L^2 EXTENSION AND EFFECTIVENESS OF L^p STRONG OPENNESS PROPERTY

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ABSTRACT. In this note, we present an L^2 extension approach to the effectiveness result of L^p strong openness property of multiplier ideal sheaves.

1. Introduction

The multiplier ideal sheaf associated to a plurisubharmonic function was widely discussed in several complex variables, complex geometry and algebraic geometry (see e.g. [26, 20, 10, 25]). Recall that the definition of multiplier ideal sheaf:

Let φ be a plurisubharmonic function (see [7]) on a complex manifold X. The multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the sheaf on X whose germs are the holomorphic functions F such that $|F|^2e^{-\varphi}$ is locally integrable.

The following strong openness property was conjectured by Demailly [8, 9] (the so-called strong openness conjecture), and established by Guan-Zhou [15].

Strong openness property:

$$\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi),$$

where $\mathcal{I}_+(\varphi) := \bigcup_{p>1} \mathcal{I}(p\varphi)$.

When $\mathcal{I}(\varphi) = \mathcal{O}$, the strong openness conjecture is called the openness conjecture, which was posed by Demailly-Kollár [10] and proved by Berndtsson [3].

The effectiveness of openness conjecture was established by Berndtsson [3], which implies the openness conjecture. In the proof of the strong openness conjecture, Ohsawa-Takegoshi L^2 extension theorem was used by Guan-Zhou [15]. After that, Guan-Zhou [17] established an effectiveness result of the strong openness property by solving the $\bar\partial$ equations with L^2 estimates. In [1], the authors of the present article gave an L^2 extension approach to an effectiveness result of the strong openness property.

In [11], Fornæss established the following L^p strong openness property by using the strong openness property [15]:

Let F be a holomorphic function on a domain $D \subset \mathbb{C}^n$ containing the origin o, φ a plurisubharmonic function on D and $p \in (0, +\infty)$. If $|F|^pe^{-\varphi}$ is L^1 on a neighborhood of o, then there exists $q > 1$ such that $|F|^pe^{-q\varphi}$ is L^1 on a neighborhood of o.

In [14], Guan-Yuan gave an effectiveness result of the L^p strong openness property by using the general concavity of minimal L^2 related to multiplier ideal sheaves in [13]. Following [1], it is natural to ask

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Question 1.1. *Can one obtain an L^2 extension approach to the effectiveness result of the L^p strong openness property for any $p>0$?*

In the present note, we give an affirmative answer to the above question.

Let D be a bounded psuedoconvex domain in $\mathbb{C}^n$ with $o\in D$. Let φ be a negative plurisubharmonic function on D such that $\varphi(o)=-\infty$. Let F be a holomorphic function on D. For any $p>0$, denote that

$$c^F_{o,p}(\varphi):=\sup\{c\geqslant 0:|F|^pe^{-c\varphi}\text{ is locally integrable near } o\}.$$

Let I be a proper ideal of $\mathcal{O}_o$, and ψ be a plurisubharmonic function on D. Denote that

$$B^{\psi}(F,I,D):=\sup_{\xi\in\ell_I}\frac{|(\xi\cdot F)(o)|^2}{K^{\psi}_{\xi,D}(o)}.$$

Here $\xi\in\ell_1:=\{\xi=(\xi_\alpha)_{\alpha\in\mathbb{N}^n}:\sum_{\alpha\in\mathbb{N}^n}|\xi_\alpha|\rho^{|\alpha|}<+\infty,\text{ for any }\rho>0\}$, and $(\xi\cdot F)(z):=\sum_{\alpha\in\mathbb{N}^n}\xi_\alpha F^{(\alpha)}(z)/\alpha!$. $\xi\in\ell_I$ means that $\xi\neq(0,\ldots,0,\ldots)$ and $(\xi\cdot f)(o)=0$ for any $(f,o)\in I$. The weighted Bergman kernel

$$K^{\psi}_{\xi,D}(z):=\sup_{f\in A^2(D,e^{-\psi})}\frac{|(\xi\cdot f)(z)|^2}{\int_D|f|^2e^{-\psi}},$$

where $A^2(D,e^{-\psi})=\{f\in\mathcal{O}(D):|f|^2e^{-\psi}\text{ is integrable on } D\}$.

Denote that $F_1=F^{\lceil\frac{p}{2}\rceil}$, $\varphi_1=(2\lceil\frac{p}{2}\rceil-p)\log|F|$, and here $\lceil m\rceil:=\min\{n\in\mathbb{Z}:n\geqslant m\}$. We obtain the following effectiveness result of the L^p strong openness theorem.

Theorem 1.2. *Let C_1 and C_2 be two positive constants. If there exists $a\geqslant 0$, such that*

(1). $\int_D|F|^pe^{-(1+a)\varphi}\leqslant C_1$;

(2). $B^{\varphi_1+a\varphi}(F_1,\mathcal{I}_+(\varphi_1+c^F_{o,p}(\varphi)\varphi)_o,D)\geqslant C_2$.

Then for any $q>a+1$ satisfying

$$\tilde{\theta}_a(q)>\frac{C_1}{C_2},$$

$|F|^pe^{-q\varphi}$ is locally L^1 integrable near o, where $\tilde{\theta}_a(q)=\frac{q-a}{q-a-1}$.

Remark 1.3. *Let $D=\Delta$, $F=z$ and $\varphi=\frac{p+2}{q}\log|z|$, then $c^F_{o,p}(\varphi)=q$. We can find that $\int_D|F|^pe^{-(1+a)\varphi}=\frac{2q\pi}{(q-a-1)(p+2)}$, and $B^{\varphi_1+a\varphi}(F_1,\mathcal{I}_+(\varphi_1+c^F_{o,p}(\varphi)\varphi)_o,D)=\frac{2q\pi}{(q-a)(p+2)}$ by straightforward calculations. then $\frac{\int_D|F|^pe^{-(1+a)\varphi}}{B^{\varphi_1+a\varphi}(F_1,\mathcal{I}_+(\varphi_1+c^F_{o,p}(\varphi)\varphi)_o,D)}=\frac{q-a}{q-a-1}$. It means that Theorem 1.2 is a sharp effectiveness result of the L^p strong openness theorem.*

When $p=2$, the above Theorem 1.2 induces the following corollary.

