{\xi,D_{k_j}}(z_{k_j}) \\ &= \limsup_{j\to+\infty} K_{\xi,D_j}(z_j). \end{aligned}$$

Then

$$\liminf_{j\to+\infty} K_{\xi,D_j}(z_j) \geqslant K_{\xi,D}(z) \geqslant \limsup_{j\to+\infty} K_{\xi,D_j}(z_j),$$

which implies

$$\lim_{j\to+\infty} K_{\xi,D_j}(z_j) = K_{\xi,D}(z).$$

□

The Bergman kernel is log-plurisubharmonic.

Lemma 2.10. *Let D be a bounded domain in $\mathbb{C}^n$, and let $\xi \in \ell_1$. Then $\log K_{\xi,D}(z)$ is plurisubharmonic on D.*

Proof. There is $\log K_{\xi.D}(z) \equiv -\infty$ when $\xi = 0$. Now we assume $\xi \neq 0$. By the definition, we have

$$\log K_{\xi,D}(z) = \sup\{2\log|(\xi \cdot F)(z)| : \int_D |F|^2 = 1, F \in \mathcal{O}(D)\}.$$

Lemma 2.2 shows that $(\xi \cdot F)(z)$ is holomorphic on D, when F is holomorphic on D. Then $\log|(\xi \cdot F)(z)|$ is plurisubharmonic. As $\log K_{\xi,D}(z)$ is upper semicontinuous according to Lemma 2.8, $\log K_{\xi,D}(z)$ is plurisubharmonic on D. □

3. Optimal L^2 extension and Guan-Zhou method

In this section, we will recall the Guan-Zhou Method, i.e. an optimal L^2 extension approach to a log-convexity property of the fibrewised Bergman kernel.

Let Ω be a pseudoconvex domain in $\mathbb{C}^{n+1}$ with coordinate (z,t), where $z \in \mathbb{C}^n$, $t \in \mathbb{C}$. Let p, q be the natural projections $p(z,t) = t$, $q(z,t) = z$ on Ω, $p(\Omega) = D$. For any $t \in D$, suppose that $\Omega_t = p^{-1}(t) \subseteq \Omega$ is bounded in $\mathbb{C}^n$. Let $K_{\xi,t}(z) = K_{\xi,\Omega_t}(z)$ be the Bergman kernels of the domains Ω_t defined as which in the above section with respect to some fixed $\xi \in \ell_1$.

We will use the following version of the optimal L^2 extension theorem.

Lemma 3.1 (Optimal L^2 extension theorem ([4], see [21, 22]))**.** *Let $D = \Delta_{t_0,r}$ be the disk in the complex plane centered on t_0 with radius r. Then for every holomorphic and L^2 integrable function f on Ω_{t_0}, there exists a holomorphic function F on Ω, such that $F|_{\Omega_{t_0}} = f$, and*

$$\frac{1}{\pi r^2}\int_{\Omega}|F|^2 \leqslant \int_{\Omega_{t_0}}|f|^2.$$

Guan-Zhou Method shows that Lemma 3.1 implies the following

Proposition 3.2. *(see [1, 17, 14]) $\log K_{\xi,t}(z)$ is a plurisubharmonic function with respect to (z,t), for any $\xi \in \ell_1$.*

Proof. Firstly, we prove that $\log K_{\xi,t}(z)$ is upper semicontinuous on Ω. Let (z_j,t_j) be a sequence of points in Ω, such that $(z_j,t_j) \to (z_0,t_0) \in \Omega$, $j \to +\infty$. Since Ω is a domain in $\mathbb{C}^{n+1}$, we know that for any compact subset of $q(\Omega_{t_0})$ denoted by K, there exists $j_K \geqslant 1$, such that $K \subseteq q(\Omega_{t_j})$ in the sense of domains in $\mathbb{C}^n$, for any $j \geqslant j_K$.

