CONCAVITY PROPERTY OF MINIMAL L^2 INTEGRALS WITH LEBESGUE MEASURABLE GAIN VII—NEGLIGIBLE WEIGHTS

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In Memory of Jean-Pierre Demailly (1957-2022)

Abstract. In this article, we present characterizations of the concavity property of minimal L^2 integrals with negligible weights degenerating to linearity on the fibrations over open Riemann surfaces and the fibrations over products of open Riemann surfaces. As applications, we obtain characterizations of the holding of equality in optimal jets L^2 extension problem with negligible weights on the fibrations over open Riemann surfaces and the fibrations over products of open Riemann surfaces.

1. introduction

Recall that the strong openness property of multiplier ideal sheaves [35], i.e. $\mathcal{I}(\varphi)=\mathcal{I}_+(\varphi):=\underset{\epsilon>0}{\cup}\mathcal{I}((1+\epsilon)\varphi)$ (conjectured by Demailly [12]) has been widely used and discussed in several complex variables, complex algebraic geometry and complex differential geometry (see e.g. [35, 40, 6, 7, 17, 8, 52, 38, 4, 53, 54, 18, 41, 9]), where multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2e^{-\varphi}$ is locally integrable (see e.g. [49, 43, 45, 13, 14, 12, 15, 42, 46, 47, 11, 39]), and φ is a plurisubharmonic function on a complex manifold M (see [10]).

When $\mathcal{I}(\varphi)=\mathcal{O}$, the strong openness property is the openness property (conjectured by Demailly-Kollár [14]). Berndtsson [3] established an effectiveness result of the openness property, and obtained the openness property. Stimulated by Berndtsson's effectiveness result, and continuing the solution of the strong openness property [35], Guan-Zhou [37] established an effectiveness result of the strong openness property by considering the minimal L^2 integral on the pseudoconvex domain D.

Considering the minimal L^2 integrals on the sub-level sets of the weight φ, Guan [21] obtained a sharp version of Guan-Zhou's effectiveness result, and established a concavity property of the minimal L^2 integrals on the sublevel sets of the weight φ (with constant gain), which was applied to give a proof of Saitoh's conjecture for conjugate Hardy H^2 kernels [22], and the sufficient and necessary condition of the existence of decreasing equisingular approximations with analytic singularities for the multiplier ideal sheaves with weights $\log(|z_1|^{a_1}+\cdots+|z_n|^{a_n})$ [23].

For smooth gain, Guan [20] (see also [24]) obtained the concavity property on Stein manifolds (weakly pseudoconvex Kähler case was obtained by Guan-Mi [25]),

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which was applied by Guan-Yuan to give an optimal support function related to the strong openness property [29] and an effectiveness result of the strong openness property in L^p [30]. For Lebesgue measurable gain, Guan-Yuan [28] obtained the concavity property on Stein manifolds (weakly pseudoconvex Kähler case was obtained by Guan-Mi-Yuan [26]), which deduced a twisted L^p strong openness property [31].

Note that the linearity is a degenerate concavity. A natural problem was posed in [32]:

Problem 1.1 ([32]). *How to characterize the concavity property degenerating to linearity?*

For open Riemann surfaces, Guan-Yuan gave an answer to Problem 1.1 for single points [28] (for the case of subharmonic weights, see Guan-Mi [24]), and gave an answer to Problem 1.1 for finite points [32]. For products of open Riemann surfaces, Guan-Yuan [33] gave an answer to Problem 1.1 for products of finite points.

For fibrations over open Riemann surfaces, Bao-Guan-Yuan [1] gave an answer to Problem 1.1 with negligible weights pulled back from the open Riemann surfaces. For fibrations over products of open Riemann surfaces, Bao-Guan-Yuan [2] gave an answer to Problem 1.1 with negligible weights vanishing identically.

In this article, for the fibrations over open Riemann surfaces and the fibrations over products of open Riemann surfaces, we give answers to Problem 1.1 with negligible weights on fibrations.

We would like to recall the definition of minimal L^2 integral as follows.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \le j \le n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M = \left(\prod_{1\le j\le n_1} \Omega_j\right) \times Y$ be an $n-$dimensional complex manifold, where $n = n_1+n_2$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\le j\le n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M.

Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \dots, n_1\}$, and denote that $Z_0 := \left(\prod_{1\le j\le n_1} Z_j\right) \times Y \subset M$.

Let $\psi < 0$ be a plurisubharmonic function on M such that $\{\psi < -t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$ and $Z_0 \subset \{\psi = -\infty\}$. Let φ_1 be a Lebesgue measurable function on $\left(\prod_{1\le j\le n_1} \Omega_j\right)$ such that $\pi_1^*(\varphi_1) + \psi$ is a plurisubharmonic function on M. Let φ_2 be a plurisubharmonic funciton on Y. Denote $\varphi := \pi_1^*(\varphi_1) + \pi_1^*(\varphi_2)$. Let $\mathcal{F}_{(z,y)} \supset \mathcal{I}(\varphi + \psi)_{(z,y)}$ be an ideal of $\mathcal{O}_{(z,y)}$ for any $(z, y) \in Z_0$. Let f be a holomorphic $(n, 0)$ form on a neighborhood U of Z_0. Let $c(t)$ be a positive Lebesgue measurable function on $(0, +\infty)$. Denote

$$\inf\Big\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \big(\tilde{f} - f, (z, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{F}\big)_{(z,y)}$$
$$\text{for any } (z, y) \in Z_0,$$
$$\& \ \tilde{f} \in H^0\big(\{\psi < -t\}, \mathcal{O}(K_M)\big).\Big\}$$

by $G(t;c,f,\varphi,\psi,\mathcal{F})$ for any $t\in[0,+\infty)$. Here $|f|^2:=(\sqrt{-1})^{n^2}f\wedge\bar{f}$ for any $(n,0)$ form f. We simply denote $G(t;c,f,\varphi,\psi,\mathcal{F})$ by $G(t)$ when there is no misunderstands and denote $G(t;c,\varphi,\psi,\mathcal{F})$ by $G(t;c)$, $G(t;\varphi)$, $G(t;\psi)$ and $G(t;\mathcal{F})$ when we focus on various choices of $c(t),\varphi,\psi$ and $\mathcal{F}$ respectively.

We generally assume that $c(t)$ is a positive function on $(0,+\infty)$ such that $\int_0^{+\infty}c(t)e^{-t}dt<+\infty$, $c(t)e^{-t}$ is decreasing with respect to t on $(0,+\infty)$ and $e^{-\varphi}c(-\psi)$ has a positive lower bound on any compact subset of $M\backslash Z_0$ in this paper (when other assumption for $c(t)$ is used, we introduce it explicitly). Then $G\big(h^{-1}(r)\big)$ is concave with respect to $r\in[0,\int_0^{+\infty}c(t_1)e^{-t_1}dt_1]$ (see [26], see also Theorem 2.1), where $h(t)=\int_t^{+\infty}c(s)e^{-s}ds$ for any $t\in[0,+\infty)$.

1.1. **Main results.** In this section, we present characterizations of the concavity property of minimal L^2 integrals with negligible weights degenerating to linearity on the fibrations over open Riemann surfaces and products of open Riemann surfaces.

1.1.1. *Linearity of the minimal L^2 integrals on fibrations over open Riemann surfaces.* In this section, we present characterizations of the concavity property of minimal L^2 integrals degenerating to linearity on the fibrations over open Riemann surface.

To state our results, we firstly recall the following notations (see [16], see also [36, 28, 26]).

Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω. A character χ on $\pi_1(\Omega)$ is a homomorphism from $\pi_1(\Omega)$ to $\mathbb{C}^*=\mathbb{C}\backslash\{0\}$ which takes values in the unit circle $\{z\in\mathbb{C};|z|=1\}$.

Let $P:\Delta\to\Omega$ be the universal covering from unit disc $\Delta\subset\mathbb{C}$ to Ω. We call the holomorphic function f (resp. holomorphic $(1,0)$ form F) on Δ is a multiplicative function $\big($resp. multiplicative differential (Prym differential)$\big)$ if there is a character χ on $\pi_1(\Omega)$, such that $g^*f=\chi(g)f$ (resp. $g^*F=\chi(g)F$) for every $g\in\pi_1(\Omega)$ which naturally acts on the universal covering of Ω. Denote the set of such kinds of f (resp. F) by $\mathcal{O}^\chi(\Omega)$ $\big($resp. $\Gamma^\chi(\Omega)\big)$.

As P is a universal covering, then for any harmonic function h on Ω, there exists a character χ_h associated to h and a multiplicative function $f_h\in\mathcal{O}^{\chi_h}(\Omega)$, such that $|f_h|=P^*e^h$. And if $g\in\mathcal{O}(\Omega)$ and g has no zero points on Ω, then we have $\chi_h=\chi_{h+\log|g|}$.

For Green function $G_\Omega(\cdot,z_0)$, one can find a χ_{z_0} and a multiplicative function $f_{z_0}\in\mathcal{O}^{\chi_{z_0}}(\Omega)$, such that $|f_{z_0}|=P^*e^{G_\Omega(\cdot,z_0)}$ (see [48]).

Now we assume that $n_1=1$ and then $M=\left(\prod_{1\le j\le n_1}\Omega_j\right)\times Y=\Omega\times Y$, where Y is an $(n-1)-$dimensional weakly pseudoconvex Kähler manifold. Denote $Z_\Omega=\{z_j:1\le j<\gamma\}$ be a subset of Ω of discrete points, where $\gamma>1$ is a positive integer or $\gamma=+\infty$. Denote $Z_0:=Z_\Omega\times Y$. Denote $Z_j:=\{z_j\}\times Y$ for any j.

Let ψ be a plurisubharmonic function on M such that $\{\psi<-t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t\in\mathbb{R}$ and $Z_0\subset\{\psi=-\infty\}$. It follows from Siu's decomposition theorem that

$$dd^c\psi=\sum_{j\ge1}2p_j[Z_j]+\sum_{i\ge1}\lambda_i[A_i]+R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $(n-1)-$dimensional analytic set, and where R is a closed positive current with the property that

$dimE_c(R) < n-1$ for every $c > 0$, where $E_c(R) = \{x \in M : v(R,x) \geq c\}$ is the upperlevel sets of Lelong number. We assume that $p_j > 0$ for any $1 \leq j < \gamma$.

Then $N := \psi - \pi_1^*\big(\sum\limits_{j\geq 1} 2p_j G_\Omega(z,z_j)\big)$ is a plurisubharmonic function on M, where $\pi_1 : M \to \Omega$ be the natural projection. We assume that $N \leq 0$ and $N|_{Z_j}$ is not identically $-\infty$ for any j.

Let φ_1 be a Lebesgue measurable function on Ω such that $\psi + \pi_1^*(\varphi)$ is a plurisubharmonic function on M. By Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j\geq 1} 2\tilde{q}_j[Z_j] + \sum_{i\geq 1}\tilde{\lambda}_i[\tilde{A}_i] + \tilde{R},$$

where $\tilde{q}_j \geq 0$ for any $1 \leq j < \gamma$.

By Weierstrass theorem on open Riemann surfaces, there exists a holomorphic function g on Ω such that $ord_{z_j}(g) = q_j := [\tilde{q}_j]$ for any $z_j \in Z_\Omega$ and $g(z) \neq 0$ for any $z \notin Z_\Omega$, where $[q]$ equals to the integral part of the nonnegative real number q. Then we know that there exists a plurisubharmonic function $\tilde{\psi}_2 \in Psh(M)$ such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2.$$

Let $\varphi_2 \in Psh(Y)$. Denote $\pi_2 : M \to Y$ be the natural projection and $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

For any $1 \leq j < \gamma$, let $\tilde{z}_j$ be a local coordinate on a neighborhood V_j of z_j satisfying $\tilde{z}_j(z_0) = 0$ and $V_j \cap V_k = \emptyset$, for any $j \neq k$. Denote $V_0 := \cup_{1\leq j<\gamma} V_j$. Let f be a holomorphic $(n,0)$ form on $V_0 \times Y$ which is a neighborhood of Z_0. Denote $\mathcal{F}_{(z_j,y)} = \mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))_{(z_j,y)}$ for any $(z_j,y) \in Z_0$. Let $G(t)$ be the minimal L^2 integral on $\{\psi < -t\}$ with respect to φ, f, $\mathcal{F}$ and c for any $t \geq 0$.

Let $\gamma = m+1$ is an positive integer, i.e., $Z_\Omega = \{z_j : 1 \leq j < \gamma\}$ contains m points. We obtain the following characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity on the fibrations over open Riemann surfaces.

Theorem 1.2. *Assume that $G(0) \in (0,+\infty)$. Then $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1). $N \equiv 0$ and $\psi = \pi_1^*\left(2\sum\limits_{j=1}^m p_j G_\Omega(\cdot,z_j)\right)$;

(2). for any $j \in \{1,2,\ldots,m\}$, $f = \pi_1^*(a_j\tilde{z}_j^{k_j}d\tilde{z}_j) \wedge \pi_2^*(f_Y) + f_j$ on $V_j \times Y$, where $a_j \in \mathbb{C}\backslash\{0\}$ is a constant, k_j is a nonnegative integer, f_Y is a holomorphic $(n-1,0)$ form on Y such that $\int_Y |f_Y|^2 e^{-\varphi_2} \in (0,+\infty)$, and $\big(f_j,(z_j,y)\big) \in \big(\mathcal{O}(K_M)\big)_{(z_j,y)} \otimes \mathcal{I}(\varphi+\psi)_{(z_j,y)}$ for any $j \in \{1,2,\ldots,m\}$ and $y \in Y$;

(3). $\varphi_1 + 2\sum\limits_{j=1}^m p_j G_\Omega(\cdot,z_j) = 2\log|g| + 2\sum\limits_{j=1}^m G_\Omega(\cdot,z_j) + 2u$, where g is a holomorphic function on Ω such that $ord_{z_j}(g) = k_j$ and u is a harmonic function on Ω;

(4). $\prod\limits_{j=1}^m \chi_{z_j} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot,z_j)$ respectively;

(5). for any $j\in\{1,2,\ldots,m\}$,

$$\lim_{z\to z_j}\frac{a_j\tilde{z}_j^{k_j}d\tilde{z}_j}{gP_*\left(f_u\left(\prod_{l=1}^m f_{z_l}\right)\left(\sum_{l=1}^m p_l\frac{df_{z_l}}{f_{z_l}}\right)\right)}=c_0, \tag{1.1}$$

where $c_0\in\mathbb{C}\setminus\{0\}$ is a constant independent of j, f_u is a holomorphic function Δ such that $|f_u|=P^*(e^u)$ and f_{z_l} is a holomorphic function on Δ such that $|f_{z_l}|=P^*\left(e^{G_\Omega(\cdot,z_l)}\right)$ for any $l\in\{1,\ldots,m\}$.

Remark 1.3. *When $N\equiv\pi_1^*(N_1)$, it follows from $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ (see Lemma 3.7) that Theorem 1.2 reduces to Theorem 1.4 in* [1] *(see also Theorem 2.5), where $N_1\le 0$ is a subharmonic function on Ω and $N_1(z_j)>-\infty$ for any j.*

Let $\gamma=+\infty$, i.e., $Z_\Omega=\{z_j:1\le j<\gamma\}$ is an infinite subset of Ω of discrete points. Assume that $2\sum_{j\ge1}p_jG_\Omega(\cdot,z_j)\not\equiv-\infty$. We present a necessary condition such that $G(h^{-1}(r))$ is linear as follows.

Proposition 1.4. *Assume that $G(0)\in(0,+\infty)$. If $G(h^{-1}(r))$ is linear with respect to $r\in(0,\int_0^{+\infty}c(t)e^{-t}dt]$, then the following statements hold:*

(1). $N\equiv 0$ and $\psi=\pi_1^*\left(2\sum_{j=1}^{\gamma}p_jG_\Omega(\cdot,z_j)\right)$;

(2). for any $j\in\mathbb{N}_+$, $f=\pi_1^*(a_j\tilde{z}_j^{k_j}d\tilde{z}_j)\wedge\pi_2^*(f_Y)+f_j$ on $V_j\times Y$, where $a_j\in\mathbb{C}\setminus\{0\}$ is a constant, k_j is a nonnegative integer, f_Y is a holomorphic $(n-1,0)$ form on Y such that $\int_Y|f_Y|^2e^{-\varphi_2}\in(0,+\infty)$, and $\big(f_j,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\big)_{(z_j,y)}\otimes\mathcal{I}(\varphi+\psi)_{(z_j,y)}$ for any $j\in\mathbb{N}_+$ and $y\in Y$;

(3). $\varphi_1+2\sum_{j=1}^{m}p_jG_\Omega(\cdot,z_j)=2\log|g|$, where g is a holomorphic function on Ω such that $ord_{z_j}(g)=k_j+1$ for any $j\in\mathbb{N}_+$;

(4). for any $j\in\mathbb{N}_+$,

$$\frac{p_j}{ord_{z_j}g}\lim_{z\to z_j}\frac{dg}{a_j\tilde{z}_j^{k_j}d\tilde{z}_j}=c_0, \tag{1.2}$$

where $c_0\in\mathbb{C}\setminus\{0\}$ is a constant independent of j;

(5). $\sum_{j\in\mathbb{N}_+}p_j<+\infty$.

Remark 1.5. *When $N\equiv\pi_1^*(N_1)$, Proposition 1.4 is Proposition 1.6 in* [1] *(see also Theorem 2.6), where $N_1\le 0$ is a subharmonic function on Ω and $N_1(z_j)>-\infty$ for any j.*

Let $\tilde{M}\subset M$ be an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0\subset\tilde{M}$. Let f be a holomorphic $(n,0)$ form on a neighborhood $\tilde{U}_0$ of Z_0 in $\tilde{M}$.

Let ψ be a plurisubharmonic function on $\tilde{M}$ such that $\{\psi<-t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t\in\mathbb{R}$. It follows from Siu's decomposition theorem that

$$dd^c\psi=\sum_{j\ge1}2p_j[Z_j]+\sum_{i\ge1}\lambda_i[A_i]+R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $(n-1)-$dimensional analytic set, and where R is a closed positive current with the property that $dimE_c(R) < n-1$ for every $c > 0$, where $E_c(R) = \{x \in \tilde{M} : v(R,x) \geq c\}$ is the upperlevel sets of Lelong number. We assume that $p_j > 0$ for any $1 \leq j < \gamma$.