Corollary 1.4. *Let D be a bounded psuedoconvex domain in $\mathbb{C}^n$ with $o\in D$. Let φ be a negative plurisubharmonic function on D such that $\varphi(o)=-\infty$. Let F be a holomorphic function on D. Let C_1 and C_2 be two positive constants. If there exists $a\geqslant 0$, such that*

(1). $\int_D|F|^2e^{-(1+a)\varphi}\leqslant C_1$;

(2). $B^{a\varphi}(F,\mathcal{I}_+(c^F_o(\varphi)\varphi)_o,D)\geqslant C_2$.

Then for any $q > a+1$ satisfying

$$\tilde{\theta}_a(q) > \frac{C_1}{C_2},$$

$|F|^p e^{-q\varphi}$ *is locally L^1 integrable near o, where $\tilde{\theta}_a(q) = \frac{q-a}{q-a-1}$.*

Here $c_o^F(\varphi)$ is the jumping number

$$c_o^F(\varphi) := \sup\{c \geqslant 0 : |F|^2 e^{-c\varphi} \text{ is locally integrable near } o\}.$$

In particular, let $a = 0$, then Corollary 1.4 induces the Theorem 1.6 in [1].

In addition, when $F \equiv 1$, note that

$$B^{a\varphi}(1, \mathcal{I}_+(c_o^F(\varphi)\varphi)_o, D) \geqslant K_{D,a\varphi}^{-1}(o).$$

Here $K_{D,a\varphi}$ is the Bergman kernel on D with respect to the weight $e^{-a\varphi}$. Then Corollary 1.4 induces the following effectiveness result of the openness property.

Corollary 1.5. *Let D be a bounded psuedoconvex domain in $\mathbb{C}^n$ with $o \in D$. Let φ be a negative plurisubharmonic function on D such that $\varphi(o) = -\infty$. Let C_1 and C_2 be two positive constants. If there exists $a \geqslant 0$, such that*

(1). $\int_D e^{-(1+a)\varphi} \leqslant C_1$;

(2). $(K_{D,a\varphi})^{-1}(o) \geqslant C_2$.

Then for any $q > a+1$ satisfying

$$\tilde{\theta}_a(q) > \frac{C_1}{C_2},$$

$|F|^p e^{-q\varphi}$ *is locally L^1 integrable near o, where $\tilde{\theta}_a(q) = \frac{q-a}{q-a-1}$.*

2. Definition and basic properties of the Bergman kernel

Firstly, we recall a linear space of sequences of complex numbers (see [1]),

$$\ell_1 := \{\xi = (\xi_\alpha)_{\alpha \in \mathbb{N}^n} : \sum_{\alpha \in \mathbb{N}^n} |\xi_\alpha| \rho^{|\alpha|} < +\infty, \text{ for any } \rho > 0\}.$$

Any element in ℓ_1 can be a linear functional over $\mathcal{O}_{z_0}$ for any $z_0 \in \mathbb{C}^n$ as follows.

For any $F(z) \in \mathcal{O}_{z_0}$, we can write that $F(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha (z - z_0)^\alpha$ near z_0. Then we define the value that ξ acts on $F(z)$ as

$$(\xi \cdot F)(z_0) := \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha \frac{F^{(\alpha)}(z_0)}{\alpha!} = \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha a_\alpha,$$

for any $\xi \in \ell_1$.

We have showed that $(\xi \cdot F)(z_0)$ is well defined in [1].

Let ψ be a plurisubharmonic function on D. For any $z \in D$, $\xi \in \ell_1$, we denote the weighted Bergman kernel by

$$K_{\xi,D}^{\psi}(z) := \sup_{F \in A^2(D, e^{-\psi})} \frac{|(\xi \cdot F)(z)|^2}{\int_D |F|^2 e^{-\psi}},$$

where D is a bounded domain in $\mathbb{C}^n$, and

$$A^2(D, e^{-\psi}) = \{f \in \mathcal{O}(D) : |f|^2 e^{-\psi} \text{ is integrable on } D\}.$$

If $A^2(D, e^{-\psi}) = \{0\}$, we denote that $K_{\xi,D}^{\psi}(z) = 0$.

Secondly, we list some lemmas. Some of them was proved in [1].

Lemma 2.1 (see [1])**.** *Let D be a bounded domain in $\mathbb{C}^n$, and let F be a holomorphic function on D. Then $(\xi \cdot F)(z)$ is also a holomorphic function on D, where $\xi \in \ell_1$.*

The following lemma shows that the weighted Bergman kernel is finite.

Lemma 2.2. *Let D be a bounded domain in $\mathbb{C}^n$, and let ψ be a plurisubharmonic function on D. Then $K^{\psi}_{\xi,D}(z_0) < +\infty$ for any $z_0 \in D$, $\xi \in \ell_1$.*

In fact, we can prove the following stronger result.

Lemma 2.3. *Let D be a bounded domain in $\mathbb{C}^n$, ψ be a plurisubharmonic function on D, and let $\xi \in \ell_1$. Then for any compact subset $K \subseteq D$, there is a finite constant $C > 0$ such that*

$$|(\xi \cdot F)(z)|^2 \leqslant C \int_D |F|^2 e^{-\psi},$$

for any $F \in A^2(D, e^{-\psi})$, and any $z \in K$.