We denote $q(\Omega_{t_j})$ by Ω_j. Then we may assume that $\{(z_{k_j},t_{k_j})\}$ is the subsequence of $\{(z_j,t_j)\}$ such that

$$\lim_{j\to+\infty} K_{\xi,\Omega_{k_j}}(z_{k_j}) = \limsup_{j\to+\infty} K_{\xi,\Omega_j}(z_j),$$

and $z_0 \in \Omega_{j_0}$. Using Lemma 2.6, we can get a sequence of holomorphic function $\{F_j\}$ on Ω_j, such that $\int_{\Omega_j}|F_j|^2 = 1$, and $|(\xi\cdot F_j)(z_j)|^2 = K_{\xi,\Omega_j}(z_j)$, for any $j \geqslant j_0$. Since for any compact subset K of $q(\Omega_{t_0})$, there exists $j_K \geqslant 1$, such that $K \subseteq \Omega_j$ for any $j \geqslant j_K$, then we can use Montel's theorem and the diagonal method to get a subsequence of $\{F_{k_j}\}$ which is uniformly convergent on every compact subset of $q(\Omega_{t_0})$. We may denote the subsequence by $\{F_{k_j}\}$ itself, and denote the limit function by F_0. Fatou's lemma and Lemma 2.7 imply that $\int_{q(\Omega_{t_0})}|F_0|^2 \leqslant 1$, and $\lim_{j\to+\infty}(\xi\cdot F_{k_j})(z_{k_j}) = (\xi\cdot F_0)(z_0)$. It follows that

$$\begin{aligned}
K_{\xi,\Omega_{t_0}}(z_0) &\geqslant \frac{|(\xi\cdot F_0)(z_0)|^2}{\int_{q(\Omega_{t_0})}|F_0|^2} \\
&\geqslant |(\xi\cdot F_0)(z_0)|^2 \\
&= \lim_{j\to+\infty}|(\xi\cdot F_{k_j})(z_{k_j})|^2 \\
&= \lim_{j\to+\infty} K_{\xi,\Omega_{k_j}}(z_{k_j}) \\
&= \limsup_{j\to+\infty} K_{\xi,\Omega_{t_j}}(z_j),
\end{aligned}$$

and

$$\limsup_{j\to+\infty}\log K_{\xi,t_j}(z_j) \leqslant \log K_{\xi,t_0}(z_0).$$

Then we obtain that $\log K_{\xi,t}(z)$ is upper semicontinuous on Ω.

Secondly we need to check that for any complex line L, $\log K_{\xi,t}(z)|_L$ is subharmonic. If the complex line lies on some Ω_t for some fixed t, we know that $\log K_{\xi,t}(z)|_L$ is subharmonic using Lemma 2.10. Then without loss of generality, we assume that L is the complex line on $\{t|(z,t)\}$ and $D = \Delta_{t_0,r} = L$.

If $\log K_{\xi,t_0}(z)=-\infty$, we are done. Then we assume that there exists $f\in A^2(\Omega_{t_0})$ such that

$$K_{\xi,t_0}(z)=\frac{|(\xi\cdot f)(z)|^2}{\int_{\Omega_{t_0}}|f|^2}.$$

Using the optimal L^2 extension theorem (Lemma 3.1), we can get a holomorphic function F on Ω such that $F(z,t_0)=f(z)$ and

$$\frac{1}{\pi r^2}\int_\Omega |F|^2\leqslant\int_{\Omega_{t_0}}|f|^2.$$

Denote that $F_t(z)=F(z,t)=F|_{\Omega_t}$. Note that the function $y=\log x$ is concave, and by Jensen's inequality, it follows from Guan-Zhou Method ([14], see also [21],[22]) that

$$\begin{aligned}
\log\left(\int_{\Omega_{t_0}}|f|^2\right)&\geqslant\log\left(\frac{1}{\pi r^2}\int_\Omega|F|^2\right)\\
&=\log\left(\frac{1}{\pi r^2}\int_{\Delta_{t_0,r}}\int_{\Omega_t}|F_t|^2\right)\\
&\geqslant\frac{1}{\pi r^2}\int_{\Delta_{t_0,r}}\log\left(\int_{\Omega_t}|F_t|^2\right)\\
&\geqslant\frac{1}{\pi r^2}\int_{\Delta_{t_0,r}}\left(\log|(\xi\cdot F_t)(z)|^2-\log K_{\xi,t}(z)\right)\\
&\geqslant\log|(\xi\cdot f)(z)|^2-\frac{1}{\pi r^2}\int_{\Delta_{t_0,r}}\log K_{\xi,t}(z).
\end{aligned}$$