Then $N := \psi - \pi_1^*(\sum\limits_{j\geq 1} 2p_j G_\Omega(z,z_j))$ is a plurisubharmonic function on $\tilde{M}$. We assume that $N \leq 0$ and $N|_{Z_j}$ is not identically $-\infty$ for any j.

Let φ_1 be a Lebesgue measurable function on Ω such that $\psi+\pi_1^*(\varphi)$ is a plurisubharmonic function on $\tilde{M}$. By Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j\geq 1} 2\tilde{q}_j[Z_j] + \sum_{i\geq 1} \tilde{\lambda}_i[\tilde{A}_i] + \tilde{R},$$

where $\tilde{q}_j \geq 0$ for any $1 \leq j < \gamma$.

By Weierstrass theorem on open Riemann surfaces, there exists a holomorphic function g on Ω such that $ord_{z_j}(g) = q_j := [\tilde{q}_j]$ for any $z_j \in Z_\Omega$ and $g(z) \neq 0$ for any $z \notin Z_\Omega$, where $[q]$ equals to the integral part of the nonnegative real number q. Then we know that there exists a plurisubharmonic function $\tilde{\psi}_2 \in Psh(\tilde{M})$ such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2.$$

Let $\varphi_2 \in Psh(Y)$. Denote $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

Let $c(t)$ be a positive function on $(0,+\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$, $c(t)e^{-t}$ is decreasing with respect to t on $(0,+\infty)$ and $e^{-\varphi}c(-\psi)$ has a positive lower bound on any compact subset of $\tilde{M}\backslash Z_0$.

Denote

$$\begin{aligned}\inf\Bigg\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi}c(-\psi) : \big(\tilde{f} - f, (z,y)\big) \in \Big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)\Big)_{(z,y)} \\ \text{for any } (z,y) \in Z_0, \\ \& \ \tilde{f} \in H^0(\{\psi < -t\}, \mathcal{O}(K_{\tilde{M}})).\Bigg\}\end{aligned}$$

by $\tilde{G}(t)$ for any $t \in [0,+\infty)$. Note that $\tilde{G}(t)$ is a minimal L^2 integrals on $\tilde{M}$, where $\tilde{M}$ is a submanifold of M.

We present a necessary condition such that $\tilde{G}\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ as follows.

Proposition 1.6. *Assume that* $\tilde{G}(0) \in (0,+\infty)$. *If* $\tilde{G}\big(h^{-1}(r)\big)$ *is linear with respect to* $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$, *then* $\tilde{M} = M$.

Remark 1.7. *When* $N = \pi_1^*(N_1)|_{\tilde{M}}$, *Proposition 1.6 is Proposition 1.7 in* [1], *where* $N_1 \leq 0$ *is a subharmonic function on* Ω *and* $N_1(z_j) > -\infty$ *for any* j.

1.1.2. *Linearity of the minimal L^2 integrals on fibrations over products of open Riemann surfaces.* In this section, we present characterizations of the concavity property of minimal L^2 integrals degenerating to linearity on the fibrations over products of open Riemann surfaces.

When $n_1 \geq 1$, $M = \Big(\prod_{1\leq j\leq n_1} \Omega_j\Big) \times Y$ is an $n-$dimensional complex manifold, where $n = n_1 + n_2$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\leq j\leq n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M. Denote $P_j : \Delta \to \Omega_j$ be the universal covering from unit disc Δ to Ω_j for $1 \le j \le n_1$.

Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \dots, n_1\}$, and denote that $Z_0 := \left(\prod_{1\le j\le n_1} Z_j\right) \times Y \subset M$. Let $N \le 0$ be a plurisubharmonic function on M satisfying $N|_{Z_0} \not\equiv -\infty$. For any $j \in \{1, \dots, n_1\}$, let φ_j be an upper semi-continuous function on Ω_j such that $\varphi_j(z) > -\infty$ for any $z \in Z_j$. Assume that $\sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j) + N$ is a plurisubharmonic function on M. Let φ_Y be a plurisubharmonic function on Y, and denote that $\varphi := \sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$.

Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$, $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $c(-\psi)$ has a positive lower bound on any compact subset of $M\backslash Z_0$. Let f be a holomorphic $(n, 0)$ form on a neighborhood of Z_0.

Assume that $Z_0 = \{z_0\} \times Y = \{(z_1, \dots, z_{n_1})\} \times Y \subset M$. Denote

$$\hat{G} := \max_{1\le j\le n_1} \left\{2p_j \pi_{1,j}^*\big(G_{\Omega_j}(\cdot, z_j)\big)\right\},$$

where p_j is positive real number for $1 \le j \le n_1$. Denote

$$\psi := \max_{1\le j\le n_1} \left\{2p_j \pi_{1,j}^*\big(G_{\Omega_j}(\cdot, z_j)\big)\right\} + N.$$

We assume that $\{\psi < -t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$. Denote $\mathcal{F}_{(z_0,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_0,y)}$ for any $(z_0, y) \in Z_0$. Let $G(t)$ be the minimal L^2 integral on $\{\psi < -t\}$ with respect to φ, f, $\mathcal{F}$ and c for any $t \ge 0$.

Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote that $V_0 := \prod_{1\le j\le n_1} V_{z_j}$, and $w := (w_1, \dots, w_{n_1})$ is a local coordinate on V_0 of $z_0 \in \prod_{1\le j\le n_1} \Omega_j$. Denote that $E := \left\{(\alpha_1, \dots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_j} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$. Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y \subset M$.

We present a characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity for the case $Z_0 = \{z_0\} \times Y$.

Theorem 1.8. *Assume that $G(0) \in (0, +\infty)$. $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1) *$N \equiv 0$ and $\psi = \max_{1\le j\le n_1} \left\{2p_j \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_j))\right\}$;*

(2) *$f = \sum_{\alpha\in E} \pi_1^*\left(w^\alpha dw_1 \wedge \dots \wedge dw_{n_1}\right) \wedge \pi_2^*(f_\alpha) + g_0$ on $V_0 \times Y$, where g_0 is a holomorphic $(n, 0)$ form on $V_0 \times Y$ satisfying $(g_0, z) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi)\big)_z$ for any point $z \in Z_0$ and f_α is a holomorphic $(n_2, 0)$ form on Y such that $\sum_{\alpha\in E} \int_Y |f_\alpha|^2 e^{-\varphi_Y} \in (0, +\infty)$;*

(3) *$\varphi_j = 2\log|g_j| + 2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j) \ne 0$ and u_j is a harmonic function on Ω_j for any $1 \le j \le n_1$;*

(4) *$\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$ for any $j \in \{1, 2, ..., n\}$ and $\alpha \in E$ satisfying $f_\alpha \not\equiv 0$, where $\chi_{j,-u_j}$ be the character associated to $-u_j$ on Ω_j and χ_{j,z_j} be the character associated to $G_{\Omega_j}(\cdot, z_j)$ on Ω_j.*

Remark 1.9. *When $N \equiv 0$ (φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$), it follows from $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ (see Lemma 3.8) that Theorem 1.8 can be referred to Theorem 1.2 in* [2] *(see also Theorem 2.9).*

Let $Z_j = \{z_{j,1}, \dots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \dots, n_1\}$, where m_j is a positive integer. Denote $Z_0 := (\prod_{1\le j\le n_1} Z_j) \times Y$. Let $N \le 0$ be a plurisubharmonic function on M satisfying $N|_{Z_0} \not\equiv -\infty$. Denote

$$\hat{G} := \max_{1\le j\le n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\le k\le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

where $p_{j,k}$ is a positive real number. Denote

$$\psi := \max_{1\le j\le n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\le k\le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\} + N.$$

We assume that $\{\psi < -t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, m_j\}$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \ne k'$. Denote that $\tilde{I}_1 := \big\{(\beta_1, \ldots, \beta_{n_1}) : 1 \le \beta_j \le m_j$ for any $j \in \{1, \ldots, n_1\}\big\}$, $V_\beta := \prod_{1\le j\le n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \ldots, \beta_{n_1}) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n_1,\beta_{n_1}}) \in \prod_{1\le j\le n_1} \Omega_j$ satisfying $w_\beta(z_\beta) = 0$.

Let $\beta^* = (1, \ldots, 1) \in \tilde{I}_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1}, \ldots, \alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\ge 0}^{n_1}$. Denote that $E' := \left\{ \alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{1\le j\le n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}} \right\}$. Let f be a holomorphic $(n, 0)$ form on $\cup_{\beta\in\tilde{I}_1} V_\beta \times Y$ satisfying $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1} \wedge \ldots \wedge dw_{n_1,1} \right) \wedge \pi_2^*\left(f_{\alpha_{\beta^*}}\right) + \sum_{\alpha\in E'} \pi_1^*(w^\alpha dw_{1,1} \wedge \ldots \wedge dw_{n_1,1}) \wedge \pi_2^*(f_\alpha)$ on $V_{\beta^*} \times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2, 0)$ forms on Y. Denote $\mathcal{F}_{(z,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z,y)}$ for any point $(z, y) \in Z_0$. Let $G(t; c)$ be the minimal L^2 integral on $\{\psi < -t\}$ with respect to φ, f and $\mathcal{F}$ for any $t \ge 0$.

We present a characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity for the case that Z_j is a set of finite points.

Theorem 1.10. *Assume that $G(0) \in (0, +\infty)$. $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$ if and only if the following statements hold:*

(1) *$N \equiv 0$ and $\psi = \max_{1\le j\le n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\le k\le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$;*

(2) *$\varphi_j = 2\log|g_j| + 2u_j$ for any $j \in \{1, \ldots, n_1\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k}) \ne 0$ for any $k \in \{1, \ldots, m_j\}$;*

(3) *There exists a nonnegative integer $\gamma_{j,k}$ for any $j \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, m_j\}$, which satisfies that $\prod_{1\le k\le m_j} \chi_{j,z_{j,k}}^{\gamma_{j,k}+1} = \chi_{j,-u_j}$ and $\sum_{1\le j\le n_1} \frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}} = 1$ for any $\beta \in \tilde{I}_1$;*

(4) *$f = \pi_1^*\left(c_\beta \left(\prod_{1\le j\le n_1} w_{j,\beta_j}^{\gamma_{j,\beta_j}} \right) dw_{1,\beta_1} \wedge \ldots \wedge dw_{n,\beta_n} \right) \wedge \pi_2^*(f_0) + g_\beta$ on $V_\beta \times Y$ for any $\beta \in \tilde{I}_1$, where c_β is a constant, $f_0 \not\equiv 0$ is a holomorphic $(n_2, 0)$ form on Y satisfying $\int_Y |f_0|^2 e^{-\varphi_2} < +\infty$, and g_β is a holomorphic $(n, 0)$ form on $V_\beta \times Y$ such that $(g_\beta, z) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi)\big)_z$ for any $z \in \{z_\beta\} \times Y$;*

(4) $c_\beta\prod_{1\le j\le n_1}\left(\lim_{z\to z_{j,\beta_j}}\frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}}dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\le k\le m_j}f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\le k\le m_j}p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)}\right)=$ c_0 *for any* $\beta\in\tilde I_1$, *where* $c_0\in\mathbb{C}\backslash\{0\}$ *is a constant independent of* β, f_{u_j} *is a holomorphic function* Δ *such that* $|f_{u_j}|=P_j^*(e^{u_j})$ *and* $f_{z_{j,k}}$ *is a holomorphic function on* Δ *such that* $|f_{z_{j,k}}|=P_j^*\left(e^{G_{\Omega_j}(\cdot,z_{j,k})}\right)$ *for any* $j\in\{1,\ldots,n_1\}$ *and* $k\in\{1,\ldots,m_j\}$.

Remark 1.11. *When* $N\equiv 0$ *(*φ_j *is a subharmonic function on* Ω_j *satisfying* $\varphi_j(z_{j,\beta_j})>-\infty$ *for any* $1\le j\le n_1$ *and* $1\le\beta_j\le m_j$*), it follows from* $\mathcal{I}(\varphi+\psi)_{(z,y)}=\mathcal{I}(\hat G+\pi_2^*(\varphi_Y))_{(z,y)}$ *for any* $(z,y)\in Z_0$ *(see Lemma 3.8) that Theorem 1.10 can be referred to Theorem 1.5 in* [2] *(see also Theorem 2.10).*

Let $Z_j=\{z_{j,k}:1\le k<\tilde m_j\}$ be a discrete subset of Ω_j for any $j\in\{1,\ldots,n_1\}$, where $\tilde m_j\in\mathbb{Z}_{\ge2}\cup\{+\infty\}$. Denote $Z_0:=(\prod_{1\le j\le n_1}Z_j)\times Y$. Let $p_{j,k}$ be a positive number for any $1\le j\le n_1$ and $1\le k<\tilde m_j$ such that $\sum_{1\le k<\tilde m_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\not\equiv-\infty$ for any j. Let

$$\hat G=\max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k<\tilde m_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\},$$

and

$$\psi=\max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k<\tilde m_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}+N.$$

We assume that $\{\psi<-t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t\in\mathbb{R}$ and $\limsup_{t\to+\infty}c(t)<+\infty$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}}\Subset\Omega_j$ of $z_{j,k}\in\Omega_j$ satisfying $w_{j,k}(z_{j,k})=0$ for any $j\in\{1,\ldots,n_1\}$ and $1\le k<\tilde m_j$, where $V_{z_{j,k}}\cap V_{z_{j,k'}}=\emptyset$ for any j and $k\ne k'$. Denote that $\tilde I_1:=\{(\beta_1,\ldots,\beta_{n_1}):1\le\beta_j<\tilde m_j$ for any $j\in\{1,\ldots,n_1\}\}$, $V_\beta:=\prod_{1\le j\le n_1}V_{z_{j,\beta_j}}$ for any $\beta=(\beta_1,\ldots,\beta_{n_1})\in\tilde I_1$ and $w_\beta:=(w_{1,\beta_1},\ldots,w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta:=(z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}})\in\prod_{1\le j\le n_1}\Omega_j$.

Let $\beta^*=(1,\ldots,1)\in\tilde I_1$, and let $\alpha_{\beta^*}=(\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1})\in\mathbb{Z}_{\ge0}^{n_1}$. Denote that $E':=\left\{\alpha\in\mathbb{Z}_{\ge0}^{n_1}:\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,1}}>\sum_{1\le j\le n_1}\frac{\alpha_{\beta^*,j}+1}{p_{j,1}}\right\}$. Let f be a holomorphic $(n,0)$ form on $\cup_{\beta\in\tilde I_1}V_\beta\times Y$ satisfying $f=\pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}}dw_{1,1}\wedge\ldots\wedge dw_{n_1,1}\right)\wedge\pi_2^*\left(f_{\alpha_{\beta^*}}\right)+\sum_{\alpha\in E'}\pi_1^*(w^\alpha dw_{1,1}\wedge\ldots\wedge dw_{n_1,1})\wedge\pi_2^*(f_\alpha)$ on $V_{\beta^*}\times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2,0)$ forms on Y. Denote $\mathcal{F}_{(z,y)}=\mathcal{I}\big(\hat G+\pi_2^*(\varphi_Y)\big)_{(z,y)}$ for any $(z,y)\in Z_0$. Let $G(t)$ be the minimal L^2 integral on $\{\psi<-t\}$ with respect to φ, f, $\mathcal{F}$ and c for any $t\ge0$.

When there exists $j_0\in\{1,\ldots,n_1\}$ such that $\tilde m_{j_0}=+\infty$, we present that $G\big(h^{-1}(r)\big)$ is not linear.

Theorem 1.12. *If* $G(0)\in(0,+\infty)$ *and there exists* $j_0\in\{1,\ldots,n_1\}$ *such that* $\tilde m_{j_0}=+\infty$, *then* $G\big(h^{-1}(r)\big)$ *is not linear with respect to* $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$.

Remark 1.13. *When* $N\equiv 0$ *(*φ_j *is a subharmonic function on* Ω_j *satisfying* $\varphi_j(z_{j,\beta_j})>-\infty$ *for any* $1\le j\le n_1$ *and* $1\le\beta_j\le m_j$*), it follows from* $\mathcal{I}(\varphi+$

$\psi)_{(z,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z,y)}$ for any $(z, y) \in Z_0$ (see Lemma 3.8) that Theorem 1.12 can be referred to Theorem 1.7 in [2] *(see also Theorem 2.11).*

Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Denote $Z_0 := (\prod_{1\le j\le n_1} Z_j) \times Y$.

Let $\tilde{M} \subset M$ be an $n-$dimensional weakly pseudoconvex Kähler manifold satisfying that $Z_0 \subset \tilde{M}$. Let f be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0.

Let $N \le 0$ be a plurisubharmonic function on $\tilde{M}$ satisfying $N|_{Z_0} \not\equiv -\infty$. For any $j \in \{1, \ldots, n_1\}$, let φ_j be an upper semi-continuous function on Ω_j such that $\varphi_j(z) > -\infty$ for any $z \in Z_j$. Assume that $\sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j) + N$ is a plurisubharmonic function on $\tilde{M}$. Let φ_Y be a plurisubharmonic function on Y, and denote that $\varphi := \sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$.

Let $p_{j,k}$ be a positive number for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$ such that $\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any j. Denote

$$\hat{G} = \max_{1\le j\le n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

and

$$\psi = \max_{1\le j\le n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\} + N.$$

We assume that $\{\psi < -t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t \in \mathbb{R}$.

Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$, $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $c(-\psi)$ has a positive lower bound on any compact subset of $\tilde{M}\backslash Z_0$. Let f be a holomorphic $(n, 0)$ form on a neighborhood of Z_0.

Denote

$$\begin{aligned}\inf\Bigg\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \big(\tilde{f} - f, (z, y)\big) \in \Big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)\Big)_{(z,y)} \\ \text{for any } (z, y) \in Z_0, \\ \& \ \tilde{f} \in H^0\big(\{\psi < -t\}, \mathcal{O}(K_{\tilde{M}})\big).\Bigg\}\end{aligned}$$

by $\tilde{G}(t)$ for any $t \in [0, +\infty)$. Note that $\tilde{G}(t)$ is a minimal L^2 integrals on $\tilde{M}$, where $\tilde{M}$ is a submanifold of M.

We present a necessary condition such that $\tilde{G}\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ as follows.

Proposition 1.14. *If $\tilde{G}(0) \in (0, +\infty)$ and $\tilde{G}\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$, we have $\tilde{M} = M$.*

Remark 1.15. *When $N \equiv 0$ (φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_{j,\beta_j}) > -\infty$ for any $1 \le j \le n_1$ and $1 \le \beta_j \le m_j$), it follows from $\mathcal{I}(\varphi + \psi)_{(z,y)} = \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))_{(z,y)}$ for any $(z, y) \in Z_0$ (see Lemma 3.8) that Proposition 1.14can be referred to Proposition 1.8 in* [2].