Proof. It is trivial when $\xi = 0$ or $A^2(D, e^{-\psi}) = \{0\}$. Now we assume $\xi \neq 0$ and $A^2(D, e^{-\psi}) \neq \{0\}$. For the compact subset K, we are able to find some $R > 0$ such that $\bigcup_{w\in K} \Delta^n_{w,R} \subset\subset D$. Then there exists a finite constant a such that $\psi(z) < a$ for any $z \in \bigcup_{w\in K} \Delta^n_{w,R}$. And for any nonzero holomorphic function $F(z)$ on D, if we write that $F(z) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha (z - z_0)^\alpha$ in $\Delta^n_{z_0,R}$ for any $z_0 \in K$, then

$$\begin{aligned}
&\int_D |F|^2 e^{-\psi} \geqslant \int_{\Delta^n_{z_0,R}} |F|^2 e^{-\psi} \geqslant \int_{\Delta^n_{z_0,R}} |F|^2 e^{-a} \\
=&e^{-a} \sum_{\alpha\in\mathbb{N}^n} \frac{\pi^n |a_\alpha|^2}{(\alpha_1 + 1)\cdots(\alpha_n + 1)} R^{2(|\alpha|+n)}.
\end{aligned}$$

By Cauchy-Schwarz's inequality,

$$|(\xi\cdot F)(z_0)|^2 \leqslant \Big(\sum_{\alpha\in\mathbb{N}^n} \frac{(\alpha_1 + 1)\cdots(\alpha_n + 1)}{\pi^n R^{2(|\alpha|+n)}} |\xi_\alpha|^2\Big)\Big(\sum_{\alpha\in\mathbb{N}^n} \frac{\pi^n |a_\alpha|^2}{(\alpha_1 + 1)\cdots(\alpha_n + 1)} R^{2(|\alpha|+n)}\Big),$$

and it implies that

$$\frac{|(\xi \cdot F)(z_0)|^2}{\int_D |F|^2 e^{-\psi}} \leqslant e^a \sum_{\alpha\in\mathbb{N}^n} \frac{(\alpha_1 + 1)\cdots(\alpha_n + 1)}{\pi^n R^{2(|\alpha|+n)}} |\xi_\alpha|^2.$$

Since $\xi \in \ell_1$, we can choose some ρ with $\rho > 1/R$ such that

$$|\xi_\alpha| \rho^{|\alpha|} < M, \ \forall \alpha \in \mathbb{N}^n,$$

for some positive constant M. Hence

$$\begin{aligned}
\frac{|(\xi \cdot F)(z_0)|^2}{\int_D |F|^2 e^{-\psi}} &\leqslant e^a \sum_{\alpha\in\mathbb{N}^n} \frac{(\alpha_1 + 1)\cdots(\alpha_n + 1)}{\pi^n R^{2(|\alpha|+n)}} |\xi_\alpha|^2 \\
&\leqslant e^a \sum_{\alpha\in\mathbb{N}^n} \frac{(\alpha_1 + 1)\cdots(\alpha_n + 1)}{\pi^n R^{2(|\alpha|+n)}} \cdot \frac{M^2}{\rho^{2|\alpha|}} \\
&= \frac{e^a M^2}{\pi^n R^{2n}} \sum_{\alpha\in\mathbb{N}^n} (\alpha_1 + 1)\cdots(\alpha_n + 1) \frac{1}{(R\rho)^{2|\alpha|}} < +\infty.
\end{aligned}$$

Now we choose

$$C = \frac{e^a M^2}{\pi^n R^{2n}} \sum_{\alpha\in\mathbb{N}^n} (\alpha_1 + 1)\cdots(\alpha_n + 1) \frac{1}{(R\rho)^{2|\alpha|}},$$

which is independent of the choice of $z_0 \in K$, and get the result. □

Now let $K = \{z_0\}$, then Lemma 2.2 can be induced by Lemma 2.3.

The following lemma was proved in [1].

Lemma 2.4 (see [1])**.** *Let D be a bounded domain in $\mathbb{C}^n$, and let $\{F_j\}$ be a sequence of holomorphic functions on D uniformly converging to F on every compact subset of D. Then for any $z \in D$, and $\xi \in \ell_1$, $\{(\xi \cdot F_j)(z)\}$ converges to $(\xi \cdot F)(z)$ uniformly on every compact subset of D.*

We prove the following lemma by the above lemma.

Lemma 2.5. *Let D be a bounded domain in $\mathbb{C}^n$, ψ be a plurisubharmonic function on D with $A^2(D, e^{-\psi}) \neq \{0\}$, and let $z \in D$. Then for any $\xi \in \ell_1$, there exists a holomorphic function F_0 on D such that*

$$K^{\psi}_{\xi,D}(z) = \frac{|(\xi \cdot F_0)(z)|^2}{\int_D |F_0|^2 e^{-\psi}}.$$

Proof. It is trivial when $\xi = 0$. Now we assume $\xi \neq 0$.

By the definition of Bergman kernel, there exists a sequence of holomorphic functions $\{F_j\}$ on D such that $\int_D |F_j|^2 e^{-\psi} = 1$, and $\lim_{j\to+\infty} |(\xi \cdot F_j)(z)|^2 = K^{\psi}_{\xi,D}(z)$. Then by Montel's theorem (see [22], Theorem 1.5), there is a subsequence of $\{F_j\}$ which is uniformly convergent on every compact subset of D. Denote the limit of the subsequence by F_0. Then Fatou's lemma and Lemma 2.4 imply that $\int_D |F_0|^2 e^{-\psi} \leqslant 1$, and $|(\xi \cdot F_0)(z)|^2 = K^{\psi}_{\xi,D}(z)$. It means that $\frac{|(\xi \cdot F_0)(z)|^2}{\int_D |F_0|^2 e^{-\psi}} \geqslant K^{\psi}_{\xi,D}(z)$. Then by the definition of $K^{\psi}_{\xi,D}(z)$, we get $\frac{|(\xi \cdot F_0)(z)|^2}{\int_D |F_0|^2 e^{-\psi}} = K^{\psi}_{\xi,D}(z)$. □

The following lemma was also proved in[1].

Lemma 2.6 (see [1])**.** *Let D be a bounded domain in $\mathbb{C}^n$, and let $\{z_j\}$ be a sequence of points in D such that $\lim_{j\to+\infty} z_j = z \in D$. Let $\{F_j\}$ be a sequence of holomorphic functions on D uniformly converging to F on every compact subset of D. Then for any $\xi \in \ell_1$, $\lim_{j\to+\infty} (\xi \cdot F_j)(z_j) = (\xi \cdot F)(z)$.*

The following lemma shows that the weighted Bergman kernel is continuous.