The last inequality above holds, since we can prove that $\log|(\xi\cdot F_t)(z)|^2$ is subharmonic with respect to t. In fact, if we write

$$F_t(w)=\sum_{\alpha\in\mathbb{N}^n}a_\alpha(t)(w-z)^\alpha,$$

then all the

$$a_\alpha(t)=\frac{1}{\alpha!}\frac{\partial^\alpha F(w,t)}{\partial w^\alpha}(z,t)$$

are holomorphic with respect to t. In addition, $(\xi\cdot F_t)(z)=\sum_{\alpha\in\mathbb{N}^n}\xi_\alpha a_\alpha(t)$ is absolutely and uniformly convergent on every compact subset of $\Delta_{t_0,r}$. Since for any compact subset of $\Delta_{t_0,r}$, denoted by K, we can find some $R>0$ such that $\Delta^n_{z,R}\subseteq q(\Omega_t)$ for any $t\in K$. Combining with $\xi\in\ell_1$, we get that

$$|\xi_\alpha a_\alpha(t)|\leqslant\frac{MM'}{(\rho R)^{|\alpha|}},\ \forall\alpha\in\mathbb{N}^n$$

for some $M,M'>0$, $\rho>1/R$, and any $t\in K$. This means that $\sum_{\alpha\in\mathbb{N}^n}\xi_\alpha a_\alpha(t)$ is absolutely and uniformly convergent on K. Then $(\xi\cdot F_t)(z)$ is holomorphic with respect to t for any fixed z, which implies that $\log|(\xi\cdot F_t)(z)|^2$ is subharmonic with respect to t. Then

$$\log K_{\xi,t_0}(z)\leqslant\frac{1}{\pi r^2}\int_{t\in\Delta_{t_0,r}}\log K_{\xi,t}(z),$$

which implies that $\log K_{\xi,t}(z)$ is plurisubharmonic with respect to (z,t).

□

4. EFFECTIVENESS RESULT OF STRONG OPENNESS PROPERTY

In this section, we complete the L^2 extension approach to the effectiveness of the strong openness property.

Let I be an ideal of $\mathcal{O}_o$ such that $I \neq \mathcal{O}_o$. We consider about a subset $\ell_I \subseteq \ell_1$ such as

$$\ell_I = \{0 \neq \xi \in \ell_1 : (\xi \cdot F)(o) = 0,\ \forall F \in I\}.$$

It is obvious that ℓ_I is always nonempty since $\xi = (1, 0, \ldots, 0, \ldots) \in \ell_I$ when $I \neq \mathcal{O}_o$.

Especially, we denote $\ell_{\mathcal{I}(\varphi)_o}$ by ℓ_φ.

In addition, denote that

$$D_t := \{z \in D : \varphi(z) < -t\},$$

and

$$K_\xi(t) := K_{\xi, D_t}(o),$$

for $t \in [0, +\infty)$. We need the following lemma.

Lemma 4.1 (see [6], Theorem 5.13, Chapter I)**.** *Let $\Omega = I + \mathrm{i}\mathbb{R}$ be a domain in $\mathbb{C}$ with the coordinate $z = x + \mathrm{i}y$, where I is an interval in $\mathbb{R}$. Let $\phi(z)$ be a subharmonic function on Ω which is independent of y. Then $\phi(x) := \phi(x + \mathrm{i}\mathbb{R})$ is a convex function with respect to $x \in I$.*

This result is also used in [1], [3] and [17].

Note that the domain

$$\{(\tau, z) : \varphi(z) - \mathrm{Re}\ \tau < 0\}$$

is pseudoconvex in $\mathbb{C}^{n+1}$, then according to Proposition 3.2, $\log K_\xi(o, \tau)$ is subharmonic for $\tau \in [0, +\infty) + \mathrm{i}\mathbb{R}$, and independent of $\mathrm{Im}\ \tau$. Lemma 4.1 shows that $\log K_\xi(t)$ is convex for $t \in [0, +\infty)$. This implies that $\log K_\xi^{-1}(t) + t$ is concave, which will be increasing if it has a lower bound. We state the following result.

Lemma 4.2. *Let D be a pseudoconvex domain in $\mathbb{C}^n$ such that $o \in D$, and let φ be a negative plurisubharmonic function on D such that $\varphi(o) = -\infty$, and $\mathcal{I}(\varphi)_o \neq \mathcal{O}_o$. Then for any fixed $\xi \in \ell_\varphi$,*

$$\log K_\xi^{-1}(t) + t \geqslant \log K_\xi^{-1}(0),\ \forall t \in [0, +\infty).$$

Lemma 4.2 can be proved by the following lemma.