1.2. **Applications.** In this section, we present characterizations of the holding of equality in optimal jets L^2 extension problem with the negligible weights on fibraions.

1.2.1. *Background: equality in optimal jets L^2 extension problem.* Let Ω be an open Riemann surface with a nontrivial Green function G_Ω. Let w be a local coordinate on a neighborhood V_{z_0} of $z_0 \in \Omega$ satisfying $w(z_0) = 0$. Let $c_\beta(z)$ be the logarithmic capacity (see [44]) on Ω, i.e.

$$c_\beta(z_0) := \exp \lim_{\xi \to z_0} G_\Omega(z, z_0) - \log |w(z)|.$$

Let $B_\Omega(z_0)$ be the Bergman kernel function on Ω. An open question was posed by Sario-Oikawa [44]: find a relation between the magnitudes of the quantities $\sqrt{\pi B_\Omega(z)}$, $c_\beta(z)$.

In [48], Suita conjectured: $\pi B_\Omega(z_0) \geq \big(c_\beta(z_0)\big)^2$ holds, and the equality holds if and only if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.

The inequality part of Suita conjecture for bounded planar domains was proved by Błocki [5], and original form of the inequality part was proved by Guan-Zhou [34]. The equality part of Suita conjecture was proved by Guan-Zhou [36], then Suita conjecture was completed proved.

It follows from the extremal property of the Bergman kernel function that the holding of the following two equalities are equivalent

(1) $\pi B_\Omega(z_0) = (c_\beta(z_0))^2$;

(2) $\inf\{\int_\Omega |F|^2 : F$ is a holomorphic $(1, 0)$ form on Ω such that $F(z_0) = dw\} = \frac{2\pi}{(c_\beta(z_0))^2}$.

Note that (2) is equivalent to the holding of the equality in optimal 0-jet L^2 extension problem for open Riemann surface Ω and single point z_0 with trivial weights $\varphi \equiv 0$ and trivial gain $c \equiv 1$. Then it is natural to ask

Problem 1.16. *How to characterize the holding of the equality in optimal k-jets L^2 extension problem, where k is a nonnegative integer?*

For open Riemann surfaces and single points, when the weights are harmonic and gain is constant, Guan-Zhou [36] gave an answer to 0-jet version of Problem 1.16, i.e. a proof of the extended Suita conjecture posed by Yamada [51].

For open Riemann surfaces and single points, Guan-Yuan [28] gave an answer to 0-jet version of Problem 1.16 (when the weights are subharmonic and gain is smooth, Guan-Mi [24] gave an answer to 0-jet version of Problem 1.16), and Guan-Mi-Yuan [26] gave an answer to Problem 1.16.

For open Riemann surfaces and analytic sets (with finite or infinite points), Guan-Yuan [32] gave an answer to Problem 1.16. For products of open Riemann surfaces and products of analytic sets, Guan-Yuan [33] gave an answer to Problem 1.16.

For fibrations over open Riemann surfaces, Bao-Guan-Yuan [1] gave an answer to Problem 1.16 with negligible weights pulled back from the open Riemann surfaces. For fibrations over products of open Riemann surfaces, Bao-Guan-Yuan [2] gave an answer to Problem 1.16 with negligible weights pulled back from the products of open Riemann surfaces.

In the following sections, for fibrations over open Riemann surfaces and fibrations over products of open Riemann surfaces, we give answers to Problem 1.16 with negligible weights on fibrations.

1.2.2. *Fibrations over open Riemann surfaces.* In this section, we give characterizations of the holding of equality in optimal jets L^2 extension problem with negligible weights from fibers over analytic subsets to fibrations over open Riemann surfaces.

Let Ω be an open Riemann surface with nontrivial Green functions. Let Y be an $(n-1)-$dimensional weakly pseudoconvex Kähler manifold. Denote $M=\Omega\times Y$. Let K_M be the canonical line bundle on M. Let π_1 and π_2 be the natural projections from M to Ω and Y respectively.

Let $Z_\Omega=\{z_j : j\in\mathbb{N}_+\&1\le j<\gamma\}$ be a subset of Ω of discrete points. Denote $Z_0:=Z_\Omega\times Y$. Denote $Z_j:=\{z_j\}\times Y$ for any $j\ge 1$.

Assume that $\tilde{M}\subset M$ is an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0\subset\tilde{M}$.

Let ψ be a plurisubharmonic function on $\tilde{M}$ such that $\{\psi<-t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t\in\mathbb{R}$ and $Z_0\subset\{\psi=-\infty\}$. It follows from Siu's decomposition theorem that

$$dd^c\psi=\sum_{j\ge1}2p_j[Z_j]+\sum_{i\ge1}\lambda_i[A_i]+R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $n-$dimensional analytic set, and where R is a closed positive current with the property that $dimE_c(R)<n$ for every $c>0$, where $E_c(R)=\{x\in M : v(R,x)\ge c\}$ is the upperlevel sets of Lelong number. We assume that $p_j\ge 0$ is a positive number for any $1\le j<\gamma$ and $2\sum_{j\ge1}p_jG_\Omega(\cdot,z_j)\not\equiv-\infty$.

Then $N:=\psi-\pi_1^*(\sum\limits_{j\ge1}2p_jG_\Omega(z,z_j))$ is a plurisubharmonic function on $\tilde{M}$. We assume that $N\le 0$.

Let k_j be a nonnegative integer for any $1\le j<\gamma$. Let φ_1 be a Lebesgue measurable function on Ω such that $\pi_1^*(\varphi_1)+\psi$ is a plurisubharmonic function on $\tilde{M}$. We also assume that there exists a holomorphic function $g\in\mathcal{O}(\Omega)$ and a plurisubharmonic function $\tilde{\psi}_2\in Psh(\tilde{M})$ such that

$$\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\log|g|)+\tilde{\psi}_2,$$

where $ord_{z_j}(g)=k_j+1$, for any $1\le j<\gamma$.

Let φ_2 be a plurisubharmonic funciton on Y. Denote $\varphi:=\pi_1^*(\varphi_1)+\pi_1^*(\varphi_2)$.

For any $1\le j<\gamma$, let $\tilde{z}_j$ be a local coordinate on a neighborhood V_j of z_j satisfying $\tilde{z}_j(z_j)=0$ and $V_j\cap V_k=\emptyset$, for any $j\ne k$. We assume that $g=d_j\tilde{z}_j^{k_j+1}h_j(\tilde{z}_j)$ on each V_j, where $h_j(z_j)=1$. Denote $V_0:=\cup_{1\le j<\gamma}V_j$.

Let $\gamma=m+1$ be a positive integer. We give an application of Theorem 1.2 as below.

Theorem 1.17. *Let ψ, φ_1 and φ_2 be as above. Let $c(t)$ be a positive measurable function on $(0,+\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$ and $\int_0^{+\infty}c(s)e^{-s}ds<+\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n,0)$ form on Y. Assume that*

$$\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}\in(0,+\infty).$$

Let f be a holomorphic $(n,0)$ form on $V_0\times Y$ satisfying that $f=\pi_1^*(a_j\tilde{z}_j^{k_j}d\tilde{z}_j)\wedge\pi_2^*(F_j)$ on $V_j\times Y$. Then there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ such that $(F-f,(z_j,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))_{(z_j,y)}$ for any $(z_j,y)\in Z_0$ and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{m}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}. \quad (1.3)$$

Moreover, equality $\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{m}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}=\inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $(\tilde{F}-f,(z_j,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))_{(z_j,y)}$ for any $(z_j,y)\in Z_0\Big\}$ holds if and only if the following statements hold:

(1). $N\equiv0$ and $\psi=\pi_1^*(2\sum_{j=1}^{m}p_jG_\Omega(\cdot,z_j))$;

(2). $\varphi_1+2\sum_{j=1}^{m}k_jG_\Omega(\cdot,z_j)=2\log|g|+2\sum_{j=1}^{m}(k_j+1)G_\Omega(\cdot,z_j)+2u$, where g is a holomorphic function on Ω such that $g(z_j)\ne0$ for any $j\in\{1,2,\dots,m\}$ and u is a harmonic function on Ω;

(3). $\prod_{j=1}^{m}\chi_{z_j}^{k_j+1}=\chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot,z_j)$ respectively;

(4). for any $j\in\{1,2,\dots,m\}$,

$$\lim_{z\to z_j}\frac{a_j\tilde{z}_j^{k_j}d\tilde{z}_j}{gP_*\left(f_u\left(\prod_{l=1}^{m}f_{z_l}^{k_l+1}\right)\left(\sum_{l=1}^{m}p_l\frac{df_{z_l}}{f_{z_l}}\right)\right)}=c_j\in\mathbb{C}\setminus\{0\}, \quad (1.4)$$

and there exist $c_0\in\mathbb{C}\setminus\{0\}$ and a holomorphic $(n-1,0)$ form F_Y on Y which are independent of j such that $c_0F_Y=c_jF_j$ for any $j\in\{1,2,\dots,m\}$;

(5). $\tilde{M}=M$.

Remark 1.18. *When $N\equiv\pi_1^*(N_1)$, it follows from $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))_{(z_j,y)}$ (see Lemma 3.7) that Theorem 1.17 can be referred to Theorem 1.8 in* [1] *(see also Theorem 2.7), where $N_1\le0$ is a subharmonic function on Ω.*

When $\gamma=+\infty$, i.e., $Z_\Omega=\{z_j:1\le j<\gamma\}$ is an infinite subset of Ω of discrete points. Let k_j be a nonnegative integer for any $j\in\mathbb{N}^+$. Assume that $p_j=k_j+1$, i.e., we have $\psi=\pi_1^*\big(\sum_{j=1}^{+\infty}2(k_j+1)G_\Omega(z,z_j)\big)+N$, where $N\le0$ is a plurisubharmonic function on $\tilde{M}$.

We give an L^2 extension result from fibers over analytic subsets to fibrations over open Riemann surfaces, where the analytic subsets are infinite subsets of discrete points on open Riemann surfaces.

Theorem 1.19. *Let ψ, φ_1 and φ_2 be as above. Let $c(t)$ be a positive measurable function on $(0,+\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$ and $\int_0^{+\infty}c(s)e^{-s}ds<+\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n,0)$ form on Y. Assume that*

$$\sum_{j=1}^{+\infty}\frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}\in(0,+\infty).$$

Let f be a holomorphic $(n,0)$ form on $V_0 \times Y$ satisfying that $f = \pi_1^*(a_j \tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(F_j)$ on $V_{z_j} \times Y$. Then there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ such that $\big(F - f, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) < \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{+\infty} \frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)}. \tag{1.5}$$

Remark 1.20. *When $N \equiv \pi_1^*(N_1)$, it follows from $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))_{(z_j,y)}$ (see Lemma 3.7) that Theorem 1.19 can be referred to Theorem 1.10 in* [1] *(see also Theorem 2.8), where $N_1 \leq 0$ is a subharmonic function on Ω.*

1.2.3. *Fibrations over products of open Riemann surfaces.* In this section, we present characterizations of the holding of equality in optimal jets L^2 extension problem with negligible weights from fibers over analytic subsets to fibrations over products of open Riemann surfaces.

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \leq j \leq n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M = \left(\prod_{1\leq j\leq n_1} \Omega_j\right) \times Y$ be an $n-$dimensional complex manifold, where $n = n_1 + n_2$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\leq j\leq n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M. Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, and denote that $Z_0 := \left(\prod_{1\leq j\leq n_1} Z_j\right) \times Y$. Let $\tilde{M} \subset M$ be an $n-$dimensional complex manifold satisfying that $Z_0 \subset \tilde{M}$, and let $K_{\tilde{M}}$ be the canonical (holomorphic) line bundle on $\tilde{M}$.

Let $Z_j = \{z_{j,k} : 1 \leq k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\geq 2} \cup \{+\infty\}$. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$. Denote that $\tilde{I}_1 := \big\{(\beta_1, \ldots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j$ for any $j \in \{1, \ldots, n_1\}\big\}$, $V_\beta := \prod_{1\leq j\leq n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n_1,\beta_{n_1}}) \in \prod_{1\leq j\leq n_1} \Omega_j$ for any $\beta = (\beta_1, \ldots, \beta_{n_1}) \in \tilde{I}_1$. Then $Z_0 = \{(z_\beta, y) : \beta \in \tilde{I}_1 \,\&\, y \in Y\} \subset \tilde{M}$.

Let $N \leq 0$ be a plrusubharmonic function on $\tilde{M}$ and let φ_j be a Lebesgue measurable function on Ω_j such that $N + \sum_{1\leq j\leq n_1} \pi_{1,j}^*(\varphi_j)$ is a plurisubharmonic function on $\tilde{M}$ satisfying $\left(N + \sum_{1\leq j\leq n_1} \pi_{1,j}^*(\varphi_j)\right)|_{Z_0} \not\equiv -\infty$. Let φ_Y be a plurisubharmonic function on Y.

Let $p_{j,k}$ be a positive number for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$, which satisfies that $\sum_{1\leq k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \leq j \leq n_1$. Denote

$$\hat{G} := \max_{1\leq j\leq n_1} \left\{ 2 \sum_{1\leq k<\tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\},$$

and denote that

$$\psi := \max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}+N.$$

We assume that $\{\psi<-t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t\in\mathbb{R}$. Let $\varphi:=\sum_{1\le j\le n_1}\pi_{1,j}^*(\varphi_j)+\pi_2^*(\varphi_Y)$ on $\tilde{M}$.

Denote that $E_\beta:=\left\{(\alpha_1,\ldots,\alpha_{n_1}):\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}=1\,\&\,\alpha_j\in\mathbb{Z}_{\ge0}\right\}$ and $\tilde{E}_\beta:=\left\{(\alpha_1,\ldots,\alpha_{n_1}):\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}\ge1\,\&\,\alpha_j\in\mathbb{Z}_{\ge0}\right\}$ for any $\beta\in\tilde{I}_1$.

Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0\subset\tilde{M}$ of Z_0 such that

$$f=\sum_{\alpha\in\tilde{E}_\beta}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(f_{\alpha,\beta}) \tag{1.6}$$

on $U_0\cap(V_\beta\times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y for any $\alpha\in E_\beta$ and $\beta\in\tilde{I}_1$.

Denote that

$$c_{j,k}:=\exp\lim_{z\to z_{j,k}}\left(\frac{\sum_{1\le k_1<\tilde{m}_j}p_{j,k_1}G_{\Omega_j}(z,z_{j,k_1})}{p_{j,k}}-\log|w_{j,k}(z)|\right)$$

for any $j\in\{1,\ldots,n\}$ and $1\le k<\tilde{m}_j$ (following from Lemma 2.18 and Lemma 2.19, we get that the above limit exists).

Let $c_j(z)$ be the logarithmic capacity (see [44]) on Ω_j, which is locally defined by

$$c_j(z_j):=\exp\lim_{z\to z_j}(G_{\Omega_j}(z,z_j)-\log|w_j(z)|).$$

For the case $Z_0=\{z_0\}\times Y\subset\tilde{M}$, where $z_0=(z_1,\ldots,z_{n_1})\in\prod_{1\le j\le n_1}\Omega_j$, we denote that $E:=\left\{(\alpha_1,\ldots,\alpha_{n_1}):\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_j}=1\,\&\,\alpha_j\in\mathbb{Z}_{\ge0}\right\}$. Let f_α be the holomorphic $(n_2,0)$ form on Y which comes from formula (1.6). We obtain a characterization of the holding of equality in optimal jets L^2 extension problem.

Theorem 1.21. *Let c be a positive function on $(0,+\infty)$ such that $\int_0^{+\infty}c(t)e^{-t}dt<+\infty$ and $c(t)e^{-t}$ is decreasing on $(0,+\infty)$. Assume that*

$$\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}\in(0,+\infty).$$

Then there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\big(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\big)\right)_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_o,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}.\end{aligned}$$

Moreover, equality $\inf\big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}\in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F}-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\big(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\big)\right)_z$ *for any* $z\in Z_0\big\}=\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\times$

$\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}$ *holds if and only if the following statements hold:*

(1) $\tilde{M}=\left(\prod_{1\le j\le n_1}\Omega_j\right)\times Y$ *and* $N\equiv 0$*;*

(2) $\varphi_j=2\log|g_j|+2u_j$*, where* g_j *is a holomorphic function on* Ω_j *such that* $g_j(z_j)\neq 0$ *and* u_j *is a harmonic function on* Ω_j *for any* $1\le j\le n_1$*;*

(3) $\chi_{j,z_j}^{\alpha_j+1}=\chi_{j,-u_j}$ *for any* $j\in\{1,2,...,n\}$ *and* $\alpha\in E$ *satisfying* $f_\alpha\not\equiv 0$*.*

Remark 1.22. *If* $(f_\alpha,y)\in\left(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y)\right)_y$ *for any* $y\in Y$ *and* $\alpha\in\tilde{E}\backslash E$*, the above result also holds when we replace the ideal sheaf* $\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)$ *by* $\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_2)\big)$*. We prove the remark in Section 7.*

Let $Z_j=\{z_{j,1},\ldots,z_{j,m_j}\}\subset\Omega_j$ for any $j\in\{1,\ldots,n_1\}$, where m_j is a positive integer. Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0\subset\tilde{M}$ of Z_0 such that

$$f=\sum_{\alpha\in\tilde{E}_\beta}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(f_{\alpha,\beta})$$

on $U_0\cap(V_\beta\times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y for any $\alpha\in E_\beta$ and $\beta\in\tilde{I}_1$. Let $\beta^*=(1,\ldots,1)\in\tilde{I}_1$, and let $\alpha_{\beta^*}=(\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1})\in E_{\beta^*}$. Denote that $E':=\left\{\alpha\in\mathbb{Z}_{\ge 0}^{n_1}:\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,1}}>1\right\}$. Assume that $f=\pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}}dw_{1,1}\wedge\ldots\wedge dw_{n_1,1}\right)\wedge\pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right)+\sum_{\alpha\in E'}\pi_1^*(w^\alpha dw_{1,1}\wedge\ldots\wedge dw_{n_1,1})\wedge\pi_2^*(f_{\alpha,\beta})$ on $U_0\cap(V_{\beta^*}\times Y)$.

We obtain a characterization of the holding of equality in optimal jets L^2 extension problem for the case that Z_j is finite.