Lemma 2.7. *Let D be a bounded domain in $\mathbb{C}^n$, ψ be a plurisubharmonic function on D. Then for any $\xi \in \ell_1$, $K^{\psi}_{\xi,D}(z)$ is a continuous function on D.*

Proof. It is trivial when $\xi = 0$ or $A^2(D, e^{-\psi}) = \{0\}$. Now we assume $\xi \neq 0$ and $A^2(D, e^{-\psi}) \neq \{0\}$. By the definition of $K^{\psi}_{\xi,D}(z)$,

$$K^{\psi}_{\xi,D}(z) = \sup\{|(\xi \cdot F)(z)| : \int_D |F|^2 e^{-\psi} = 1, F \in \mathcal{O}(D)\}.$$

Combining with Lemma 2.1, we know that $K^{\psi}_{\xi,D}(z)$ is lower semicontinuous.

Next we prove that $K^{\psi}_{\xi,D}(z)$ is also upper continuous. Let $\{z_j\}$ be a sequence of points in D such that $\lim_{j\to+\infty} z_j = z_0 \in D$. And we may assume that $\{z_{k_j}\}$ is the subsequence of $\{z_j\}$ such that

$$\lim_{j\to+\infty} K^{\psi}_{\xi,D}(z_{k_j}) = \limsup_{j\to+\infty} K^{\psi}_{\xi,D}(z_j).$$

Using Lemma 2.5, we can get a sequence of holomorphic function $\{F_j\}$ on D, such that $\int_D |F_j|^2 e^{-\psi} = 1$, and $|(\xi \cdot F_j)(z_j)|^2 = K^{\psi}_{\xi,D}(z_j)$, for any $j \geqslant 1$. Then using Montel's theorem and the diagonal method, we can select a subsequence of $\{F_{k_j}\}$ which is uniformly convergent on every compact subset of D. We may denote the subsequence by $\{F_{k_j}\}$ itself, and denote the limit function by F_0. It follows from Fatou's lemma and Lemma 2.6 that $\int_D |F_0|^2 e^{-\psi} \leqslant 1$, $\lim_{j\to+\infty}(\xi \cdot F_{k_j})(z_{k_j}) = (\xi \cdot F_0)(z_0)$. Then

$$\begin{aligned}
K^{\psi}_{\xi,D}(z_0) &\geqslant \frac{|(\xi \cdot F_0)(z_0)|^2}{\int_D |F_0|^2 e^{-\psi}} \\
&\geqslant |(\xi \cdot F_0)(z_0)|^2 \\
&= \lim_{j\to+\infty} |(\xi \cdot F_{k_j})(z_{k_j})|^2 \\
&= \lim_{j\to+\infty} K^{\psi}_{\xi,D}(z_{k_j}) \\
&= \limsup_{j\to+\infty} K^{\psi}_{\xi,D}(z_j).
\end{aligned}$$

We get that $K^{\psi}_{\xi,D}(z)$ is upper semicontinuous.

It is known that $K^{\psi}_{\xi,D}(z)$ is lower semicontinuous, which implies that $K^{\psi}_{\xi,D}(z)$ is continuous. □

The weighted Bergman kernel is also log-plurisubharmonic.

Lemma 2.8. *Let D be a bounded domain in $\mathbb{C}^n$, ψ be a plurisubharmonic function on D, and let $\xi \in \ell_1$. Then $\log K^{\psi}_{\xi,D}(z)$ is plurisubharmonic on D.*

Proof. There is $\log K^{\psi}_{\xi.D}(z) \equiv -\infty$ when $\xi = 0$ or $A^2(D, e^{-\psi}) = \{0\}$. Now we assume that $\xi \neq 0$ and $A^2(D, e^{-\psi}) \neq \{0\}$. By the definition, we have

$$\log K^{\psi}_{\xi,D}(z) = \sup\{2\log|(\xi \cdot F)(z)| : \int_D |F|^2 e^{-\psi} = 1, F \in \mathcal{O}(D)\}.$$

Lemma 2.1 shows that $(\xi \cdot F)(z)$ is holomorphic on D, when F is holomorphic on D. Then $\log|(\xi \cdot F)(z)|$ is plurisubharmonic. As $\log K^{\psi}_{\xi,D}(z)$ is upper semicontinuous according to Lemma 2.7, $\log K^{\psi}_{\xi,D}(z)$ is plurisubharmonic on D. □

3. Optimal L^2 extension and Guan-Zhou method

In this section, we will recall the Guan-Zhou Method, i.e. an optimal L^2 extension approach to a log-convexity property of the fibrewised Bergman kernel.

Let Ω be a pseudoconvex domain in $\mathbb{C}^{n+1}$ with coordinate (z,t), where $z \in \mathbb{C}^n$, $t \in \mathbb{C}$. Let p, q be the natural projections $p(z,t) = t$, $q(z,t) = z$ on Ω, $p(\Omega) = D$. For any $t \in D$, suppose that $\Omega_t = p^{-1}(t) \subseteq \Omega$ is bounded in $\mathbb{C}^n$. Let ψ be a plurisubharmonic function on Ω, and let $K^{\psi}_{\xi,t}(z) = K^{\psi}_{\xi,\Omega_t}(z)$ be the Bergman kernels of the domains Ω_t defined as which in the above section with respect to some fixed $\xi \in \ell_1$.

We will use the following version of the optimal L^2 extension theorem.