Lemma 4.3 (see [19])**.** *Let D be a pseudoconvex domain in $\mathbb{C}^n$ such that $o \in D$, and let φ be a negative plurisubharmonic function on D such that $\varphi(o) = -\infty$. Let F be an L^2 integrable holomorphic function on $\{\varphi < -t_0\}$. Then there exists a holomorphic function F_{t_0} on D, such that*

$$(F_{t_0} - F, o) \in \mathcal{I}(\varphi)_o$$

and

$$\int_D |F_{t_0} - (1 - b_{t_0}(\varphi))F|^2 \leqslant C_D \int_{\{-t_0 - 1 < \varphi < -t_0\}} |F|^2 e^{-\varphi},$$

where $b_{t_0}(t) = \int_{-\infty}^{t} \mathbb{I}_{\{-t_0-1<s<-t_0\}} \mathrm{d}s$ and $t_0 \geqslant 0$, and here C_D is a positive constant only dependent of D.

Lemma 4.3 can also be referred to [14, 11, 13, 10].

Proof of Lemma 4.2. We only need to prove that there is a lower bound of $\log K_\xi^{-1}(t)+t$.

Lemma 2.6 shows that for any $t\in[0,+\infty)$, there exists $F\in\mathcal{O}(\{\varphi<-t\})$, such that

$$K_\xi^{-1}(t)=\frac{|(\xi\cdot F)(o)|^2}{\int_{\{\varphi<-t\}}|F|^2}.$$

It follows from Lemma 4.3 that there exists a holomorphic function F_t on D such that

$$(F_t-F,o)\in\mathcal{I}(\varphi)_o,$$

and

$$\int_D|F_t-(1-b_t(\varphi))F|^2\leqslant C_D\int_{\{-t-1<\varphi<-t\}}|F|^2e^{-\varphi}.$$

Note that $\xi\in\ell_\varphi$, then $(F_t-F,o)\in\mathcal{I}(\varphi)_o$ induces that $(\xi\cdot F_t)(o)=(\xi\cdot F)(o)$. On the one hand,

$$\begin{aligned}&\left(\int_D|F_t-(1-b_t(\varphi))F|^2\right)^{\frac{1}{2}}\\ \geqslant&\left(\int_D|F_t|^2\right)^{\frac{1}{2}}-\left(\int_D|(1-b_t(\varphi))F|^2\right)^{\frac{1}{2}}\\ \geqslant&K_{\xi,D}^{-\frac{1}{2}}(o)|(\xi\cdot F_t)(o)|-\left(\int_{\{\varphi<-t\}}|F|^2\right)^{\frac{1}{2}}.\\ =&K_{\xi,D}^{-\frac{1}{2}}(o)|(\xi\cdot F)(o)|-\left(\int_{\{\varphi<-t\}}|F|^2\right)^{\frac{1}{2}}.\end{aligned}$$

On the other hand, we have

$$\int_{\{-t-1<\varphi<-t\}}|F|^2e^{-\varphi}\leqslant e^{t+1}\int_{\{-t-1<\varphi<-t\}}|F|^2\leqslant e^{t+1}\int_{\{\varphi<-t\}}|F|^2.$$

Then

$$(C_D^{\frac{1}{2}}e^{\frac{t+1}{2}}+1)^2\int_{\{\varphi<-t\}}|F|^2\geqslant K_{\xi,D}^{-1}(o)|(\xi\cdot F)(o)|^2,$$

which means

$$\int_{\{\varphi<-t\}}|F|^2\geqslant Ce^{-t}|(\xi\cdot F)(o)|^2,$$

where $C=(2(e+1)K_{\xi,D}(o)\max\{C_D,1\})^{-1}$ is a positive constant independent of the choises of F and t.

Then we get that

$$\log K_\xi^{-1}(t)+t\geqslant\log C,$$

for any $t\in[0,+\infty)$. Then $\log K_\xi^{-1}(t)+t$ has a lower bound, inducing that it is increasing, and

$$\log K_\xi^{-1}(t)+t\geqslant\log K_\xi^{-1}(0).$$

□

By Lemma 4.2, we get the following proposition.