Theorem 1.23. *Let* c *be a positive function on* $(0,+\infty)$ *such that* $\int_0^{+\infty}c(t)e^{-t}dt<+\infty$ *and* $c(t)e^{-t}$ *is decreasing on* $(0,+\infty)$*. Assume that*

$$\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\in(0,+\infty).$$

Then there exists a holomorphic $(n,0)$ *form* F *on* $\tilde{M}$ *satisfying that* $(F-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\big(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\big)\right)_z$ *for any* $z\in Z_0$ *and*

$$\begin{aligned}&\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}$$

Moreover, equality $\inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}\in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F}-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\big(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\big)\right)_z$ *for any* $z\in Z_0\Big\}=\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ *holds if and only if the following statements hold:*

(1) $\tilde{M}=\left(\prod_{1\le j\le n_1}\Omega_j\right)\times Y$ *and* $N\equiv 0$*;*

(2) $\varphi_j=2\log|g_j|+2u_j$ *for any* $j\in\{1,\ldots,n_1\}$*, where* u_j *is a harmonic function on* Ω_j *and* g_j *is a holomorphic function on* Ω_j *satisfying* $g_j(z_{j,k})\neq 0$ *for any* $k\in\{1,\ldots,m_j\}$*;*

(3) *There exists a nonnegative integer* $\gamma_{j,k}$ *for any* $j\in\{1,\ldots,n_1\}$ *and* $k\in\{1,\ldots,m_j\}$*, which satisfies that* $\prod_{1\le k\le m_j}\chi_{j,z_{j,k}}^{\gamma_{j,k}+1}=\chi_{j,-u_j}$ *and* $\sum_{1\le j\le n_1}\frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}}=1$ *for any* $\beta\in\tilde{I}_1$*;*

(4) $f_{\alpha,\beta}=c_\beta f_0$ *holds for* $\alpha=(\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})$ *and* $f_{\alpha,\beta}\equiv 0$ *holds for any* $\alpha\in E_\beta\backslash\{(\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})\}$*, where* $\beta\in\tilde{I}_1$*,* c_β *is a constant and* $f_0\not\equiv 0$ *is a holomorphic* $(n_2,0)$ *form on* Y *satisfying* $\int_Y|f_0|^2e^{-\varphi_2}<+\infty$*;*

(5) $c_\beta\prod_{1\le j\le n_1}\left(\lim_{z\to z_{j,\beta_j}}\frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}}dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\le k\le m_j}f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\le k\le m_j}p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)}\right)=$ c_0 *for any* $\beta\in\tilde{I}_1$*, where* $c_0\in\mathbb{C}\backslash\{0\}$ *is a constant independent of* β*,* f_{u_j} *is a holomorphic function* Δ *such that* $|f_{u_j}|=P_j^*(e^{u_j})$ *and* $f_{z_{j,k}}$ *is a holomorphic function on* Δ *such that* $|f_{z_{j,k}}|=P_j^*\left(e^{G_{\Omega_j}(\cdot,z_{j,k})}\right)$ *for any* $j\in\{1,\ldots,n_1\}$ *and* $k\in\{1,\ldots,m_j\}$*.*

Remark 1.24. *If* $(f_{\alpha,\beta},y)\in(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y))_y$ *holds for any* $y\in Y$*,* $\alpha\in\tilde{E}_\beta\backslash E_\beta$ *and* $\beta\in\tilde{I}_1$*, the above result also holds when we replace the ideal sheaf* $\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)$ *by* $\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_2)\big)$*. We prove the remark in Section 7.*

Let $Z_j=\{z_{j,k}:1\le k<\tilde{m}_j\}$ be a discrete subset of Ω_j for any $j\in\{1,\ldots,n_1\}$, where $\tilde{m}_j\in\mathbb{Z}_{\ge 2}\cup\{+\infty\}$. Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0\subset\tilde{M}$ of Z_0 such that

$$f=\sum_{\alpha\in\tilde{E}_\beta}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(f_{\alpha,\beta})$$

on $U_0\cap(V_\beta\times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y for any $\alpha\in E_\beta$ and $\beta\in\tilde{I}_1$. Let $\beta^*=(1,\ldots,1)\in\tilde{I}_1$, and let $\alpha_{\beta^*}=(\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1})\in E_{\beta^*}$. Denote that $E':=\left\{\alpha\in\mathbb{Z}_{\ge 0}^{n_1}:\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,1}}>1\right\}$. Assume that $f=\pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}}dw_{1,1}\wedge\ldots\wedge dw_{n_1,1}\right)\wedge\pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right)+\sum_{\alpha\in E'}\pi_1^*(w^\alpha dw_{1,1}\wedge\ldots\wedge dw_{n_1,1})\wedge\pi_2^*(f_{\alpha,\beta})$ on $U_0\cap(V_{\beta^*}\times Y)$.

When there exists $j_0\in\{1,\ldots,n_1\}$ such that $\tilde{m}_{j_0}=+\infty$, we obtain that the equality in optimal jets L^2 extension problem *could not hold.*

Theorem 1.25. *Let* c *be a positive function on* $(0,+\infty)$ *such that* $\int_0^{+\infty}c(t)e^{-t}dt<+\infty$ *and* $c(t)e^{-t}$ *is decreasing on* $(0,+\infty)$*. Assume that*

$$\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\in(0,+\infty)$$

and there exists $j_0\in\{1,\ldots,n_1\}$ *such that* $\tilde{m}_{j_0}=+\infty$*.*

Then there exists a holomorphic $(n,0)$ *form* F *on* $\tilde{M}$ *satisfying that* $(F-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ *for any* $z\in Z_0$

and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$$
$$<\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*\left(\sum_{1\le j\le n_1}\varphi_j\right)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Remark 1.26. *If $(f_{\alpha,\beta},y)\in(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y))_y$ holds for any $y\in Y$, $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in\tilde{I}_1$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)$ by $\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_2)\big)$. We prove the remark in Section 7.*

Remark 1.27. *We note that, by using our recent progress (see* [27]*) on minimal L^2 integrals related to boundary points, the main results in the present article hold without assuming the condtion "$\{\psi<-t\}\backslash Z_0$ is a weakly pseudoconvex Kähler manifold for any $t\in\mathbb{R}$".*

2. Preparations I: Minimal L^2 integrals

In this section, we recall and present some lemmas related to minimal L^2 integrals.

2.1. Minimal L^2 integrals on weakly pseudoconvex Kähler manifolds.

In this section, we recall some results about the concavity property of minimal L^2 integrals on weakly pseudoconvex Kähler manifolds in [26].

Let M be an $n-$dimensional complex manifold. Let X and Z be closed subsets of M. A triple (M,X,Z) satisfies condition (A), if the following statements hold:

(1). X is a closed subset of M and X is locally neligible with respect to L^2 holomorphic functions, i.e., for any coordinated neighborhood $U\subset M$ and for any L^2 holomorphic function f on $U\setminus X$, there exists an L^2 holomorphic function $\tilde{f}$ on U such that $\tilde{f}|_{U\setminus X}=f$ with the same L^2 norm;

(2). Z is an analytic subset of M and $M\setminus(X\cup Z)$ is a weakly pseudoconvex Kähler manifold.

Let (M,X,Z) be a triple satisfying condition (A). Let K_M be the canonical line bundle on M. Let ψ be a plurisubharmonic function on M such that $\{\psi<-t\}\backslash(X\cup Z)$ is a weakly pseudoconvex Kähler manifold for any $t\in\mathbb{R}$. Let φ be a Lebesgue measurable function on M such that $\varphi+\psi$ is a plurisubharmonic function on M. Denote $T=-\sup_M\psi$.

We recall the concept of "gain" in [26]. A positive measurable function c on $(T,+\infty)$ is in the class $\mathcal{P}_{T,M}$ if the following two statements hold:

(1). $c(t)e^{-t}$ is decreasing with respect to t;

(2). there is a closed subset E of M such that $E\subset Z\cap\{\psi=-\infty\}$ and for any compact subset $K\subset M\setminus E$, $e^{-\varphi}c(-\psi)$ has a positive lower bound on K.

Let Z_0 be a subset of M such that $Z_0\cap\mathrm{Supp}(\mathcal{O}/\mathcal{I}(\varphi+\psi))\ne\emptyset$. Let $U\supset Z_0$ be an open subset of M, and let f be a holomorphic $(n,0)$ form on U. Let $\mathcal{F}_{z_0}\supset\mathcal{I}(\varphi+\psi)_{z_0}$ be an ideal of $\mathcal{O}_{z_0}$ for any $z_0\in Z_0$.

Denote that

$$G(t;c) := \inf\left\{ \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) : \tilde{f} \in H^0(\{\psi<-t\}, \mathcal{O}(K_M)) \right. \\ \left. \&(\tilde{f}-f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0}) \right\}, \tag{2.1}$$

and

$$\mathcal{H}^2(c,t) := \left\{ \tilde{f} : \int_{\{\psi<-t\}} |\tilde{f}|^2 e^{-\varphi} c(-\psi) < +\infty, \tilde{f} \in H^0(\{\psi<-t\}, \mathcal{O}(K_M)) \right. \\ \left. \&(\tilde{f}-f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0}) \right\}, \tag{2.2}$$

where $t \in [T,+\infty)$, and c is a nonnegative measurable function on $(T,+\infty)$. Here $|\tilde{f}|^2 := \sqrt{-1}^{n^2} \tilde{f} \wedge \bar{\tilde{f}}$ for any $(n,0)$ form $\tilde{f}$, and $(\tilde{f}-f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ means that $(\tilde{f}-f, z_0) \in (\mathcal{O}(K_M)\otimes\mathcal{F})_{z_0}$ for any $z_0 \in Z_0$. If there is no holomorphic $(n,0)$ form $\tilde{f}$ on $\{\psi<-t\}$ satisfying $(\tilde{f}-f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$, we set $G(t;c) = +\infty$.

In [26], Guan-Mi-Yuan obtained the following concavity of $G(t;c)$.

Theorem 2.1 ([26]). *Let $c \in \mathcal{P}_{T,M}$ such that $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. If there exists $t \in [T,+\infty)$ satisfying that $G(t) < +\infty$, then $G(h^{-1}(r))$ is concave with respect to $r \in (0, \int_T^{+\infty} c(t)e^{-t}dt)$, $\lim_{t\to T+0} G(t) = G(T)$ and $\lim_{t\to+\infty} G(t) = 0$, where $h(t) = \int_t^{+\infty} c(t_1)e^{-t_1}dt_1$.*

Lemma 2.2 ([26]). *Let $c \in \mathcal{P}_{T,M}$ satisfying $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. Assume that $G(t) < +\infty$ for some $t \in [T,+\infty)$. Then there exists a unique holomorphic $(n,0)$ form F_t on $\{\psi<-t\}$ satisfying $(F_t-f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} c(-\psi) = G(t)$. Furthermore, for any holomorphic $(n,0)$ form $\hat{F}$ on $\{\psi<-t\}$ satisfying $(\hat{F}-f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$ and $\int_{\{\psi<-t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi) < +\infty$, we have the following equality*

$$\int_{\{\psi<-t\}} |F_t|^2 e^{-\varphi} c(-\psi) + \int_{\{\psi<-t\}} |\hat{F}-F_t|^2 e^{-\varphi} c(-\psi) \\ = \int_{\{\psi<-t\}} |\hat{F}|^2 e^{-\varphi} c(-\psi). \tag{2.3}$$

Guan-Mi-Yuan also obtained the following corollary of Theorem 2.1, which is a necessary condition for the concavity degenerating to linearity.

Lemma 2.3 ([26]). *Let $c(t) \in \mathcal{P}_{T,M}$ such that $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$. If $G(t) \in (0,+\infty)$ for some $t \geq T$ and $G(h^{-1}(r))$ is linear with respect to $r \in (0, \int_T^{+\infty} c(s)e^{-s}ds)$, where $h(t) = \int_t^{+\infty} c(l)e^{-l}dl$, then there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f) \in H^0(Z_0, (\mathcal{O}(K_M)\otimes\mathcal{F})|_{Z_0})$, and $G(t;c) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq T$.*

Furthermore, we have

$$\int_{\{-t_2\leq\psi<-t_1\}} |F|^2 e^{-\varphi} a(-\psi) = \frac{G(T_1;c)}{\int_{T_1}^{+\infty} c(t)e^{-t}dt} \int_{t_1}^{t_2} a(t)e^{-t}dt, \tag{2.4}$$

for any nonnegative measurable function a on $(T,+\infty)$, where $T \le t_1 < t_2 \le +\infty$.

Especially, if $\mathcal{H}^2(\tilde{c},t_0) \subset \mathcal{H}^2(c,t_0)$ for some $t_0 \ge T$, where $\tilde{c}$ is a nonnegative measurable function on $(T,+\infty)$, we have

$$G(t_0;\tilde{c}) = \int_{\{\psi<-t_0\}} |F|^2 e^{-\varphi}\tilde{c}(-\psi) = \frac{G(T_1;c)}{\int_{T_1}^{+\infty} c(s)e^{-s}ds}\int_{t_0}^{+\infty}\tilde{c}(s)e^{-s}ds. \tag{2.5}$$

Remark 2.4 ([26]). *Let $c(t) \in \mathcal{P}_{T,M}$. If $\mathcal{H}^2(\tilde{c},t_1) \subset \mathcal{H}^2(c,t_1)$, then $\mathcal{H}^2(\tilde{c},t_2) \subset \mathcal{H}^2(c,t_2)$, where $t_1 > t_2 > T$. In the following, we give some sufficient conditions of $\mathcal{H}^2(\tilde{c},t_0) \subset \mathcal{H}^2(c,t_0)$ for $t_0 > T$:*

(1). $\tilde{c} \in \mathcal{P}_{T,M}$ and $\lim\limits_{t\to+\infty}\frac{\tilde{c}(t)}{c(t)} > 0$. Especially, $\tilde{c} \in \mathcal{P}_{T,M}$, c and $\tilde{c}$ are smooth on $(T,+\infty)$ and $\frac{d}{dt}(\log\tilde{c}(t)) \ge \frac{d}{dt}(\log c(t))$;

(2). $\tilde{c} \in \mathcal{P}_{T,M}$, $\mathcal{H}^2(c,t_0) \neq \emptyset$ and there exists $t > t_0$ such that $\{\psi < -t\} \Subset \{\psi < -t_0\}$, $\{z \in \overline{\{\psi < -t\}} : \mathcal{I}(\varphi+\psi)_z \neq \mathcal{O}_z\} \subset Z_0$ and $\mathcal{F}|_{\overline{\{\psi<-t\}}} = \mathcal{I}(\varphi+\psi)|_{\overline{\{\psi<-t\}}}$.

2.2. **The sufficient and necessary conditions of the concavity of $G(h^{-1}(r))$ degenerating to linearity.** In this section, we recall some result on the characterizations of the concavity of $G(h^{-1}(r))$ degenerating to linearity on the fibrations over open Riemann surfaces and products of open Riemann surfaces.

The following result can be referred to [1].

Let $Z_0^1 := \{z_j : j \in \mathbb{N}\&1 \le j \le m\}$ be a finite subset of the open Riemann surface Ω. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega \times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y and $Z_0 := \pi_1^{-1}(Z_0^1)$. Let ψ_1 be a subharmonic function on Ω such that $p_j = \frac{1}{2}v(dd^c\psi_1, z_j) > 0$, and let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1+\psi_1$ is subharmonic on Ω. Let φ_2 be a plurisubharmonic function on Y. Denote that $\psi := \pi_1^*(\psi_1)$, $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j} \Subset \Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j \in Z_0^1$, where $V_{z_j} \cap V_{z_k} = \emptyset$ for any j,k, $j \neq k$. Denote that $V_0 := \bigcup_{1\le j\le m} V_{z_j}$. Let f be a holomorphic $(n,0)$ form on $V_0 \times Y$. Denote $\mathcal{F}_{(z_j,y)} = \mathcal{I}(\psi+\varphi)_{(z_j,y)}$ for any $(z_j,y) \in Z_0$. Let $G(t;c)$ be the minimal L^2 integral on $\{\psi < -t\}$ with respect to φ, f and $\mathcal{F}$ for any $t \ge 0$.

We recall a characterization of the concavity of $G(h^{-1}(r))$ degenerating to linearity for the fibers over sets of finite points as follows.

Theorem 2.5 ([1]). *Assume that $G(0) \in (0,+\infty)$ and $(\psi_1 - 2p_jG_\Omega(\cdot,z_j))(z_j) > -\infty$, where $p_j = \frac{1}{2}v(dd^c(\psi_1),z_j) > 0$ for any $j \in \{1,2,\ldots,m\}$. Then $G(h^{-1}(r))$ is linear with respect to $r \in (0,\int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1). $\psi_1 = 2\sum\limits_{j=1}^m p_jG_\Omega(\cdot,z_j)$;

(2). for any $j \in \{1,2,\ldots,m\}$, $f = \pi_1^*(a_jw_j^{k_j}dw_j)\wedge\pi_2^*(f_Y) + f_j$ on $V_{z_j}\times Y$, where $a_j \in \mathbb{C}\backslash\{0\}$ is a constant, k_j is a nonnegative integer, f_Y is a holomorphic $(n-1,0)$ form on Y such that $\int_Y |f_Y|^2e^{-\varphi_2} \in (0,+\infty)$, and $(f_j,(z_j,y)) \in (\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi))_{(z_j,y)}$ for any $j \in \{1,2,\ldots,m\}$ and $y \in Y$;

(3). $\varphi_1+\psi_1 = 2\log|g| + 2\sum\limits_{j=1}^m G_\Omega(\cdot,z_j) + 2u$, where g is a holomorphic function on Ω such that $ord_{z_j}(g) = k_j$ and u is a harmonic function on Ω;

(4). $\prod_{j=1}^{m}\chi_{z_j}=\chi_{-u}$, *where* χ_{-u} *and* χ_{z_j} *are the characters associated to the functions* $-u$ *and* $G_\Omega(\cdot,z_j)$ *respectively;*

(5). for any $j\in\{1,2,\ldots,m\}$,

$$\lim_{z\to z_j}\frac{a_jw_j^{k_j}dw_j}{gP_*\left(f_u\left(\prod_{l=1}^{m}f_{z_l}\right)\left(\sum_{l=1}^{m}p_l\frac{df_{z_l}}{f_{z_l}}\right)\right)}=c_0, \tag{2.6}$$

where $c_0\in\mathbb{C}\setminus\{0\}$ *is a constant independent of* j, f_u *is a holomorphic function* Δ *such that* $|f_u|=P^*(e^u)$ *and* $f_{z_{j,k}}$ *is a holomorphic function on* Δ *such that* $|f_{z_l}|=P^*\left(e^{G_\Omega(\cdot,z_l)}\right)$ *for any* $l\in\{1,\ldots,m\}$ *.*

When $Z_0^1:=\{z_j:j\in\mathbb{N}\&1\le j<+\infty\}$ is an infinite subset of the open Riemann surface Ω of discrete set. Assume that $2\sum_{j=1}^{+\infty}p_jG_\Omega(\cdot,z_j)\not\equiv-\infty$. We recall the following necessary condition such that $G\big(h^{-1}(r)\big)$ is linear.