Lemma 3.1 (Optimal L^2 extension theorem ([5], see [23, 24]))**.** *Let $D = \Delta_{t_0,r}$ be the disk in the complex plane centered on t_0 with radius r. Then for any f in*

$A^2(\Omega_{t_0}, e^{-\psi})$, there exists a holomorphic function F on Ω, such that $F|_{\Omega_{t_0}} = f$, and

$$\frac{1}{\pi r^2}\int_{\Omega}|F|^2e^{-\psi} \leqslant \int_{\Omega_{t_0}}|f|^2e^{-\psi}.$$

Guan-Zhou Method shows that Lemma 3.1 implies the following

Proposition 3.2. *(see [1, 2, 19, 18]) $\log K^{\psi}_{\xi,t}(z)$ is a plurisubharmonic function with respect to (z,t), for any $\xi \in \ell_1$.*

Proof. Firstly, we prove that $\log K^{\psi}_{\xi,t}(z)$ is upper semicontinuous on Ω. Let (z_j, t_j) be a sequence of points in Ω, such that $(z_j, t_j) \to (z_0, t_0) \in \Omega$, $j \to +\infty$. Since Ω is a domain in $\mathbb{C}^{n+1}$, we know that for any compact subset of $q(\Omega_{t_0})$ denoted by K, there exists $j_K \geqslant 1$, such that $K \subseteq q(\Omega_{t_j})$ in the sense of domains in $\mathbb{C}^n$, for any $j \geqslant j_K$.

We denote $q(\Omega_{t_j})$ by Ω_j. Then we may assume that $\{(z_{k_j}, t_{k_j})\}$ is the subsequence of $\{(z_j, t_j)\}$ such that

$$\lim_{j\to+\infty} K^{\psi}_{\xi,\Omega_{k_j}}(z_{k_j}) = \limsup_{j\to+\infty} K_{\xi,\Omega_j}(z_j),$$

and $z_0 \in \Omega_{j_0}$. Using Lemma 2.5, we can get a sequence of holomorphic function $\{F_j\}$ on Ω_j, such that $\int_{\Omega_j}|F_j|^2e^{-\psi} = 1$, and $|(\xi\cdot F_j)(z_j)|^2 = K^{\psi}_{\xi,\Omega_j}(z_j)$, for any $j \geqslant j_0$. Since for any compact subset K of $q(\Omega_{t_0})$, there exists $j_K \geqslant 1$, such that $K \subseteq \Omega_j$ for any $j \geqslant j_K$, then we can use Montel's theorem and the diagonal method to get a subsequence of $\{F_{k_j}\}$ which is uniformly convergent on every compact subset of $q(\Omega_{t_0})$. We may denote the subsequence by $\{F_{k_j}\}$ itself, and denote the limit function by F_0. Fatou's lemma and Lemma 2.6 imply that $\int_{q(\Omega_{t_0})}|F_0|^2e^{-\psi} \leqslant 1$, and $\lim_{j\to+\infty}(\xi\cdot F_{k_j})(z_{k_j}) = (\xi\cdot F_0)(z_0)$. It follows that

$$\begin{aligned}
K^{\psi}_{\xi,\Omega_{t_0}}(z_0) &\geqslant \frac{|(\xi\cdot F_0)(z_0)|^2}{\int_{q(\Omega_{t_0})}|F_0|^2e^{-\psi}}\\
&\geqslant |(\xi\cdot F_0)(z_0)|^2\\
&= \lim_{j\to+\infty}|(\xi\cdot F_{k_j})(z_{k_j})|^2\\
&= \lim_{j\to+\infty} K^{\psi}_{\xi,\Omega_{k_j}}(z_{k_j})\\
&= \limsup_{j\to+\infty} K^{\psi}_{\xi,\Omega_{t_j}}(z_j),
\end{aligned}$$

and

$$\limsup_{j\to+\infty}\log K^{\psi}_{\xi,t_j}(z_j) \leqslant \log K^{\psi}_{\xi,t_0}(z_0).$$

Then we obtain that $\log K^{\psi}_{\xi,t}(z)$ is upper semicontinuous on Ω.

Secondly we need to check that for any complex line L, $\log K^{\psi}_{\xi,t}(z)|_L$ is subharmonic. If the complex line lies on some Ω_t for some fixed t, we know that $\log K^{\psi}_{\xi,t}(z)|_L$ is subharmonic using Lemma 2.8. Then without loss of generality, we assume that L is the complex line on $\{t|(z,t)\}$ and $D = \Delta_{t_0,r} = L$.

If $\log K^{\psi}_{\xi,t_0}(z) = -\infty$, we are done. Then we assume that there exists $f \in A^2(\Omega_{t_0}, e^{-\psi})$ such that
$$K^{\psi}_{\xi,t_0}(z) = \frac{|(\xi \cdot f)(z)|^2}{\int_{\Omega_{t_0}} |f|^2 e^{-\psi}}.$$
Using the optimal L^2 extension theorem (Lemma 3.1), we can get a holomorphic function F on Ω such that $F(z,t_0) = f(z)$ and
$$\frac{1}{\pi r^2}\int_{\Omega} |F|^2 e^{-\psi} \leqslant \int_{\Omega_{t_0}} |f|^2 e^{-\psi}.$$
Denote that $F_t(z) = F(z,t) = F|_{\Omega_t}$. Note that the function $y = \log x$ is concave, and by Jensen's inequality, it follows from Guan-Zhou Method ([18], see also [23],[24]) that
$$\begin{aligned}
\log\left(\int_{\Omega_{t_0}} |f|^2 e^{-\psi}\right) &\geqslant \log\left(\frac{1}{\pi r^2}\int_{\Omega} |F|^2 e^{-\psi}\right)\\
&= \log\left(\frac{1}{\pi r^2}\int_{\Delta_{t_0,r}}\int_{\Omega_t} |F_t|^2 e^{-\psi}\right)\\
&\geqslant \frac{1}{\pi r^2}\int_{\Delta_{t_0,r}} \log\left(\int_{\Omega_t} |F_t|^2 e^{-\psi}\right)\\
&\geqslant \frac{1}{\pi r^2}\int_{\Delta_{t_0,r}} \left(\log|(\xi\cdot F_t)(z)|^2 - \log K^{\psi}_{\xi,t}(z)\right)\\
&\geqslant \log|(\xi\cdot f)(z)|^2 - \frac{1}{\pi r^2}\int_{\Delta_{t_0,r}} \log K^{\psi}_{\xi,t}(z).
\end{aligned}$$