Proposition 4.4. *Let D be a pseudoconvex domain in $\mathbb{C}^n$ such that $o \in D$, and let φ be a negative plurisubharmonic function on D such that $\varphi(o) = -\infty$, and $\mathcal{I}(\varphi)_o \neq \mathcal{O}_o$. Let F be a holomorphic function on D. Then for any $p > 1$, and any $\xi \in \ell_\varphi$,*

$$\int_D |F|^2 e^{-\frac{\varphi}{p}} \geqslant \frac{p}{p-1} \cdot \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)}.$$

Proof. Using Lemma 4.2, we have that $K_\xi^{-1}(t) \geqslant e^{-t} K_\xi^{-1}(0)$ for any $t \in [0, +\infty)$ if $\xi \in \ell_\varphi$. And It is known that

$$\int_D |F|^2 e^{-\frac{\varphi}{p}} = \int_{-\infty}^{+\infty} \left(\int_{\{\frac{\varphi}{p} < -t\}} |F|^2 \right) e^t \mathrm{d}t$$

(this equality can be referred in [10]), which implies that

$$\begin{aligned}
&\int_0^{+\infty} \left(\int_{\{\varphi < -pt\}} |F|^2 \right) e^t \mathrm{d}t \\
\geqslant &\int_0^{+\infty} |(\xi \cdot F)(o)|^2 K_\xi^{-1}(pt) e^t \mathrm{d}t \\
\geqslant &|(\xi \cdot F)(o)|^2 K_\xi^{-1}(0) \int_0^{+\infty} e^{(1-p)t} \mathrm{d}t = \frac{1}{p-1} \cdot \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)},
\end{aligned}$$

and

$$\int_{-\infty}^0 \left(\int_{\{\varphi < -pt\}} |F|^2 \right) e^t \mathrm{d}t \geqslant \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)} \int_{-\infty}^0 e^t \mathrm{d}t = \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)}.$$

Then

$$\int_D |F|^2 e^{-\frac{\varphi}{p}} \geqslant \frac{p}{p-1} \cdot \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)}.$$

□

Let D be a bounded domain in $\mathbb{C}^n$, and let I be an ideal of $\mathcal{O}_o$ such that $I \neq \mathcal{O}_o$. Let (F, o) be an element in $\mathcal{O}_o$. Then we denote that

$$B(F, I, D) = \sup_{\xi \in \ell_I} \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)}.$$

Especially, we denote $B(F, \mathcal{I}(\varphi)_o, D)$ by $B(F, \varphi, D)$. Then we get the following corollary of Proposition 4.4.

Corollary 4.5. *Let D be a pseudoconvex domain in $\mathbb{C}^n$ such that $o \in D$, and let φ be a negative plurisubharmonic function on D such that $\varphi(o) = -\infty$, and $\mathcal{I}(\varphi)_o \neq \mathcal{O}_o$. Let F be a holomorphic function on D such that $F \notin \mathcal{I}(\varphi)_o$. Then for any $p > 1$,*

$$\int_D |F|^2 e^{-\frac{\varphi}{p}} \geqslant \frac{p}{p-1} B(F, \varphi, D).$$

The following proposition shows that if $(F, o) \notin I$, then $B(F, I, D) > 0$.

Proposition 4.6. *Let D be a bounded domain in $\mathbb{C}^n$ such that $o \in D$, and let I be an ideal of $\mathcal{O}_o$ such that $I \neq \mathcal{O}_o$. Let $(F, o) \in \mathcal{O}_o$ such that $(F, o) \notin I$. Then we are able to find some $\xi \in \ell_I$ such that $(\xi \cdot F)(o) \neq 0$, or in other words, $B(F, I, D) > 0$.*

We will prove it with the separating theorem (Theorem 3.5 in [23]) in functional analysis theory, which is a corollary of the Hahn-Banach theorem.

Lemma 4.7 (The separating theorem)**.** *Let M be a subspace of a locally convex space X over the complex field, and $x_0 \in X$. If x_0 is not in the closure of M, then there exists $\Lambda \in X^*$ (the dual space of X) such that $\Lambda x_0 = 1$ but $\Lambda x = 0$ for every $x \in M$.*

We give the proof of Proposition 4.6 as follows.

Proof of Proposition 4.6. We consider the analytic Krull topology on $\mathcal{O}_o$, which induced by the separating family of seminorms $\sum a_\alpha z^\alpha \mapsto |a_\alpha|$. Then $\mathcal{O}_o$ is a locally convex space under the analytic Krull topology. And the ideal I is closed in $\mathcal{O}_o$ under the analytic Krull topology (see chapter IX of [6]).