Proposition 2.6 ([1])**.** *Assume that* $G(0)\in(0,+\infty)$ *and* $\big(\psi_1-2p_jG_\Omega(\cdot,z_j)\big)(z_j)>-\infty$, *where* $p_j=\frac{1}{2}v(dd^c(\psi_1),z_j)>0$ *for any* $j\in\mathbb{N}_+$. *Assume that* $G\big(h^{-1}(r)\big)$ *is linear with respect to* $r\in(0,\int_0^{+\infty}c(t)e^{-t}dt]$, *then the following statements hold:*

(1). $\psi_1=2\sum_{j=1}^{\infty}p_jG_\Omega(\cdot,z_j)$;

(2). for any $j\in\mathbb{N}_+$, $f=\pi_1^*(a_jw_j^{k_j}dw_j)\wedge\pi_2^*(f_Y)+f_j$ *on* $V_{z_j}\times Y$, *where* $a_j\in\mathbb{C}\setminus\{0\}$ *is a constant,* k_j *is a nonnegative integer,* f_Y *is a holomorphic* $(n-1,0)$ *form on* Y *such that* $\int_Y|f_Y|^2e^{-\varphi_2}\in(0,+\infty)$, *and* $\big(f_j,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi)\big)_{(z_j,y)}$ *for any* $j\in\mathbb{N}_+$ *and* $y\in Y$;

(3). $\varphi_1+\psi_1=2\log|g|$, *where* g *is a holomorphic function on* Ω *such that* $ord_{z_j}(g)=k_j+1$ *for any* $j\in\mathbb{N}_+$;

(4). for any $j\in\mathbb{N}_+$,

$$\frac{p_j}{ord_{z_j}g}\lim_{z\to z_j}\frac{dg}{a_jw_j^{k_j}dw_j}=c_0, \tag{2.7}$$

where $c_0\in\mathbb{C}\setminus\{0\}$ *is a constant independent of* j;

(5). $\sum_{j\in\mathbb{N}_+}p_j<+\infty$.

Let $Z_0^1:=\{z_j:j\in\mathbb{N}\&1\le j\le m\}$ be a finite subset of the open Riemann surface Ω. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M=\Omega\times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y, and $Z_0:=\pi_1^{-1}(Z_0^1)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j}\Subset Omega$ of z_j satisfying $w_j(z_j)=0$ for $z_j\in Z_0^1$, where $V_{z_j}\cap V_{z_k}=\emptyset$ for any j,k, $j\ne k$. Let $c_\beta(z)$ be the logarithmic capacity (see [44]) on Ω which is defined by

$$c_\beta(z_j):=\exp\lim_{z\to z_0}(G_\Omega(z,z_j)-\log|w_j(z)|).$$

Denote that $V_0:=\bigcup_{1\le j\le m}V_{z_j}$. Assume that $\tilde{M}\subset M$ is an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0\subset\tilde{M}$.

We recall the following characterization of the holding of the equality in optimal L^2 extension from fibers over analytic subsets to fibrations over open Riemann surfaces (see [1]).

Theorem 2.7 ([1]). *Let k_j be a nonnegative integer for any $j \in \{1,2,...,m\}$. Let ψ_1 be a negative subharmonic function on Ω satisfying that $\frac{1}{2}v(dd^c\psi_1, z_j) = p_j > 0$ for any $j \in \{1,2,...,m\}$. Denote $\psi := \pi_1^*(\psi_1)$. Let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1+\psi_1$ is subharmonic on Ω, $\frac{1}{2}v(dd^c(\varphi_1+\psi_1), z_j) = k_j+1$ and $\alpha_j := \big(\varphi_1+\psi_1-2(k_j+1)G_\Omega(\cdot, z_j)\big)(z_j) > -\infty$ for any j. Let φ_2 be a plurisubharmonic function on Y. Let $c(t)$ be a positive measurable function on $(0,+\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$ and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n-1,0)$ form on Y such that $\int_Y |F_j|^2 e^{-\varphi_2} < +\infty$ for any j.*

Let f be a holomorphic $(n,0)$ form on $V_0\times Y$ satisfying that $f = \pi_1^*(a_j w_j^{k_j} dw_j)\wedge \pi_2^*(F_j)$ on $V_{z_j}\times Y$. Then there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ such that $(F-f,(z_j,y)) \in (\mathcal{O}(K_{\tilde{M}})\otimes \mathcal{I}(\varphi+\psi))_{(z_j,y)}$ for any $(z_j,y)\in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \le \left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^m \frac{2\pi |a_j|^2 e^{-\alpha_j}}{p_j c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}. \quad (2.8)$$

Moreover, equality $\left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^m \frac{2\pi|a_j|^2e^{-\alpha_j}}{p_jc_\beta(z_j)^{2(k_j+1)}}\int_Y|F_j|^2e^{-\varphi_2} = \inf\Big\{ \int_M |\tilde{F}|^2 e^{-\varphi}c(-\psi) :$ $\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $\big(\tilde{F}-f,(z_j,y)\big) \in \big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\varphi+\psi)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0\Big\}$ holds if and only if the following statements hold:

(1). $\psi_1 = 2\sum\limits_{j=1}^m p_j G_\Omega(\cdot, z_j)$;

(2). $\varphi_1+\psi_1 = 2\log|g| + 2\sum\limits_{j=1}^m (k_j+1)G_\Omega(\cdot,z_j) + 2u$, where g is a holomorphic function on Ω such that $g(z_j)\neq 0$ for any $j\in\{1,2,\ldots,m\}$ and u is a harmonic function on Ω;

(3). $\prod\limits_{j=1}^m \chi_{z_j}^{k_j+1} = \chi_{-u}$, where χ_{-u} and χ_{z_j} are the characters associated to the functions $-u$ and $G_\Omega(\cdot,z_j)$ respectively;

(4). for any $j\in\{1,2,\ldots,m\}$,

$$\lim_{z\to z_j} \frac{a_j w_j^{k_j} dw_j}{gP_*\left(f_u\left(\prod\limits_{l=1}^m f_{z_l}^{k_l+1}\right)\left(\sum\limits_{l=1}^m p_l \frac{df_{z_l}}{f_{z_l}}\right)\right)} = c_j \in \mathbb{C}\setminus\{0\}, \quad (2.9)$$

and there exist $c_0\in\mathbb{C}\setminus\{0\}$ and a holomorphic $(n-1,0)$ form F_Y on Y which are independent of j such that $c_0F_Y = c_jF_j$ for any $j\in\{1,2,\ldots,m\}$;

(5). $\tilde{M} = M$.

Let $Z_0^1 := \{z_j : 1\le j < +\infty\}$ be an infinite discrete subset of the open Riemann surface Ω. Let Y be an $n-1$ dimensional weakly pseudoconvex Kähler manifold. Let $M = \Omega\times Y$ be a complex manifold, and K_M be the canonical line bundle on M. Let π_1, π_2 be the natural projections from M to Ω and Y, and $Z_0 := \pi_1^{-1}(Z_0^1)$.

Let w_j be a local coordinate on a neighborhood $V_{z_j}\Subset\Omega$ of z_j satisfying $w_j(z_j) = 0$ for $z_j\in Z_0^1$, where $V_{z_j}\cap V_{z_k} = \emptyset$ for any j,k, $j\neq k$. Denote that $V_0 := \bigcup_{j=1}^\infty V_{z_j}$.

We recall the following L^2 extension result from fibers over analytic subsets to fibrations over open Riemann surfaces, where the analytic subsets are infinite points on open Riemann surfaces (see [1]).

Theorem 2.8 ([1])**.** *Let k_j be a nonnegative integer for any $j \in \mathbb{N}_+$. Let ψ_1 be a negative subharmonic function on Ω satisfying that $\frac{1}{2}v(dd^c\psi_1, z_j) = k_j + 1 > 0$ for any $j \in \mathbb{N}_+$. Denote $\psi := \pi_1^*(\psi_1)$. Let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1 + \psi_1$ is subharmonic on Ω, $\frac{1}{2}v(dd^c(\varphi_1 + \psi_1), z_j) = k_j + 1$ and $\alpha_j := \big(\varphi_1 + \psi_1 - 2(k_j + 1)G_\Omega(\cdot, z_j)\big)(z_j) > -\infty$ for any j. Let φ_2 be a plurisubharmonic function on Y. Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$. Let a_j be a constant for any j. Let F_j be a holomorphic $(n-1, 0)$ form on Y such that $\int_Y |F_j|^2 e^{-\varphi_2} < +\infty$ for any j.*

Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y$ satisfying that $f = \pi_1^*(a_j w_j^{k_j} dw_j) \wedge \pi_2^*(F_j)$ on $V_{z_j} \times Y$. If

$$\sum_{j=1}^{\infty} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j + 1)c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2} < +\infty,$$

then there exists a holomorphic $(n, 0)$ form F on M such that $(F - f, (z_j, y)) \in (\mathcal{O}(K_M) \otimes \mathcal{I}(\varphi + \psi))_{(z_j,y)}$ for any $(z_j, y) \in Z_0$ and

$$\int_M |F|^2 e^{-\varphi} c(-\psi) < \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j=1}^{\infty} \frac{2\pi |a_j|^2 e^{-\alpha_j}}{(k_j + 1)c_\beta(z_j)^{2(k_j+1)}} \int_Y |F_j|^2 e^{-\varphi_2}. \tag{2.10}$$

The following results can be referred to [2].

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1 \le j \le n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M = \left(\prod_{1 \le j \le n_1} \Omega_j\right) \times Y$ be an $n-$dimensional complex manifold, where $n = n_1 + n_2$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1 \le j \le n_1} \Omega_j$, Ω_j and Y respectively. Let K_M be the canonical (holomorphic) line bundle on M. Let Z_j be a (closed) analytic subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, and denote that $Z_0 := \left(\prod_{1 \le j \le n_1} Z_j\right) \times Y$.

Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \ne k'$. Denote that $\tilde{I}_1 := \big\{(\beta_1, \ldots, \beta_{n_1}) : 1 \le \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \ldots, n_1\}\big\}$, $V_\beta := \prod_{1 \le j \le n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n_1,\beta_{n_1}}) \in \prod_{1 \le j \le n_1} \Omega_j$ for any $\beta = (\beta_1, \ldots, \beta_{n_1}) \in \tilde{I}_1$. Then $Z_0 = \{(z_\beta, y) : \beta \in \tilde{I}_1 \,\&\, y \in Y\} \subset \tilde{M}$.

Let φ_j be a subharmonic function on Ω_j such that $\varphi_j(z_{j,k}) > -\infty$ for any $1 \le k \le \tilde{m}_j$. Let φ_Y be a plurisubharmonic function on Y.

Let $p_{j,k}$ be a positive number for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$, which satisfies that $\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \le j \le n_1$. Denote that

$$\psi := \max_{1\le j\le n_1} \left\{ 2 \sum_{1\le k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}$$

and $\varphi := \sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j) + \pi_2^*(\varphi_Y)$ on M.

Denote that $E_\beta := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ and $\tilde{E}_\beta := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \ge 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ for any $\beta \in \tilde{I}_1$.

Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in E_\beta$ and $\beta \in \tilde{I}_1$.

Let $Z_0 = \{z_0\} \times Y = \{(z_1, \ldots, z_{n_1})\} \times Y \subset M$. Let

$$\psi = \max_{1\le j\le n_1} \left\{2p_j\pi_{1,j}^*\big(G_{\Omega_j}(\cdot, z_j)\big)\right\},$$

where p_j is positive real number for $1 \le j \le n_1$. Let w_j be a local coordinate on a neighborhood V_{z_j} of $z_j \in \Omega_j$ satisfying $w_j(z_j) = 0$. Denote that $V_0 := \prod_{1\le j\le n_1} V_{z_j}$, and $w := (w_1, \ldots, w_{n_1})$ is a local coordinate on V_0 of $z_0 \in \prod_{1\le j\le n_1} \Omega_j$. Denote that $E := \left\{(\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_j} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$. Let f be a holomorphic $(n, 0)$ form on $V_0 \times Y \subset M$.

We recall a characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity for the case $Z_0 = \{z_0\} \times Y$.

Theorem 2.9 ([2]). *Assume that $G(0) \in (0, +\infty)$. $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(t)e^{-t}dt]$ if and only if the following statements hold:*

(1) $f = \sum_{\alpha\in E} \pi_1^*(w^\alpha dw_1 \wedge \ldots \wedge dw_{n_1}) \wedge \pi_2^*(f_\alpha) + g_0$ on $V_0 \times Y$, where g_0 is a holomorphic $(n, 0)$ form on $V_0\times Y$ satisfying $(g_0, z) \in \big(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi)\big)_z$ for any $z \in Z_0$ and f_α is a holomorphic $(n_2, 0)$ form on Y such that $\sum_{\alpha\in E} \int_Y |f_\alpha|^2 e^{-\varphi_Y} \in (0, +\infty)$;

(2) $\varphi_j = 2 \log |g_j| + 2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j) \ne 0$ and u_j is a harmonic function on Ω_j for any $1 \le j \le n_1$;

(3) $\chi_{j,z_j}^{\alpha_j+1} = \chi_{j,-u_j}$ for any $j \in \{1, 2, ..., n\}$ and $\alpha \in E$ satisfying $f_\alpha \not\equiv 0$.

Let $Z_j = \{z_{j,1}, \ldots, z_{j,m_j}\} \subset \Omega_j$ for any $j \in \{1, \ldots, n_1\}$, where m_j is a positive integer. Let

$$\psi = \max_{1\le j\le n_1} \left\{ \pi_{1,j}^* \left(2 \sum_{1\le k\le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\},$$

where $p_{j,k}$ is a positive real number. Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $j \in \{1, \ldots, n_1\}$ and $k \in \{1, \ldots, m_j\}$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \ne k'$. Denote that $\tilde{I}_1 := \big\{(\beta_1, \ldots, \beta_{n_1}) : 1 \le \beta_j \le m_j$ for any $j \in \{1, \ldots, n_1\}\big\}$, $V_\beta := \prod_{1\le j\le n_1} V_{z_{j,\beta_j}}$

for any $\beta=(\beta_1,\ldots,\beta_{n_1})\in\tilde{I}_1$ and $w_\beta:=(w_{1,\beta_1},\ldots,w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta:=(z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}})\in\prod_{1\le j\le n_1}\Omega_j$ satisfying $w_\beta(z_\beta)=0$.

Let $\beta^*=(1,\ldots,1)\in\tilde{I}_1$, and let $\alpha_{\beta^*}=(\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1})\in\mathbb{Z}^{n_1}_{\ge0}$. Denote that $E':=\left\{\alpha\in\mathbb{Z}^{n_1}_{\ge0}:\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,1}}>\sum_{1\le j\le n_1}\frac{\alpha_{\beta^*,j}+1}{p_{j,1}}\right\}$. Let f be a holomorphic $(n,0)$ form on $\cup_{\beta\in\tilde{I}_1}V_\beta\times Y$ satisfying $f=\pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}}dw_{1,1}\wedge\ldots\wedge dw_{n_1,1}\right)\wedge\pi_2^*\left(f_{\alpha_{\beta^*}}\right)+\sum_{\alpha\in E'}\pi_1^*(w^\alpha dw_{1,1}\wedge\ldots\wedge dw_{n_1,1})\wedge\pi_2^*(f_\alpha)$ on $V_{\beta^*}\times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2,0)$ forms on Y.

We recall a characterization of the concavity of $G\big(h^{-1}(r)\big)$ degenerating to linearity for the case Z_j is a set of finite points.

Theorem 2.10 ([2]). *Assume that $G(0)\in(0,+\infty)$. $G\big(h^{-1}(r)\big)$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$ if and only if the following statements hold:*

(1) $\varphi_j=2\log|g_j|+2u_j$ for any $j\in\{1,\ldots,n_1\}$, where u_j is a harmonic function on Ω_j and g_j is a holomorphic function on Ω_j satisfying $g_j(z_{j,k})\neq0$ for any $k\in\{1,\ldots,m_j\}$;

(2) There exists a nonnegative integer $\gamma_{j,k}$ for any $j\in\{1,\ldots,n_1\}$ and $k\in\{1,\ldots,m_j\}$, which satisfies that $\prod_{1\le k\le m_j}\chi_{j,z_{j,k}}^{\gamma_{j,k}+1}=\chi_{j,-u_j}$ and $\sum_{1\le j\le n_1}\frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}}=1$ for any $\beta\in\tilde{I}_1$;

(3) $f=\pi_1^*\left(c_\beta\left(\prod_{1\le j\le n_1}w_{j,\beta_j}^{\gamma_{j,\beta_j}}\right)dw_{1,\beta_1}\wedge\ldots\wedge dw_{n,\beta_n}\right)\wedge\pi_2^*(f_0)+g_\beta$ on $V_\beta\times Y$ for any $\beta\in\tilde{I}_1$, where c_β is a constant, $f_0\not\equiv0$ is a holomorphic $(n_2,0)$ form on Y satisfying $\int_Y|f_0|^2e^{-\varphi_2}<+\infty$, and g_β is a holomorphic $(n,0)$ form on $V_\beta\times Y$ such that $(g_\beta,z)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\varphi+\psi)\big)_z$ for any $z\in\{z_\beta\}\times Y$;

(4) $c_\beta\prod_{1\le j\le n_1}\left(\lim_{z\to z_{j,\beta_j}}\frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}}dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\le k\le m_j}f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\le k\le m_j}p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)}\right)=c_0$ *for any $\beta\in\tilde{I}_1$, where $c_0\in\mathbb{C}\backslash\{0\}$ is a constant independent of β, f_{u_j} is a holomorphic function Δ such that $|f_{u_j}|=P_j^*(e^{u_j})$ and $f_{z_{j,k}}$ is a holomorphic function on Δ such that $|f_{z_{j,k}}|=P_j^*\left(e^{G_{\Omega_j}(\cdot,z_{j,k})}\right)$ for any $j\in\{1,\ldots,n_1\}$ and $k\in\{1,\ldots,m_j\}$.*

Let $Z_j=\{z_{j,k}:1\le k<\tilde{m}_j\}$ be a discrete subset of Ω_j for any $j\in\{1,\ldots,n_1\}$, where $\tilde{m}_j\in\mathbb{Z}_{\ge2}\cup\{+\infty\}$. Let $p_{j,k}$ be a positive number for any $1\le j\le n_1$ and $1\le k<\tilde{m}_j$ such that $\sum_{1\le k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\not\equiv-\infty$ for any j. Let

$$\psi=\max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}.$$

Assume that $\limsup_{t\to+\infty}c(t)<+\infty$.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}}\Subset\Omega_j$ of $z_{j,k}\in\Omega_j$ satisfying $w_{j,k}(z_{j,k})=0$ for any $j\in\{1,\ldots,n_1\}$ and $1\le k<\tilde{m}_j$, where $V_{z_{j,k}}\cap V_{z_{j,k'}}=\emptyset$ for any j and $k\neq k'$. Denote that $\tilde{I}_1:=\{(\beta_1,\ldots,\beta_{n_1}):1\le\beta_j<\tilde{m}_j$ for any $j\in\{1,\ldots,n_1\}\}$, $V_\beta:=\prod_{1\le j\le n_1}V_{z_{j,\beta_j}}$ for any $\beta=(\beta_1,\ldots,\beta_{n_1})\in\tilde{I}_1$ and $w_\beta:=(w_{1,\beta_1},\ldots,w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta:=(z_{1,\beta_1},\ldots,z_{n_1,\beta_{n_1}})\in\prod_{1\le j\le n_1}\Omega_j$.