The last inequality above holds, since we can prove that $\log|(\xi\cdot F_t)(z)|^2$ is subharmonic with respect to t. In fact, if we write
$$F_t(w) = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(t)(w-z)^\alpha,$$
then all the
$$a_\alpha(t) = \frac{1}{\alpha!}\frac{\partial^\alpha F(w,t)}{\partial w^\alpha}(z,t)$$
are holomorphic with respect to t. In addition, $(\xi\cdot F_t)(z) = \sum_{\alpha\in\mathbb{N}^n}\xi_\alpha a_\alpha(t)$ is absolutely and uniformly convergent on every compact subset of $\Delta_{t_0,r}$. Since for any compact subset of $\Delta_{t_0,r}$, denoted by K, we can find some $R > 0$ such that $\Delta^n_{z,R} \subseteq q(\Omega_t)$ for any $t \in K$. Combining with $\xi \in \ell_1$, we get that
$$|\xi_\alpha a_\alpha(t)| \leqslant \frac{MM'}{(\rho R)^{|\alpha|}},\ \forall \alpha \in \mathbb{N}^n$$
for some $M, M' > 0$, $\rho > 1/R$, and any $t \in K$. This means that $\sum_{\alpha\in\mathbb{N}^n}\xi_\alpha a_\alpha(t)$ is absolutely and uniformly convergent on K. Then $(\xi\cdot F_t)(z)$ is holomorphic with respect to t for any fixed z, which implies that $\log|(\xi\cdot F_t)(z)|^2$ is subharmonic with respect to t. Then
$$\log K^{\psi}_{\xi,t_0}(z) \leqslant \frac{1}{\pi r^2}\int_{t\in\Delta_{t_0,r}} \log K^{\psi}_{\xi,t}(z),$$
which implies that $\log K^{\psi}_{\xi,t}(z)$ is plurisubharmonic with respect to (z,t).

□

4. EFFECTIVENESS RESULT OF L^p STRONG OPENNESS PROPERTY

In this section, we complete the L^2 extension approach to the L^p effectiveness of the strong openness property.

Let I be an ideal of $\mathcal{O}_o$ such that $I \neq \mathcal{O}_o$. We consider about a subset $\ell_I \subseteq \ell_1$ such as

$$\ell_I = \{0 \neq \xi \in \ell_1 : (\xi \cdot F)(o) = 0,\ \forall F \in I\}.$$

It is obvious that ℓ_I is always nonempty since $\xi = (1, 0, \ldots, 0, \ldots) \in \ell_I$ when $I \neq \mathcal{O}_o$.

Especially, we denote $\ell_{\mathcal{I}(\varphi)_o}$ by ℓ_φ.

Let φ be a negative plurisubharmonic function on a bounded pseudoconvex domain $D \subseteq \mathbb{C}^n$, and let ψ be a plurisubharmonic function on D. Denote that

$$D_t := \{z \in D : \varphi(z) < -t\},$$

and

$$K_\xi^\psi(t) := K_{\xi,D_t}^\psi(o),$$

for $t \in [0, +\infty)$. We need the following lemma.

Lemma 4.1 (see [7], Theorem 5.13, Chapter I). *Let $\Omega = I + \mathrm{i}\mathbb{R}$ be a domain in $\mathbb{C}$ with the coordinate $z = x + \mathrm{i}y$, where I is an interval in $\mathbb{R}$. Let $\phi(z)$ be a subharmonic function on Ω which is independent of y. Then $\phi(x) := \phi(x + \mathrm{i}\mathbb{R})$ is a convex function with respect to $x \in I$.*

This result is also used in [2], [4] and [19].

Note that the domain

$$\{(\tau, z) : \varphi(z) - \mathrm{Re}\ \tau < 0\}$$

is pseudoconvex in $\mathbb{C}^{n+1}$, then according to Proposition 3.2, $\log K_\xi^\psi(o, \tau)$ is subharmonic for $\tau \in [0, +\infty) + \mathrm{i}\mathbb{R}$, and independent of $\mathrm{Im}\ \tau$. Lemma 4.1 shows that $\log K_\xi^\psi(t)$ is convex for $t \in [0, +\infty)$. This implies that $-\log K_\xi^\psi(t) + t$ is concave, which will be increasing if it has a lower bound. We state the following result.

Lemma 4.2. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ such that $o \in D$, and let φ be a negative plurisubharmonic function on D such that $\varphi(o) = -\infty$, and ψ be a plurisubharmonic function on D such that $\mathcal{I}(\varphi + \psi)_o \neq \mathcal{O}_o$. Then for any fixed $\xi \in \ell_{\varphi+\psi}$,*

$$-\log K_\xi^\psi(t) + t \geqslant -\log K_\xi^\psi(0),\ \forall t \in [0, +\infty).$$

Lemma 4.2 can be proved by the following lemma.

Lemma 4.3 (see [21]). *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ such that $o \in D$, and let φ be a negative plurisubharmonic function on D such that $\varphi(o) = -\infty$, and ψ be a plurisubharmonic function on D. Let F be an holomorphic function on $\{\varphi < -t_0\}$ such that $\int_{\{\varphi < -t_0\}} |F|^2 e^{-\psi} < +\infty$. Then there exists a holomorphic function F_{t_0} on D, such that*

$$(F_{t_0} - F, o) \in \mathcal{I}(\varphi + \psi)_o$$

and

$$\int_D |F_{t_0} - (1 - b_{t_0}(\varphi))F|^2 e^{-\psi} \leqslant C_D \int_{\{-t_0-1 < \varphi < -t_0\}} |F|^2 e^{-\varphi-\psi},$$

where $b_{t_0}(t)=\int_{-\infty}^{t}\mathbb{I}_{\{-t_0-1<s<-t_0\}}\mathrm{d}s$ *and* $t_0\geqslant 0$*, and here* C_D *is a positive constant only dependent of* D*.*

Lemma 4.3 can also be referred to [18, 16, 17, 12].

Proof of Lemma 4.2. We only need to prove that there is a lower bound of $-\log K_{\xi}^{\psi}(t)+t$.