Then using the separating theorem (Lemma 4.7), we can find some $\eta \in \mathcal{O}_o^{dual}$ such that $\eta \cdot F = 1$ and $\eta \cdot g = 0$ for every $g \in I$, since $F \notin I$. Here $\mathcal{O}_o^{dual}$ is the dual space of $\mathcal{O}_o$ under the analytic Krull topology.

Now we prove that $\mathcal{O}_o^{dual}$ equals to ℓ_1 as sets. Suppose $\eta \in \mathcal{O}_o^{dual}$, and $\eta \cdot z^\alpha = \eta_\alpha$ for any $\alpha \in \mathbb{N}^n$. Since η is linear, we have

$$\eta \cdot (\sum_{|\alpha| \leqslant k} a_\alpha z^\alpha) = \sum_{|\alpha| \leqslant k} \eta_\alpha a_\alpha.$$

And since η is continuous, we have

$$\eta \cdot (\sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha) = \sum_{\alpha \in \mathbb{N}^n} \eta_\alpha a_\alpha.$$

Then we need to ensure that $\sum_{\alpha \in \mathbb{N}^n} \eta_\alpha a_\alpha$ is convergent for any $g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha \in \mathcal{O}_o$. Take

$$g = \sum_{\alpha \in \mathbb{N}^n} \frac{e^{-i \arg(\eta_\alpha)}}{R^{|\alpha|}} z^\alpha \in \mathcal{O}_o,$$

for any $R > 0$. Then

$$\eta \cdot g = \sum_{\mathbb{N}^n} |\eta_\alpha| \frac{1}{R^{|\alpha|}}.$$

If we denote that $\eta^* = (\eta_\alpha)_{\alpha \in \mathbb{N}^n}$, then $\eta^* \in \ell_1$ by the above computation. Moreover, we have that $\eta \cdot g = (\eta^* \cdot g)(o)$. Thus each element in $\mathcal{O}_o^{dual}$ can be seen as an element in ℓ_1.

In addition, each element in ℓ_1 can be seen as an element in $\mathcal{O}_o^{dual}$. For any $\xi = (\xi_\alpha)_{\alpha \in \mathbb{N}^n} \in \ell_1$, and $g = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$, let $\tilde{\xi} \cdot g := \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha a_\alpha$, then the summation is absolutely convergent. It is also clear that $\tilde{\xi}$ is linear, and $\tilde{\xi}$ is continuous under each seminorm $\sum a_\alpha z^\alpha \mapsto |a_\alpha|$. It means that $\tilde{\xi} \in \mathcal{O}_o^{dual}$ with respect to the analytic Krull topology on $\mathcal{O}_o$.

Then we get that $\mathcal{O}_o^{dual}$ equals to ℓ_1 as sets. For any ideal I of $\mathcal{O}_o$ and $(F, o) \in \mathcal{O}_o$ with $(F, o) \notin I$, since we can find some $\eta \in \mathcal{O}_o^{dual}$ such that $\eta \cdot F = 1$ and $\eta \cdot g = 0$, there exists $\xi \in \ell_1$ with $\xi \in \ell_I$ and $(\xi \cdot F)(o) \neq 0$. This means that $B(F, I, D) > 0$. $\square$

If F is holomorphic and L^2 integrable on D, then $B(F,I,D) \leqslant \int_D |F|^2$, since

$$B(F,I,D) = \sup_{\xi\in\ell_I,\ (\xi\cdot F)(o)\neq 0} \frac{|(\xi\cdot F)(o)|^2}{K_{\xi,D}(o)}$$
$$\leqslant \sup_{\xi\in\ell_I,\ (\xi\cdot F)(o)\neq 0} \frac{|(\xi\cdot F)(o)|^2}{|(\xi\cdot F)(o)|^2/\int_D |F|^2} = \int_D |F|^2.$$

It is clear that if there are two ideals I_1 and I_2 of $\mathcal{O}_o$ such that $I_1 \subseteq I_2 \neq \mathcal{O}_o$, then $B(F,I_1,D) \geqslant B(F,I_2,D)$.

Now we prove Theorem 1.6.