Let $\beta^* = (1,\ldots,1) \in I_1$, and let $\alpha_{\beta^*} = (\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1}) \in \mathbb{Z}_{\geq 0}^{n_1}$. Denote that $E' := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,1}} > \sum_{1\leq j\leq n_1} \frac{\alpha_{\beta^*,j}+1}{p_{j,1}}\right\}$. Let f be a holomorphic $(n,0)$ form on $\cup_{\beta\in I_1} V_\beta \times Y$ satisfying $f = \pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}} dw_{1,1}\wedge\ldots\wedge dw_{n_1,1}\right)\wedge \pi_2^*\left(f_{\alpha_{\beta^*}}\right) + \sum_{\alpha\in E'} \pi_1^*(w^\alpha dw_{1,1}\wedge\ldots\wedge dw_{n_1,1})\wedge\pi_2^*(f_\alpha)$ on $V_{\beta^*}\times Y$, where $f_{\alpha_{\beta^*}}$ and f_α are holomorphic $(n_2,0)$ forms on Y.

We recall that $G\big(h^{-1}(r)\big)$ is not linear when there exists $j_0 \in \{1,\ldots,n_1\}$ such that $\tilde{m}_{j_0} = +\infty$ as follows.

Theorem 2.11 ([2])**.** *If $G(0)\in(0,+\infty)$ and there exists $j_0\in\{1,\ldots,n_1\}$ such that $\tilde{m}_{j_0}=+\infty$, then $G\big(h^{-1}(r)\big)$ is not linear with respect to $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$.*

Let $\tilde{M}\subset M$ be an $n-$dimensional complex manifold satisfying that $Z_0\subset\tilde{M}$, and let $K_{\tilde{M}}$ be the canonical (holomorphic) line bundle on $\tilde{M}$.

Let $\Psi\leq 0$ be a plurisubharmonic function on $\prod_{1\leq j\leq n_1}\Omega_j$, and let φ_j be a Lebesgue measurable function on Ω_j such that $\Psi+\sum_{1\leq j\leq n_1}\tilde{\pi}_j^*(\varphi_j)$ is plurisubharmonic on $\prod_{1\leq j\leq n_1}\Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\leq j\leq n_1}\Omega_j$ to Ω_j. Let φ_Y be a plurisubharmonic function on Y. Let $p_{j,k}$ be a positive number for any $1\leq j\leq n_1$ and $1\leq k<\tilde{m}_j$, which satisfies that $\sum_{1\leq k<\tilde{m}_j} p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\not\equiv-\infty$ for any $1\leq j\leq n_1$. Denote that

$$\psi := \max_{1\leq j\leq n_1}\left\{2\sum_{1\leq k<\tilde{m}_j} p_{j,k}\pi_{1,j}^*\big(G_{\Omega_j}(\cdot,z_{j,k})\big)\right\}+\pi_1^*(\Psi)$$

and $\varphi := \sum_{1\leq j\leq n_1}\pi_{1,j}^*(\varphi_j)+\pi_2^*(\varphi_Y)$ on M.

Denote that $E_\beta := \left\{(\alpha_1,\ldots,\alpha_{n_1}) : \sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \,\&\, \alpha_j\in\mathbb{Z}_{\geq0}\right\}$ and $\tilde{E}_\beta := \left\{(\alpha_1,\ldots,\alpha_{n_1}) : \sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}} \geq 1 \,\&\, \alpha_j\in\mathbb{Z}_{\geq0}\right\}$ for any $\beta\in\tilde{I}_1$. Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0\subset\tilde{M}$ of Z_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(f_{\alpha,\beta})$$

on $U_0\cap(V_\beta\times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y for any $\alpha\in E_\beta$ and $\beta\in\tilde{I}_1$. Denote that

$$c_{j,k} := \exp\lim_{z\to z_{j,k}}\left(\frac{\sum_{1\leq k_1<\tilde{m}_j} p_{j,k_1}G_{\Omega_j}(z,z_{j,k_1})}{p_{j,k}} - \log|w_{j,k}(z)|\right)$$

for any $j\in\{1,\ldots,n\}$ and $1\leq k<\tilde{m}_j$ (following from Lemma 2.18 and Lemma 2.19, we get that the above limit exists).

When $Z_0=\{z_0\}\times Y\subset\tilde{M}$, where $z_0=(z_1,\ldots,z_{n_1})\in\prod_{1\leq j\leq n_1}\Omega_j$.

We recall a characterization of the holding of equality in optimal jets L^2 extension problem for the case $Z_0=\{z_0\}\times Y$.

Theorem 2.12 ([2])**.** *Let c be a positive function on $(0,+\infty)$ such that $\int_0^{+\infty}c(t)e^{-t}dt<+\infty$ and $c(t)e^{-t}$ is decreasing on $(0,+\infty)$. Assume that*

$$\sum_{\alpha\in E}\frac{(2\pi)^{n_1}e^{-\left(\Psi+\sum_{1\leq j\leq n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_0)}\int_Y|f_\alpha|^2e^{-\varphi_Y}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}\in(0,+\infty).$$

Then there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)\right)_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}e^{-\left(\Psi+\sum_{1\le j\le n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_0)}\int_Y|f_\alpha|^2e^{-\varphi_Y}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}.\end{aligned}$$

Moreover, equality $\inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}\in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F}-f,z)\in\left(\mathcal{O}\left(K_{\tilde{M}}\right)\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)\right)_z$ *for any* $z\in Z_0\Big\}=\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\times\sum_{\alpha\in E}\frac{(2\pi)^{n_1}e^{-\left(\Psi+\sum_{1\le j\le n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_0)}\int_Y|f_\alpha|^2e^{-\varphi_Y}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}$ *holds if and only if the following statements hold:*

(1) *$\tilde{M}=\left(\prod_{1\le j\le n_1}\Omega_j\right)\times Y$ and $\Psi\equiv0$;*

(2) *$\varphi_j=2\log|g_j|+2u_j$, where g_j is a holomorphic function on Ω_j such that $g_j(z_j)\neq0$ and u_j is a harmonic function on Ω_j for any $1\le j\le n_1$;*

(3) *$\chi_{j,z_j}^{\alpha_j+1}=\chi_{j,-u_j}$ for any $j\in\{1,2,...,n\}$ and $\alpha\in E$ satisfying $f_\alpha\not\equiv0$.*

Remark 2.13 ([2])**.** *If $(f_\alpha,y)\in(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y))_y$ for any $y\in Y$ and $\alpha\in\tilde{E}\backslash E$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)$ by $\mathcal{I}(\varphi+\psi)$.*

Let $Z_j=\{z_{j,1},\ldots,z_{j,m_j}\}\subset\Omega_j$ for any $j\in\{1,\ldots,n_1\}$, where m_j is a positive integer. Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0\subset\tilde{M}$ of Z_0 such that

$$f=\sum_{\alpha\in\tilde{E}_\beta}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(f_{\alpha,\beta})$$

on $U_0\cap(V_\beta\times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y for any $\alpha\in E_\beta$ and $\beta\in\tilde{I}_1$. Let $\beta^*=(1,\ldots,1)\in\tilde{I}_1$, and let $\alpha_{\beta^*}=(\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1})\in E_{\beta^*}$. Assume that $f=\pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}}dw_{1,1}\wedge\ldots\wedge dw_{n_1,1}\right)\wedge\pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right)+\sum_{\alpha\in E'}\pi_1^*(w^\alpha dw_{1,1}\wedge\ldots\wedge dw_{n_1,1})\wedge\pi_2^*(f_{\alpha,\beta})$ on $U_0\cap(V_{\beta^*}\times Y)$.

We recall a characterization of the holding of equality in optimal jets L^2 extension problem for the case that Z_j is finite.

Theorem 2.14 ([2])**.** *Let c be a positive function on $(0,+\infty)$ such that $\int_0^{+\infty}c(t)e^{-t}dt<+\infty$ and $c(t)e^{-t}$ is decreasing on $(0,+\infty)$. Assume that*

$$\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\left(\Psi+\sum_{1\le j\le n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\in(0,+\infty).$$

Then there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ for any $z\in Z_0$

and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$$
$$\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\left(\Psi+\sum_{1\leq j\leq n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Moreover, equality $\inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}\in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F}-f,z)\in\Big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\Big(\max_{1\leq j\leq n_1}\Big\{2\sum_{1\leq k\leq m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\Big\}\Big)\Big)_z$ *for any* $z\in Z_0\Big\}=\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\left(\Psi+\sum_{1\leq j\leq n_1}\tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ *holds if and only if the following statements hold:*

(1) $\tilde{M}=\left(\prod_{1\leq j\leq n_1}\Omega_j\right)\times Y$ *and* $\Psi\equiv0$;

(2) $\varphi_j=2\log|g_j|+2u_j$ *for any* $j\in\{1,\ldots,n_1\}$, *where* u_j *is a harmonic function on* Ω_j *and* g_j *is a holomorphic function on* Ω_j *satisfying* $g_j(z_{j,k})\neq0$ *for any* $k\in\{1,\ldots,m_j\}$;

(3) *There exists a nonnegative integer* $\gamma_{j,k}$ *for any* $j\in\{1,\ldots,n_1\}$ *and* $k\in\{1,\ldots,m_j\}$, *which satisfies that* $\prod_{1\leq k\leq m_j}\chi_{j,z_{j,k}}^{\gamma_{j,k}+1}=\chi_{j,-u_j}$ *and* $\sum_{1\leq j\leq n_1}\frac{\gamma_{j,\beta_j}+1}{p_{j,\beta_j}}=1$ *for any* $\beta\in\tilde{I}_1$;

(4) $f_{\alpha,\beta}=c_\beta f_0$ *holds for* $\alpha=(\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})$ *and* $f_{\alpha,\beta}\equiv0$ *holds for any* $\alpha\in E_\beta\backslash\{(\gamma_{1,\beta_1},\ldots,\gamma_{n_1,\beta_{n_1}})\}$, *where* $\beta\in\tilde{I}_1$, c_β *is a constant and* $f_0\not\equiv0$ *is a holomorphic* $(n_2,0)$ *form on* Y *satisfying* $\int_Y|f_0|^2e^{-\varphi_2}<+\infty$;

(5) $c_\beta\prod_{1\leq j\leq n_1}\left(\lim_{z\to z_{j,\beta_j}}\frac{w_{j,\beta_j}^{\gamma_{j,\beta_j}}dw_{j,\beta_j}}{g_j(P_j)_*\left(f_{u_j}\left(\prod_{1\leq k\leq m_j}f_{z_{j,k}}^{\gamma_{j,k}+1}\right)\left(\sum_{1\leq k\leq m_j}p_{j,k}\frac{df_{z_{j,k}}}{f_{z_{j,k}}}\right)\right)}\right)=c_0$ *for any* $\beta\in\tilde{I}_1$, *where* $c_0\in\mathbb{C}\backslash\{0\}$ *is a constant independent of* β, f_{u_j} *is a holomorphic function* Δ *such that* $|f_{u_j}|=P_j^*(e^{u_j})$ *and* $f_{z_{j,k}}$ *is a holomorphic function on* Δ *such that* $|f_{z_{j,k}}|=P_j^*\left(e^{G_{\Omega_j}(\cdot,z_{j,k})}\right)$ *for any* $j\in\{1,\ldots,n_1\}$ *and* $k\in\{1,\ldots,m_j\}$.

Remark 2.15 ([2]). *If* $(f_{\alpha,\beta},y)\in(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y))_y$ *holds for any* $y\in Y$, $\alpha\in\tilde{E}_\beta\backslash E_\beta$ *and* $\beta\in\tilde{I}_1$, *the above result also holds when we replace the ideal sheaf* $\mathcal{I}\left(\max_{1\leq j\leq n_1}\left\{2\sum_{1\leq k\leq m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)$ *by* $\mathcal{I}(\varphi+\psi)$.

Let $Z_j=\{z_{j,k}:1\leq k<\tilde{m}_j\}$ be a discrete subset of Ω_j for any $j\in\{1,\ldots,n_1\}$, where $\tilde{m}_j\in\mathbb{Z}_{\geq2}\cup\{+\infty\}$. Let f be a holomorphic $(n,0)$ form on a neighborhood $U_0\subset\tilde{M}$ of Z_0 such that

$$f=\sum_{\alpha\in\tilde{E}_\beta}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(f_{\alpha,\beta})$$

on $U_0\cap(V_\beta\times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2,0)$ form on Y for any $\alpha\in E_\beta$ and $\beta\in\tilde{I}_1$. Let $\beta^*=(1,\ldots,1)\in\tilde{I}_1$, and let $\alpha_{\beta^*}=(\alpha_{\beta^*,1},\ldots,\alpha_{\beta^*,n_1})\in E_{\beta^*}$. Assume that $f=\pi_1^*\left(w_{\beta^*}^{\alpha_{\beta^*}}dw_{1,1}\wedge\ldots\wedge dw_{n_1,1}\right)\wedge\pi_2^*\left(f_{\alpha_{\beta^*},\beta^*}\right)+\sum_{\alpha\in E'}\pi_1^*(w^\alpha dw_{1,1}\wedge\ldots\wedge dw_{n_1,1})\wedge\pi_2^*(f_{\alpha,\beta})$ on $U_0\cap(V_{\beta^*}\times Y)$.

We recall that the equality in optimal jets L^2 extension problem could not hold when there exists $j_0 \in \{1, \ldots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$.

Theorem 2.16 ([2]). *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Assume that*

$$\sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \le j \le n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \le j \le n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} \in (0, +\infty)$$

and there exists $j_0 \in \{1, \ldots, n_1\}$ such that $\tilde{m}_{j_0} = +\infty$.

Then there exists a holomorphic $(n, 0)$ form F on $\tilde{M}$ satisfying that $(F - f, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1 \le j \le n_1} \left\{2 \sum_{1 \le k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)\right)_z$ for any $z \in Z_0$ and

$$\begin{aligned} &\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi) \\ &< \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\left(\Psi + \sum_{1 \le j \le n_1} \tilde{\pi}_j^*(\varphi_j)\right)(z_\beta)} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y}}{\prod_{1 \le j \le n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}}. \end{aligned}$$

Remark 2.17 ([2]). *If $(f_{\alpha,\beta}, y) \in (\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y))_y$ holds for any $y \in Y$, $\alpha \in \tilde{E}_\beta \backslash E_\beta$ and $\beta \in \tilde{I}_1$, the above result also holds when we replace the ideal sheaf $\mathcal{I}\left(\max_{1 \le j \le n_1} \left\{2 \sum_{1 \le k < \tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}\right)$ by $\mathcal{I}(\varphi + \psi)$.*

2.3. **Basic properties of the Green functions.** In this section, we recall some basic properties of the Green functions. Let Ω be an open Riemann surface, which admits a nontrivial Green function G_Ω, and let $z_0 \in \Omega$.

Lemma 2.18 (see [44], see also [50]). *Let w be a local coordinate on a neighborhood of z_0 satisfying $w(z_0) = 0$. $G_\Omega(z, z_0) = \sup_{v \in \Delta_\Omega^*(z_0)} v(z)$, where $\Delta_\Omega^*(z_0)$ is the set of negative subharmonic function on Ω such that $v - \log |w|$ has a locally finite upper bound near z_0. Moreover, $G_\Omega(\cdot, z_0)$ is harmonic on $\Omega \backslash \{z_0\}$ and $G_\Omega(\cdot, z_0) - \log |w|$ is harmonic near z_0.*

Lemma 2.19 (see [32]). *Let $K = \{z_j : j \in \mathbb{Z}_{\ge 1} \,\&\, j < \gamma\}$ be a discrete subset of Ω, where $\gamma \in \mathbb{Z}_{>1} \cup \{+\infty\}$. Let ψ be a negative subharmonic function on Ω such that $\frac{1}{2} v(dd^c\psi, z_j) \ge p_j$ for any j, where $p_j > 0$ is a constant. Then $2 \sum_{1 \le j < \gamma} p_j G_\Omega(\cdot, z_j)$ is a subharmonic function on Ω satisfying that $2 \sum_{1 \le j < \gamma} p_j G_\Omega(\cdot, z_j) \ge \psi$ and $2 \sum_{1 \le j < \gamma} p_j G_\Omega(\cdot, z_j)$ is harmonic on $\Omega \backslash K$.*

Lemma 2.20 (see [28]). *For any open neighborhood U of z_0, there exists $t > 0$ such that $\{G_\Omega(z, z_0) < -t\}$ is a relatively compact subset of U.*

Lemma 2.21 (see [32]). *There exists a sequence of open Riemann surfaces $\{\Omega_l\}_{l \in \mathbb{Z}^+}$ such that $z_0 \in \Omega_l \Subset \Omega_{l+1} \Subset \Omega$, $\cup_{l \in \mathbb{Z}^+} \Omega_l = \Omega$, Ω_l has a smooth boundary $\partial \Omega_l$ in Ω and $e^{G_{\Omega_l}(\cdot, z_0)}$ can be smoothly extended to a neighborhood of $\overline{\Omega_l}$ for any $l \in \mathbb{Z}^+$, where G_{Ω_l} is the Green function of Ω_l. Moreover, $\{G_{\Omega_l}(\cdot, z_0) - G_\Omega(\cdot, z_0)\}$ is decreasingly convergent to 0 on Ω with respect to l.*

Let $M = (\prod_{1 \le j \le n_1} \Omega_j) \times Y$, where Ω_j is an open Riemann surface and Y is an n_2-dimensional complex manifold and $n_1 + n_2 = n$. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1 \le j \le n_1} \Omega_j$, Ω_j and Y respectively. Let

$Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $1 \le j \le n_1$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Denote that $Z_0 := \left(\prod_{1\le j\le n_1} Z_j\right) \times Y$.