Lemma 2.5 shows that for any $t\in[0,+\infty)$, there exists $F\in\mathcal{O}(\{\varphi<-t\})$, such that

$$K_{\xi}^{\psi}(t)=\frac{|(\xi\cdot F)(o)|^2}{\int_{\{\varphi<-t\}}|F|^2e^{-\psi}}.$$

It follows from Lemma 4.3 that there exists a holomorphic function F_t on D such that

$$(F_t-F,o)\in\mathcal{I}(\varphi+\psi)_o$$

and

$$\int_D|F_t-(1-b_t(\varphi))F|^2e^{-\psi}\leqslant C_D\int_{\{-t-1<\varphi<-t\}}|F|^2e^{-\varphi-\psi}.$$

Note that $\xi\in\ell_{\varphi+\psi}$, then $(F_t-F,o)\in\mathcal{I}(\varphi+\psi)_o$ induces that $(\xi\cdot F_t)(o)=(\xi\cdot F)(o)$. On the one hand,

$$\begin{aligned}&\left(\int_D|F_t-(1-b_t(\varphi))F|^2e^{-\psi}\right)^{\frac{1}{2}}\\ \geqslant&\left(\int_D|F_t|^2e^{-\psi}\right)^{\frac{1}{2}}-\left(\int_D|(1-b_t(\varphi))F|^2e^{-\psi}\right)^{\frac{1}{2}}\\ \geqslant&(K_{\xi,D}^{\psi}(o))^{-\frac{1}{2}}|(\xi\cdot F_t)(o)|-\left(\int_{\{\varphi<-t\}}|F|^2e^{-\psi}\right)^{\frac{1}{2}}.\\ =&(K_{\xi,D}^{\psi}(o))^{-\frac{1}{2}}|(\xi\cdot F)(o)|-\left(\int_{\{\varphi<-t\}}|F|^2e^{-\psi}\right)^{\frac{1}{2}}.\end{aligned}$$

On the other hand, we have

$$\int_{\{-t-1<\varphi<-t\}}|F|^2e^{-\varphi-\psi}\leqslant e^{t+1}\int_{\{-t-1<\varphi<-t\}}|F|^2e^{-\psi}\leqslant e^{t+1}\int_{\{\varphi<-t\}}|F|^2e^{-\psi}.$$

Then

$$(C_D^{\frac{1}{2}}e^{\frac{t+1}{2}}+1)^2\int_{\{\varphi<-t\}}|F|^2e^{-\psi}\geqslant(K_{\xi,D}^{\psi}(o))^{-1}|(\xi\cdot F)(o)|^2,$$

which means

$$\int_{\{\varphi<-t\}}|F|^2\geqslant Ce^{-t}|(\xi\cdot F)(o)|^2,$$

where $C=(2(e+1)K_{\xi,D}^{\psi}(o)\max\{C_D,1\})^{-1}$ is a positive constant independent of the choises of F and t.

Then we get that

$$-\log K_{\xi}^{\psi}(t)+t\geqslant\log C$$

for any $t\in[0,+\infty)$. Then $-\log K_{\xi}^{\psi}(t)+t$ has a lower bound, inducing that it is increasing, and

$$-\log K_{\xi}^{\psi}(t)+t\geqslant-\log K_{\xi}^{\psi}(0).$$

□

By Lemma 4.2, we get the following proposition.

Proposition 4.4. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ such that $o \in D$, and let φ be a negative plurisubharmonic function on D such that $\varphi(o) = -\infty$, and ψ be a plurisubharmonic function on D such that $\mathcal{I}(\varphi+\psi)_o \neq \mathcal{O}_o$. Let F be a holomorphic function on D. Then for any $q > 1$, and any $\xi \in \ell_{\varphi+\psi}$,*

$$\int_D |F|^2 e^{-\frac{\varphi}{q}-\psi} \geqslant \frac{q}{q-1} \cdot \frac{|(\xi \cdot F)(o)|^2}{K^{\psi}_{\xi,D}(o)}.$$

Proof. Using Lemma 4.2, we have that $K^{\psi}_{\xi}(t) \leqslant e^t K^{\psi}_{\xi}(0)$ for any $t \in [0, +\infty)$ if $\xi \in \ell_{\varphi+\psi}$. And It is known that

$$\int_D |F|^2 e^{-\frac{\varphi}{q}-\psi} = \int_{-\infty}^{+\infty} \left(\int_{\{\frac{\varphi}{q}<-t\}} |F|^2 e^{-\psi} \right) e^t \mathrm{d}t,$$

(this equality can be referred in [12]), which implies that

$$\begin{aligned} &\int_0^{+\infty} \left(\int_{\{\varphi<-qt\}} |F|^2 e^{-\psi} \right) e^t \mathrm{d}t \\ \geqslant &\int_0^{+\infty} \frac{|(\xi \cdot F)(o)|^2}{K^{\psi}_{\xi}(qt)} e^t \mathrm{d}t \\ \geqslant &\frac{|(\xi \cdot F)(o)|^2}{K^{\psi}_{\xi}(0)} \int_0^{+\infty} e^{(1-q)t} \mathrm{d}t = \frac{1}{q-1} \cdot \frac{|(\xi \cdot F)(o)|^2}{K^{\psi}_{\xi,D}(o)}, \end{aligned}$$

and

$$\int_{-\infty}^0 \left(\int_{\{\varphi<-qt\}} |F|^2 e^{-\psi} \right) e^t \mathrm{d}t \geqslant \frac{|(\xi \cdot F)(o)|^2}{K^{\psi}_{\xi,D}(o)} \int_{-\infty}^0 e^t \mathrm{d}t = \frac{|(\xi \cdot F)(o)|^2}{K^{\psi}_{\xi,D}(o)},$$

$$\int_D |F|^2 e^{-\frac{\varphi}{q}-\psi} \geqslant \frac{q}{q-1} \cdot \frac{|(\xi \cdot F)(o)|^2}{K^{\psi}_{\xi,D}(o)}.$$

□

Let D be a bounded domain in $\mathbb{C}^n$, and let I be an ideal of $\mathcal{O}_o$ such that $I \neq \mathcal{O}_o$. Let ψ be a plurisubharmonic function on D. Let (F, o) be an element in $\mathcal{O}_o$. Then we denote that

$$B^{\psi}(F, I, D) = \sup_{\xi \in \ell_I} \frac{|(\xi \cdot F)(o)|^2}{K^{\psi}_{\xi,D}(o)}.$$

Then we get the following corollary of Proposition 4.4.