Proof of Theorem 1.6: For $p > 2c_o^F(\varphi)$, $|F|^2e^{-p\varphi}$ is not integrable near o, and $B(F,p\varphi,D) \geqslant B(F,\mathcal{I}_+(2c_o^F(\varphi)\varphi)_o,D)$. Then Corollary 4.5 shows that

$$\int_D |F|^2e^{-\varphi} \geqslant \frac{p}{p-1}B(F,\mathcal{I}_+(2c_o^F(\varphi)\varphi)_o,D).$$

Let $p \to 2c_o^F(\varphi)^+$, the above inequality also holds for $p \geqslant 2c_o^F(\varphi)$. Then if $p > 1$ satisfing

$$\int_D |F|^2e^{-\varphi} < \frac{p}{p-1}B(F,\mathcal{I}_+(2c_o^F(\varphi)\varphi)_o,D),$$

we get $p < 2c_o^F(\varphi)$, which means that $|F|^2e^{-p\varphi}$ is integrable near o.

Since $(F,o) \notin \mathcal{I}_+(2c_o^F(\varphi)\varphi)_o$, we know that

$$0 < B(F,\mathcal{I}_+(2c_o^F(\varphi)\varphi)_o,D) \leqslant \int_D |F|^2 < \int_D |F|^2e^{-\varphi}.$$

Then the proof is done. □

5. The sharpness of Proposition 1.4 and Theorem 1.6

Firstly, we show that the inequality in Proposition 1.4 is sharp. Assume that D is the unit disk $\Delta \subseteq \mathbb{C}$, $F \equiv 1$ and $\varphi = 2\log|z|$. Let $\xi_0 = (1,0,\cdots,0,\cdots) \in \ell_\varphi$, and for any $p > 1$,

$$\int_\Delta |F|^2e^{-\frac{\varphi}{p}} = \frac{p}{p-1}\pi = \frac{p}{p-1}\cdot\frac{|\xi_0\cdot F)(o)|^2}{K_{\xi_0,D}(o)}.$$

This implies that Proposition 1.4 is sharp.

Secondly, we show that the effectiveness result in Theorem 1.6 is also sharp. Assume that D is the unit disk $\Delta \subseteq \mathbb{C}$, $F \equiv 1$ and $\varphi = \frac{2}{p}\log|z|$, $p > 1$. Then $\int_D |F|^2e^{-\varphi} = \frac{p}{p-1}\pi$, and $\ell_{\mathcal{I}_+(2c_o^F(\varphi)\varphi)_o} = \mathbb{C}^*\xi_0$, inducing that $B(F,\mathcal{I}_+(2c_o^F(\varphi)\varphi)_o,D) = \pi$, and

$$\frac{\int_D |F|^2e^{-\varphi}}{B(F,\mathcal{I}_+(2c_o^F(\varphi)\varphi)_o,D)} = \frac{p}{p-1}.$$

In addition, we know that for $q > 1$, it is equivalent to $q < p$ that $|F|^2e^{-q\varphi}$ is locally integrable near o, and $q < p \Leftrightarrow \frac{q}{q-1} > \frac{p}{p-1}$ for $p,q > 1$. This means that Theorem 1.6 is sharp.

At last, we show that Theorem 1.6 implies the sharp effectiveness result of the openness conjecture ([10]).

Corollary 5.1. *([10]) Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$, $o \in D$, and let φ be a negative plurisubharmonic function on D, $\varphi(o) = -\infty$. Assume that $\int_D e^{-\varphi} < +\infty$, then for $p > 1$ satisfying*

$$\frac{p}{p-1} > K_D^{-1}(o) \int_D e^{-\varphi},$$

$|F|^2 e^{-p\varphi}$ is locally integrable near o, where K_D is the original Bergman kernel on D.

Proof. Let $F \equiv 1$, and $\xi_0 = (1, 0, \cdots, 0, \cdots) \in \ell_{\mathcal{I}_+(2c_o^F(\varphi)\varphi)_o}$. Then we have

$$\begin{aligned} B(F, \mathcal{I}_+(2c_o^F(\varphi)\varphi)_o, D) &= \sup_{\xi \in \ell_{\mathcal{I}_+(2c_o^F(\varphi)\varphi)_o}} \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)} \\ &\geqslant \frac{|(\xi_0 \cdot 1)(o)|^2}{K_{\xi_0,D}(o)} \\ &= K_D^{-1}(o). \end{aligned}$$

Then according to Theorem 1.6, for any $p > 1$ satisfying

$$\frac{p}{p-1} > K_D^{-1}(o) \int_D e^{-\varphi},$$

$|F|^2 e^{-p\varphi}$ is locally integrable near o. □

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