Let $G = \max_{1\le j\le n_1} \left\{\tilde{\pi}_j^* \left(2\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$ be a plurisubharmonic function on $\prod_{1\le j\le n_1} \Omega_j$, where $\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $j \in \{1, .., n\}$ and $\tilde{\pi}_j$ is the natural projection from $\prod_{1\le j\le n_1} \Omega_j$ to Ω_j. Let $\tilde{M} \subset M$ be an $n-$dimensional weakly pseudoconvex submanifold satisfying that $Z_0 \subset \tilde{M}$.

Let $N \le 0$ be a plurisubharmonic function on $\tilde{M}$. Denote $\psi = \pi_1^*(G) + N$.

Lemma 2.22. *Let $l(t)$ be a positive Lebesgue measurable function on $(0, +\infty)$ satisfying that l is decreasing on $(0, +\infty)$ and $\int_0^{+\infty} l(t)dt < +\infty$.*

If $N \not\equiv 0$, then there exists a Lebesgue measurable subset $\tilde{V}$ of $\tilde{M}$ such that $l(-\psi) < l\big(-\pi_1^*(G)\big)$ on $\tilde{V}$ and $\mu(\tilde{V}) > 0$, where μ is the Lebesgue measure on $\tilde{M}$.

Proof. Let $V_1 \Subset \prod_{1\le j\le n_1} \Omega_j \backslash \{z_\beta : \exists j \in \{1, 2, \dots, n_1\} \ s.t. 2 \le \beta_j \le \tilde{m}_j\}$ be an open neighborhood of z_{β^*}, where $\beta^* = (1, 1, \dots, 1)$. It follows from Lemma 2.20 that there exists $t_0 > 0$ such that $V_1 \cap \{G < -t_0\} \Subset V_1$.

As $l(t)$ is decreasing and $\int_0^{+\infty} l(t)dt < +\infty$, there exists $t_1 > t_0$ such that $l(t) < l(t_1)$ holds for any $t > t_1$.

For any $w \in Y$, let $U \Subset Y$ be an open neighborhood of w in Y such that $V_1 \times U \subset \tilde{M}$.

As ψ is upper semi-continuous function and $N \not\equiv 0$ is a plurisubharmonic function on $\tilde{M}$, we have

$$\sup_{(z,w)\in(V_1\cap\{G\le -t_1\})\times U} \psi(z) < -t_1,$$

which implies that there exists $t_2 \in (t_0, t_1)$ such that

$$\sup_{(z,w)\in(V_1\cap\{G\le -t_2\})\times \tilde{U}} \psi(z) < -t_1,$$

where $\tilde{U} \Subset U$ is open.

Denote $t_3 = -\sup_{(z,w)\in(V_1\cap\{G\le -t_2\})\times \tilde{U}} \psi(z)$. Then we know $-t_3 < -t_1$.

Let $V = \{-t_1 < G < -t_2\} \cap V_1$, then $\mu(V \times \tilde{U}) > 0$. As $l(t)$ is decreasing with respect to t, for any $(z, w) \in V \times \tilde{U}$, we have

$$l(-\psi) \le l(t_3) < l(t_1) \le l\big(\pi_1^*(-G)\big).$$

Lemma 2.22 is proved. □

2.4. **Other lemmas.** We call a positive measurable function c on $(S, +\infty)$ in class $\tilde{\mathcal{P}}_S$ if $\int_S^s c(l)e^{-l}dl < +\infty$ for some $s > S$ and $c(t)e^{-t}$ is decreasing with respect to t.

Lemma 2.23 (see [26])**.** *Let $B \in (0, +\infty)$ and $t_0 \ge S$ be arbitrarily given. Let (M, ω) be an $n-$dimensional weakly pseudoconvex Kähler manifold. Let $\psi < -S$ be a plurisubharmonic function on M. Let φ be a plurisubharmonic function on M. Let F be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\}$ such that*

$$\int_{K\cap\{\psi<-t_0\}} |F|^2 < +\infty,$$

for any compact subset K of M, and

$$\int_M \frac{1}{B}\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}|F|^2e^{-\varphi}\le C<+\infty.$$

Then there exists a holomorphic $(n,0)$ form $\widetilde{F}$ on X, such that

$$\int_M |\widetilde{F}-(1-b_{t_0,B}(\psi))F|^2e^{-\varphi+v_{t_0,B}(\psi)}c(-v_{t_0,B}(\psi))\le C\int_S^{t_0+B}c(t)e^{-t}dt. \quad (2.11)$$

where $b_{t_0,B}(t)=\int_{-\infty}^t\frac{1}{B}\mathbb{I}_{\{-t_0-B<s<-t_0\}}ds$, $v_{t_0,B}(t)=\int_{-t_0}^t b_{t_0,B}(s)ds-t_0$ and $c(t)\in\tilde{\mathcal{P}}_S$.

Lemma 2.24 (see [26])**.** *Let M be a complex manifold. Let S be an analytic subset of M. Let $\{g_j\}_{j=1,2,\ldots}$ be a sequence of nonnegative Lebesgue measurable functions on M, which satisfies that g_j are almost everywhere convergent to g on M when $j\to+\infty$, where g is a nonnegative Lebesgue measurable function on M. Assume that for any compact subset K of $M\backslash S$, there exist $s_K\in(0,+\infty)$ and $C_K\in(0,+\infty)$ such that*

$$\int_K {g_j}^{-s_K}dV_M\le C_K$$

for any j, where dV_M is a continuous volume form on M.

Let $\{F_j\}_{j=1,2,\ldots}$ be a sequence of holomorphic $(n,0)$ form on M. Assume that $\liminf_{j\to+\infty}\int_M|F_j|^2g_j\le C$, where C is a positive constant. Then there exists a subsequence $\{F_{j_l}\}_{l=1,2,\ldots}$, which satisfies that $\{F_{j_l}\}$ is uniformly convergent to a holomorphic $(n,0)$ form F on M on any compact subset of M when $l\to+\infty$, such that

$$\int_M|F|^2g\le C.$$

Lemma 2.25 (see [19])**.** *Let N be a submodule of $\mathcal{O}^q_{\mathbb{C}^n,o}$, $1\le q<\infty$, let $f_j\in\mathcal{O}_{\mathbb{C}^n}(U)^q$ be a sequence of q-tuples holomorphic function in an open neighborhood U of the origin o. Assume that the f_j converges uniformly in U towards a $q-$tuples $f\in\mathcal{O}_{\mathbb{C}^n}(U)^q$, assume furthermore that all germs (f_j,o) belong to N. Then $(f,o)\in N$.*

Lemma 2.26 (see [33])**.** *Let $\psi=\max_{1\le j\le n}\{2p_j\log|w_j|\}$ be a plurisubharmonic function on $\mathbb{C}^n$, where $p_j>0$. Let $f=\sum_{\alpha\in\mathbb{Z}^n_{\ge0}}b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $\{\psi<-t_0\}$, where $t_0>0$. Denote that $q_\alpha:=\sum_{1\le j\le n}\frac{\alpha_j+1}{p_j}-1$ for any $\alpha\in\mathbb{Z}^n_{\ge0}$ and $E_1:=\{\alpha\in\mathbb{Z}^n_{\ge0}:q_\alpha=0\}$. Let k_1 be a constant satisfying $k_1+1>0$. Then*

$$\begin{aligned}\int_{\{-t-1-k_1<\psi<-t\}}|f|^2e^{-\psi}d\lambda_n=&\sum_{\alpha\in E_1}\frac{|b_\alpha|^2\pi^n(1+k_1)}{\prod_{1\le j\le n}(\alpha_j+1)}\\&+\sum_{\alpha\notin E_1}\frac{|b_\alpha|^2\pi^n(q_\alpha+1)(e^{-q_\alpha t}-e^{-q_\alpha(t+1+k_1)})}{q_\alpha\prod_{1\le j\le n}(\alpha_j+1)}\end{aligned}$$

for any $t>t_0$.

Remark 2.27. *Lemma 2.26 is stated in* [33] *in the case $k_1=0$, the same proof as in* [33] *shows that Lemma 2.26 holds for k_1 which satisfying $k_1+1>0$.*

Lemma 2.28. *Let $\psi = \max_{1\le j\le n}\{2p_j\log|w_j|\}$ be a plurisubharmonic function on $\mathbb{C}^n$, where $p_j>0$. Let $f=\sum_{\alpha\in\mathbb{Z}^n_{\ge0}} b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $\{\psi<-t_0\}$, where $t_0>0$. Let $c(t)$ be a nonnegative measurable function on $(t_0,+\infty)$. Denote that $q_\alpha := \sum_{1\le j\le n}\frac{\alpha_j+1}{p_j}-1$ for any $\alpha\in\mathbb{Z}^n_{\ge0}$. Let A be a real constant such that $t_0+A>0$. Then*

$$\int_{\{\psi+A<-t\}}|f|^2c(-\psi-A)d\lambda_n = \sum_{\alpha\in\mathbb{Z}^n_{\ge0}}\left(\int_t^{+\infty}c(s)e^{-(q_\alpha+1)s}ds\right)e^{-(q_\alpha+1)A}\frac{(q_\alpha+1)|b_\alpha|^2\pi^n}{\prod_{1\le j\le n}(\alpha_j+1)}$$

holds for any $t\ge t_0$.

Proof. For any $t\ge t_0$, by direct calculations, we obtain that (note that $t+A>0$)

$$\begin{aligned}
&\int_{\{\psi+A<-t\}}|w^\alpha|^2c(-\psi-A)d\lambda_n\\
=&(2\pi)^n\int_{\left\{\max_{1\le j\le n}\left\{s_j^{p_j}\right\}<e^{-\frac{t+A}{2}}\ \&\ s_j>0\right\}}\prod_{1\le j\le n}s_j^{2\alpha_j+1}\cdot c\left(-\log\max_{1\le j\le n}\left\{s_j^{2p_j}\right\}-A\right)ds_1ds_2\dots ds_n\\
=&(2\pi)^n\frac{1}{\prod_{1\le j\le n}p_j}\\
&\times\int_{\left\{\max_{1\le j\le n}\{r_j\}<e^{-\frac{t+A}{2}}\ \&\ r_j>0\right\}}\prod_{1\le j\le n}r_j^{\frac{2\alpha_j+2}{p_j}-1}\cdot c\left(-\log\max_{1\le j\le n}\left\{r_j^2\right\}-A\right)dr_1dr_2\dots dr_n.
\end{aligned}\tag{2.12}$$

By the Fubini's theorem, we have

$$\begin{aligned}
&\int_{\left\{\max_{1\le j\le n}\{r_j\}<e^{-\frac{t+A}{2}}\ \&\ r_j>0\right\}}\prod_{1\le j\le n}r_j^{\frac{2\alpha_j+2}{p_j}-1}\cdot c\left(-\log\max_{1\le j\le n}\left\{r_j^2\right\}-A\right)dr_1dr_2\dots dr_n\\
=&\sum_{j'=1}^n\int_0^{e^{-\frac{t+A}{2}}}\left(\int_{\{0\le r_j<r_{j'},j\ne j'\}}\prod_{j\ne j'}r_j^{\frac{2\alpha_j+2}{p_j}-1}\cdot\wedge_{j\ne j'}dr_j\right)r_{j'}^{\frac{2\alpha_{j'}+2}{p_{j'}}-1}c(-2\log r_{j'}-A)dr_{j'}\\
=&\sum_{j'=1}^n\left(\prod_{j\ne j'}\frac{p_j}{2\alpha_j+2}\right)\int_0^{e^{-\frac{t+A}{2}}}r_{j'}^{\sum_{1\le k\le n}\frac{2\alpha_k+2}{p_k}-1}c(-2\log r_{j'}-A)dr_{j'}\\
=&(q_\alpha+1)e^{-(q_\alpha+1)A}\left(\int_t^{+\infty}c(s)e^{-(q_\alpha+1)s}ds\right)\prod_{1\le j\le n}\frac{p_j}{2\alpha_j+2}.
\end{aligned}\tag{2.13}$$

Following from $\int_{\{\psi<-t\}}|f|^2c(-\psi)d\lambda_n=\sum_{\alpha\in\mathbb{Z}^n_{\ge0}}|b_\alpha|^2\int_{\{\psi<-t\}}|w^\alpha|^2c(-\psi)d\lambda_n$, equality (2.12) and equality (2.13), we obtain that

$$\int_{\{\psi<-t\}}|f|^2d\lambda_n=\sum_{\alpha\in\mathbb{Z}^n_{\ge0}}\left(\int_t^{+\infty}c(s)e^{-(q_\alpha+1)s}ds\right)e^{-(q_\alpha+1)A}\frac{(q_\alpha+1)|b_\alpha|^2\pi^n}{\prod_{1\le j\le n}(\alpha_j+1)}.$$

□

Lemma 2.29. *Let $\Delta^n\subset\mathbb{C}^n$ be a polydisc. Let φ be a bounded subharmonic function on Δ^n. Assume that v is a nonnegative continuous real function on Δ^n.*

Denote

$$I_t:=\int_{\{z\in\Delta^n:-t-1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t+k_1\}}v(z)\frac{\prod_{j=1}^n|z_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}}e^{-\varphi(z_1,\ldots,z_n)}dz\wedge d\bar{z},$$

$$J_t:=\int_{\{z\in\Delta^n:-t-1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t+k_1\}}v(z)\frac{\prod_{j=1}^n|z_j|^{2\beta_j}}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}}e^{-\varphi(z_1,\ldots,z_n)}dz\wedge d\bar{z},$$

where $p_j,\alpha_j,\beta_j>0$ are constants for any j satisfying $\sum_{1\le j\le n}\frac{\alpha_j+1}{p_j}=1$ and $\sum_{1\le j\le n}\frac{\beta_j+1}{p_j}>1$, k_1,k_2 are constants satisfying $k_1+k_2+1>0$.

Then

$$\limsup_{t\to+\infty}I_t\le\frac{(2\pi)^n(1+k_1+k_2)}{\prod_{1\le j\le n}(\alpha_j+1)}v(0)e^{-\varphi(0)}. \tag{2.14}$$

and

$$\lim_{t\to+\infty}J_t=0.$$

Proof. The idea of the proof can be referred to Lemma 3.3 in [53]. For the convenience of the readers, we give a proof below.

Denote

$$I_{t+k_1}:=\int_{\{z\in\Delta^n:-t-1-k_1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t\}}v(z)\frac{\prod_{j=1}^n|z_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}}e^{-\varphi(z_1,\ldots,z_n)}dz\wedge d\bar{z},$$

Note that

$$\limsup_{t\to+\infty}I_t=\limsup_{t\to+\infty}I_{t+k_1},$$

then we only need to prove (2.14) for I_{t+k_1}.

Denote

$$B_{\delta,t}:=\{z\in\Delta^n:\varphi(e^{\frac{-t}{2p_1}}z_1,e^{\frac{-t}{2p_2}}z_2,\ldots,e^{\frac{-t}{2p_n}}z_n)<(1+\delta)\varphi(0)\},$$

where $\delta\in(0,+\infty)$ and $t\in(0,+\infty)$.

Let $\lambda(B_{\delta,t})$ be the $2n-$dimensional Lebesgue measure of $B_{\delta,t}$.

Since the computation is local, we may assume that φ is a negative upper semicontinuous function on Δ^n. Note that $\varphi(0)>-\infty$. For any $\epsilon\in(0,1)$, there exists $t_\epsilon>0$ such that

$$\varphi(e^{\frac{-t}{2p_1}}z_1,e^{\frac{-t}{2p_2}}z_2,\ldots,e^{\frac{-t}{2p_n}}z_n)\le(1-\epsilon)\varphi(0)$$

for any $z\in\Delta^n$, when $t\ge t_\epsilon$. We denote $\varphi(e^{\frac{-t}{2p_1}}z_1,e^{\frac{-t}{2p_2}}z_2,\ldots,e^{\frac{-t}{2p_n}}z_n)$ by $\varphi(e^{\frac{-t}{2p}}z)$ when there is no misunderstanding.

Note that for fixed t, $\varphi(e^{\frac{-t}{2p}}z)$ is subharmonic on Δ^n with respect to z. It follows from mean value inequality that, for all $t \geq t_\epsilon$, we have

$$\begin{aligned}
\varphi(0) &\leq \frac{1}{\pi}\int_{z_1\in\Delta}\varphi(e^{\frac{-t}{2p_1}}z_1,0\dots,0)d\lambda_{z_1}\\
&\leq \frac{1}{\pi^n}\int_{z\in\Delta^n}\varphi(e^{\frac{-t}{2p_1}}z_1,e^{\frac{-t}{2p_2}}z_2,\dots,e^{\frac{-t}{2p_n}}z_n)d\lambda_z\\
&= \frac{1}{\pi^n}\int_{\Delta^n\setminus B_{\delta,t}}\varphi(e^{\frac{-t}{2p}}z)d\lambda_z + \frac{1}{\pi}\int_{B_{\delta,t}}\varphi(e^{\frac{-t}{2p}}z)d\lambda_z\\
&\leq \frac{(1-\epsilon)\varphi(0)}{\pi^n}(\pi^n-\lambda(B_{\delta,t})) + \frac{(1+\delta)\varphi(0)}{\pi^n}\lambda(B_{\delta,t})\\
&= \varphi(0)(1-\epsilon+\frac{\delta+\epsilon}{\pi^n}\lambda(B_{\delta,t})).
\end{aligned}$$

As $\varphi(0)<0$, we have

$$\lambda(B_{\delta,t}) \leq \frac{\pi^n\epsilon}{\delta+\epsilon} \leq \frac{\pi^n\epsilon}{\delta}$$

for any $t \geq t_\epsilon$. Hence

$$\lim_{t\to+\infty}\lambda(B_{\delta,t})=0.$$

Since φ is bounded, we have $e^{-\varphi}\leq C_1$ for some $C_1>0$. As $v(t)$ is continuous, when t is large enough, we have

$$\sup_{\{z\in\Delta^n:-t-1-k_1-k_2<\max\limits_{1\leq j\leq n}\{2p_j\log|z_j|\}<-t\}} v(z)\leq C_2,$$

where $C_2>0$ is a constant independent of t. We denote $v(e^{\frac{-t}{2p_1}}z_1,e^{\frac{-t}{2p_2}}z_2,\dots,e^{\frac{-t}{2p_n}}z_n)$ by $v(e^{\frac{-t}{2p}}z)$ when there is no misunderstandings.