Corollary 4.5. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ such that $o \in D$, and let φ be a negative plurisubharmonic function on D such that $\varphi(o) = -\infty$, and ψ be a plurisubharmonic function on D such that $\mathcal{I}(\varphi+\psi)_o \neq \mathcal{O}_o$. Let F be a holomorphic function on D. Then for any $q > 1$, and any $\xi \in \ell_{\varphi+\psi}$,*

$$\int_D |F|^2 e^{-\frac{\varphi}{q}-\psi} \geqslant \frac{q}{q-1} B^{\psi}(F, \mathcal{I}(\varphi+\psi)_o, D).$$

The following proposition shows that $B^{\psi}(F,I,D)>0$ if $(F,o)\notin I$ and $A^2(D,e^{-\psi})\neq\{0\}$.

Proposition 4.6 (see [1]). *Let D be a bounded domain in $\mathbb{C}^n$ such that $o\in D$, and let I be an ideal of $\mathcal{O}_o$ such that $I\neq\mathcal{O}_o$. Let $(F,o)\in\mathcal{O}_o$ such that $(F,o)\notin I$. Then we are able to find some $\xi\in\ell_I$ such that $(\xi\cdot F)(o)\neq 0$.*

Moreover, if $F\in A^2(D,e^{-\psi})$, then $B^{\psi}(F,I,D)\leqslant\int_D|F|^2e^{-\psi}$, since

$$\begin{aligned}B^{\psi}(F,I,D)=&\sup_{\xi\in\ell_I,\ (\xi\cdot F)(o)\neq 0}\frac{|(\xi\cdot F)(o)|^2}{K^{\psi}_{\xi,D}(o)}\\ \leqslant&\sup_{\xi\in\ell_I,\ (\xi\cdot F)(o)\neq 0}\frac{|(\xi\cdot F)(o)|^2}{|(\xi\cdot F)(o)|^2/\int_D|F|^2e^{-\psi}}=\int_D|F|^2e^{-\psi}.\end{aligned}$$

It is clear that if there are two ideals I_1 and I_2 of $\mathcal{O}_o$ such that $I_1\subseteq I_2\neq\mathcal{O}_o$, then $B^{\psi}(F,I_1,D)\geqslant B^{\psi}(F,I_2,D)$.

Now we can prove the following theorem.

Theorem 4.7. *Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ such that $o\in D$, and let φ be a negative plurisubharmonic function on D such that $\varphi(o)=-\infty$, and ψ be a plurisubharmonic function on D. Let F be a holomorphic function on D with $\int_D|F|^2e^{-\varphi-\psi}<+\infty$. Then for any $q>1$ satisfying*

$$\frac{q}{q-1}>\frac{\int_D|F|^2e^{-\varphi-\psi}}{B^{\psi}(F,\mathcal{I}_+(\psi+c_o^F(\varphi,\psi)\varphi)_o,D)},$$

$|F|^2e^{-q\varphi-\psi}$ is locally integrable near o.

Here $c_o^F(\varphi,\psi):=\sup\{c\geqslant 0\ :\ |F|^2e^{-c\varphi-\psi}$ is locally integrable near $o\}$.

Proof. For $q>c_o^F(\varphi,\psi)$, $|F|^2e^{-q\varphi-\psi}$ is not integrable near o, and $B^{\psi}(F,\mathcal{I}(\psi+q\varphi)_o,D)\geqslant B^{\psi}(F,\mathcal{I}_+(\psi+c_o^F(\varphi,\psi)\varphi)_o,D)$. Then Corollary 4.5 shows that

$$\int_D|F|^2e^{-\varphi-\psi}\geqslant\frac{q}{q-1}B^{\psi}(F,\mathcal{I}_+(\psi+c_o^F(\varphi,\psi)\varphi)_o,D).$$

Let $q\to c_o^F(\varphi,\psi)^+$, then the above inequality also holds for $q\geqslant c_o^F(\varphi,\psi)$. Then if $q>1$ satisfing

$$\int_D|F|^2e^{-\varphi-\psi}<\frac{q}{q-1}B^{\psi}(F,\mathcal{I}_+(\psi+c_o^F(\varphi,\psi)\varphi)_o,D),$$

we get $q<c_o^F(\varphi,\psi)$, which means that $|F|^2e^{-q\varphi-\psi}$ is integrable near o.

Since $F\notin\mathcal{I}_+(\psi+c_o^F(\varphi,\psi)\varphi)_o$, we know that

$$0<B^{\psi}(F,\mathcal{I}_+(\psi+c_o^F(\varphi,\psi)\varphi)_o,D)\leqslant\int_D|F|^2e^{-\psi}<\int_D|F|^2e^{-\varphi-\psi}.$$

Then the proof is done. □

Then we give the proof of Theorem 1.2.

Proof of Theorem 1.2: In Theorem 4.7, let $\psi=\varphi_1+a\varphi$, then $\int_D|F_1|e^{-\varphi-\psi}=\int_D|F|^pe^{-(1+a)\varphi}$, and $\mathcal{I}_+(\psi+c_o^{F_1}(\varphi,\psi)\varphi)_o=\mathcal{I}_+(\varphi_1+c_{o,p}^F(\varphi)\varphi)_o$. Thus for any $q'>1$ satisfying

$$\frac{q'}{q'-1}>\frac{\int_D|F|^pe^{-(1+a)\varphi}}{B^{\varphi_1+a\varphi}(F_1,\mathcal{I}_+(\varphi_1+c_{o,p}^F(\varphi)\varphi)_o,D)},$$

$|F|^p e^{-(q'+a)\varphi}$ is locally integrable near o. Let $q = q' + a$, then the proof is done. $\square$

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