Let $z_i = e^{\frac{-t}{2p_i}}w_i$. We denote $v(e^{\frac{-t}{2p_1}}w_1,e^{\frac{-t}{2p_2}}w_2,\dots,e^{\frac{-t}{2p_n}}w_n)$ by $v(e^{\frac{-t}{2p}}w)$ and denote $\varphi(e^{\frac{-t}{2p_1}}w_1,e^{\frac{-t}{2p_2}}w_2,\dots,e^{\frac{-t}{2p_n}}w_n)$ by $\varphi(e^{\frac{-t}{2p}}w)$ for simplicity. By Lemma 2.26, we have

$$\begin{aligned}
I_{t+k_1} &= \int_{\{z\in\Delta:-t-1-k_1-k_2<\max\limits_{1\leq j\leq n}\{2p_j\log|z_j|\}<-t\}} v(z)\frac{\prod_{j=1}^n|z_j|^{2\alpha_j}}{\max\limits_{1\leq j\leq n}\{|z_j|^{2p_j}\}}e^{-\varphi(z)}dz\wedge d\bar z\\
&= \int_{\{w\in\Delta^n:e^{-1-k_1-k_2}<\max\limits_{1\leq j\leq n}\{|w_j|^{2p_j}\}<1\}} v(e^{\frac{-t}{2p}}w)\frac{\prod_{j=1}^n|w_j|^{2\alpha_j}}{\max\limits_{1\leq j\leq n}\{|w_j|^{2p_j}\}}e^{-\varphi(e^{\frac{-t}{2p}}w)}dw\wedge d\bar w\\
&= \int_{\{w\in\Delta^n:e^{-1-k_1-k_2}<\max\limits_{1\leq j\leq n}\{|w_j|^{2p_j}\}<1\}\cap B_{\delta,t}} v(e^{\frac{-t}{2p}}w)\frac{\prod_{j=1}^n|w_j|^{2\alpha_j}}{\max\limits_{1\leq j\leq n}\{|w_j|^{2p_j}\}}e^{-\varphi(e^{\frac{-t}{2p}}w)}dw\wedge d\bar w+\\
&\quad\int_{\{w\in\Delta^n:e^{-1-k_1-k_2}<\max\limits_{1\leq j\leq n}\{|w_j|^{2p_j}\}<1\}\setminus B_{\delta,t}} v(e^{\frac{-t}{2p}}w)\frac{\prod_{j=1}^n|w_j|^{2\alpha_j}}{\max\limits_{1\leq j\leq n}\{|w_j|^{2p_j}\}}e^{-\varphi(e^{\frac{-t}{2p}}w)}dw\wedge d\bar w\\
&\leq CC_1C_2\lambda(B_{\delta,t}) + \left(\sup_{\{e^{-1-k_1-k_2}<\max\limits_{1\leq j\leq n}\{|w_j|^{2p_j}\}<1\}} v(e^{\frac{-t}{2p}}w)\right)e^{-(1+\delta)\varphi(0)}\frac{(2\pi)^n(1+k_1+k_2)}{\prod_{1\leq j\leq n}(\alpha_j+1)}.
\end{aligned}\tag{2.15}$$

Hence we know

$$\limsup_{t\to+\infty} I_{t+k_1} = \frac{(2\pi)^n(1+k_1+k_2)}{\prod_{1\le j\le n}(\alpha_j+1)} e^{-(1+\delta)\varphi(0)}v(0).$$

By the arbitrariness of δ, we know (2.14) holds for I_{t+k_1}.

For J_t, we know when t is large enough, the function $v(z)$ and $e^{-\varphi(z)}$ are uniformly bounded by some constant $M>0$ with respect to t. Then it follows from Lemma 2.26 that

$$\begin{aligned}\limsup_{t\to+\infty} J_{t+k_1} \le & M \limsup_{t\to+\infty}\int_{\{z\in\Delta^n:-t-1-k_1-k_2<\max_{1\le j\le n}\{2p_j\log|z_j|\}<-t\}} \frac{\prod_{j=1}^n |z_j|^{2\beta_j}}{\max_{1\le j\le n}\{|z_j|^{2p_j}\}} dz\wedge d\bar{z} \\ = & M\lim_{t\to+\infty}\frac{\pi^n(q_\alpha+1)(e^{-q_\alpha t}-e^{-q_\alpha(t+1+k_1+k_2)})}{q_\alpha\prod_{1\le j\le n}(\alpha_j+1)},\end{aligned}$$

where $q_\alpha=\frac{\beta_j+1}{p_j}-1$. As $k_1+k_2+1>0$, we have $\lim_{t\to+\infty} J_t=0$.

Lemma 2.29 is proved. □

Lemma 2.30. *Let $\Delta^n\subset\mathbb{C}^n$ be a polydisc. Let $f=\sum_{\alpha\in E} b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on Δ^n, where $E:=\left\{\alpha=(\alpha_1,\ldots,\alpha_n):\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_j}=1\right\}$. Assume that v is a nonnegative continuous real function on Δ^n. Denote*

$$S_t := \int_{\{z\in\Delta^n:-t-1-k_2<\max_{1\le j\le n}\{2p_j\log|z_j|\}<-t+k_1\}} v(z)\frac{|f|^2}{\max_{1\le j\le n}\{|z_j|^{2p_j}\}}e^{-\varphi(z_1,\ldots,z_n)}dz\wedge d\bar{z}.$$

Then we have

$$\limsup_{t\to+\infty} S_t \le \sum_{\alpha\in E}\frac{|b_\alpha|^2(2\pi)^n(1+k_1+k_2)}{\prod_{1\le j\le n}(\alpha_j+1)}v(0)e^{-\varphi(0)}.$$

Proof. In the proof of Lemma 2.30, we follow the notations we used in the proof of Lemma 2.29. Let $z_i = e^{\frac{-t}{2p_i}} w_i$, by Lemma 2.26, we have

$$\begin{aligned}
S_t =& \int_{\{z\in\Delta^n:-t-1-k_2<\max\limits_{1\le j\le n}\{2p_j\log|z_j|\}<-t+k_1\}} v(z)\frac{|f|^2}{\max\limits_{1\le j\le n}\{|z_j|^{2p_j}\}}e^{-\varphi(z_1,\ldots,z_n)}dz\wedge d\bar{z}\\
=& \int_{\{w\in\Delta^n:e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}\cap B_{\delta,t}} v(e^{\frac{-t}{2p}}w)\frac{|f(w)|^2}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}}e^{-\varphi(e^{\frac{-t}{2p}}w)}dw\wedge d\bar{w}+\\
& \int_{\{w\in\Delta^n:e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}\setminus B_{\delta,t}} v(e^{\frac{-t}{2p}}w)\frac{|f(w)|^2}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}}e^{-\varphi(e^{\frac{-t}{2p}}w)}dw\wedge d\bar{w}\\
\le& CC_1C_2\lambda(B_{\delta,t}) + \left(\sup_{\{e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} v(e^{\frac{-t}{2p}}w)\right)e^{-(1+\delta)\varphi(0)}\times\\
& \int_{\{e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} \frac{|f(w)|^2}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}}dw\wedge d\bar{w}\\
=& CC_1C_2\lambda(B_{\delta,t}) + \left(\sup_{\{e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} v(e^{\frac{-t}{2p}}w)\right)e^{-(1+\delta)\varphi(0)}\times\\
& \sum_{\alpha\in E}\int_{\{e^{-1-k_1-k_2}<\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}<1\}} \frac{|b_\alpha|^2\prod_{j=1}^{n_1}|w_j|^{2\alpha_j}}{\max\limits_{1\le j\le n}\{|w_j|^{2p_j}\}}dw\wedge d\bar{w}.
\end{aligned}$$

Hence we know

$$\limsup_{t\to+\infty} S_t \le \sum_{\alpha\in E}\frac{|b_\alpha|^2(2\pi)^n(1+k_1+k_2)}{\prod_{1\le j\le n}(\alpha_j+1)}v(0)e^{-\varphi(0)}.$$

□

Lemma 2.31 (see [33])**.** *Let $c(t)$ be a positive measurable function on $(0,+\infty)$, and let $a\in\mathbb{R}$. Assume that $\int_t^{+\infty}c(s)e^{-s}ds\in(0,+\infty)$ when t near $+\infty$. Then we have*

(1) $\lim\limits_{t\to+\infty}\frac{\int_t^{+\infty}c(s)e^{-as}ds}{\int_t^{+\infty}c(s)e^{-s}ds}=1$ *if and only if* $a=1$,

(2) $\lim\limits_{t\to+\infty}\frac{\int_t^{+\infty}c(s)e^{-as}ds}{\int_t^{+\infty}c(s)e^{-s}ds}=0$ *if and only if* $a>1$,

(3) $\lim\limits_{t\to+\infty}\frac{\int_t^{+\infty}c(s)e^{-as}ds}{\int_t^{+\infty}c(s)e^{-s}ds}=+\infty$ *if and only if* $a<1$.

Lemma 2.32 (see [32])**.** *If $c(t)$ is a positive measurable function on $(T,+\infty)$ such that $c(t)e^{-t}$ is decreasing on $(T,+\infty)$ and $\int_{T_1}^{+\infty}c(s)e^{-s}ds<+\infty$ for some $T_1>T$, then there exists a positive measurable function $\tilde{c}$ on $(T,+\infty)$ satisfying the following statements:*

(1). $\tilde{c}\ge c$ *on* $(T,+\infty)$*;*

(2). $\tilde{c}(t)e^{-t}$ *is strictly decreasing on* $(T,+\infty)$ *and* $\tilde{c}$ *is increasing on* $(a,+\infty)$*, where* $a>T$ *is a real number;*

(3). $\int_{T_1}^{+\infty}\tilde{c}(s)e^{-s}ds<+\infty$.

Moreover, if $\int_T^{+\infty} c(s)e^{-s}ds < +\infty$ *and* $c \in \mathcal{P}_T$*, we can choose* $\tilde{c}$ *satisfying the above conditions,* $\int_T^{+\infty} \tilde{c}(s)e^{-s}ds < +\infty$ *and* $\tilde{c} \in \mathcal{P}_T$*.*

3. Preparations II: multiplier ideal sheaves and optimal L^2 extensions

In this section, we recall and present some lemmas related to multiplier ideal sheaves and optimal L^2 extensions.

3.1. **Multiplier ideal sheaves.** We need the following lemmas in the local cases.

Let $\Delta \subset \mathbb{C}$ be the unit disc. Let $X = \Delta^m$ and let $Y = \Delta^{n-m}$. Denote $M = X \times Y$. Let π_1 and π_2 be the natural projections from M to X and Y respectively.

Let ψ_1 be a plurisubharmonic function on X. Let $\psi = \pi_1^*(\psi_1)$. Let φ_1 be a Lebesgue measurable function on X and $\varphi_2 \in Psh(Y)$. Denote $\varphi = \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

Lemma 3.1. *Assume that*

$$\int_M |f|^2 e^{-\varphi} c(-\psi) < +\infty.$$

Then for any $w \in \Delta^{n-m}$*,*

$$\int_{z\in\Delta^m} |f(z,w)|^2 e^{-\varphi_1} c(-\psi_1) < +\infty.$$

Proof. According to the Fubini's Theorem, we have

$$\begin{aligned}
&\int_{z\in\Delta^m} |f(z,w)|^2 e^{-\varphi_1} c(-\psi_1)\\
\leq &\frac{1}{(\pi r^2)^{n-m}} \int_{w'\in\Delta^{n-m}(w,r)} \left(\int_{z\in\Delta^m} |f(z,w')|^2 e^{-\varphi_1} c(-\psi_1)\right)\\
\leq &\frac{e^T}{(\pi r^2)^{n-m}} \int_{w'\in\Delta^{n-m}(w,r)} \left(\int_{z\in\Delta^m} |f(z,w')|^2 e^{-\varphi_1} c(-\psi_1)\right) e^{-\varphi_2}\\
\leq &\frac{e^T}{(\pi r^2)^{n-m}} \int_{\Delta^n} |f|^2 e^{-\varphi} c(-\psi) < +\infty,
\end{aligned}$$

where $r > 0$ such that $\Delta^{n-m}(w,r) \Subset \Delta^{n-m}$, and $T := -\sup_{w'\in\Delta^{n-m}(w,r)} \varphi_2(w')$. $\square$

Lemma 3.2. *Let* $f_1(z)$ *be a holomorphic function on* X *such that* $(f_1, o) \in \mathcal{I}(\varphi_1 + \psi_1)_o$*, and* $f_2(w)$ *be a holomorphic function on* Y *such that* $(f_2, w) \in \mathcal{I}(\varphi_2)_w$ *for any* $w \in Y$*. Let* $\tilde{f}(z,w) = f_1(z)f_2(w)$ *on* M*, then* $\big(\tilde{f}, (o,w)\big) \in \mathcal{I}(\varphi + \psi)_{(o,w)}$ *for any* $(o,w) \in Y$*.*

Proof. According to $(f_1, o) \in \mathcal{I}(\varphi_1 + \psi_1)_o$ and $(f_2, w) \in \mathcal{I}(\varphi_2)_w$, we can find some $r > 0$ such that

$$\int_{\Delta^m(o,r)} |f_1|^2 e^{-\varphi_1-\psi_1} < +\infty,$$

and

$$\int_{\Delta^{n-m}(w,r)} |f_2|^2 e^{-\varphi_2} < +\infty.$$

Then using Fubini's Theorem, we get

$$\int_{\Delta^n((o,w),r)} |\tilde{f}|^2 e^{-\varphi-\psi} = \int_{\Delta^m(o,r)} |f_1|^2 e^{-\varphi_1-\psi_1} \int_{\Delta^{n-m}(w,r)} |f_2|^2 e^{-\varphi_2} < +\infty,$$

which means that $\big(\tilde{f},(o,w)\big)\in\mathcal{I}(\varphi+\psi)_{(o,w)}$. □

Let $\Delta^{n_1}=\big\{w\in\mathbb{C}^{n_1}:|w_j|<1 \text{ for any } j\in\{1,\ldots,n_1\}\big\}$ be product of the unit disks. Let Y be an n_2-dimensional complex manifold, and let $M=\Delta^{n_1}\times Y$. Denote $n=n_1+n_2$. Let π_1 and π_2 be the natural projections from M to Δ^{n_1} and Y respectively. Let ρ_1 be a nonnegative Lebesgue measurable function on Δ^{n_1} satisfying that $\rho_1(w)=\rho_1(|w_1|,\ldots,|w_{n_1}|)$ for any $w\in\Delta^{n_1}$ and the Lebesgue measure of $\{w\in\Delta^{n_1}:\rho_1(w)>0\}$ is positive. Let ρ_2 be a nonnegative Lebesgue measurable function on Y, and denote that $\rho=\pi_1^*(\rho_1)\times\pi_2^*(\rho_2)$ on M.

Lemma 3.3 (see [2])**.** *For any holomorphic $(n,0)$ form F on M, there exists a unique sequence of holomorphic $(n_2,0)$ forms $\{F_\alpha\}_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that*

$$F=\sum_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}}\pi_1^*(w^\alpha dw_1\wedge\ldots\wedge dw_{n_1})\wedge\pi_2^*(F_\alpha), \tag{3.1}$$

where the right term of the above equality is uniformly convergent on any compact subset of M. Moreover, if $\int_M|F|^2\rho<+\infty$, we have

$$\int_Y|F_\alpha|^2\rho_2<+\infty \tag{3.2}$$

for any $\alpha\in\mathbb{Z}_{\geq 0}^{n_1}$.

Let $\tilde{M}\subset M$ be an $n-$dimensional complex manifold satisfying that $\{o\}\times Y\subset\tilde{M}$, where o is the origin in Δ^{n_1}.

Lemma 3.4 (see [2])**.** *For any holomorphic $(n,0)$ form F on $\tilde{M}$, there exist a unique sequence of holomorphic $(n_2,0)$ forms $\{F_\alpha\}_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}}$ on Y and a neighborhood $M_2\subset\tilde{M}$ of $\{o\}\times Y$, such that*

$$F=\sum_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}}\pi_1^*(w^\alpha dw_1\wedge\ldots\wedge dw_{n_1})\wedge\pi_2^*(F_\alpha)$$

on M_2, where the right term of the above equality is uniformly convergent on any compact subset of M_2. Moreover, if $\int_{\tilde{M}}|F|^2\rho<+\infty$, we have

$$\int_K|F_\alpha|^2\rho_2<+\infty$$

for any compact subset K of Y and $\alpha\in\mathbb{Z}_{\geq 0}^{n_1}$.

Let $f=\sum_{\alpha\in\mathbb{Z}_{\geq 0}^{n}}b_\alpha w^\alpha$ (Taylor expansion) be a holomorphic function on $D=\{w\in\mathbb{C}^n:|w_j|<r_0 \text{ for any } j\in\{1,\ldots,n\}\}$, where $r_0>0$. Let

$$\psi=\max_{1\leq j\leq n_1}\{2p_j\log|w_j|\}$$

be a plurisubharmonic function on $\mathbb{C}^n$, where $n_1\leq n$ and $p_j>0$ is a constant for any $j\in\{1,\ldots,n_1\}$. We recall a characterization of $\mathcal{I}(\psi)_o$, where o is the origin in $\mathbb{C}^n$.

Lemma 3.5 (see [2])**.** *$(f,o)\in\mathcal{I}(\psi)_o$ if and only if $\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_j}>1$ for any $\alpha\in\mathbb{Z}_{\geq 0}^{n}$ satisfying $b_\alpha\neq 0$.*

Let $\Omega=\Delta\subset\mathbb{C}$ be an unit disk. Let $Y=\Delta^n$. Denote $M=\Omega\times Y$. Let π_1 and π_2 be the natural projections from M to Ω and Y respectively.

Let $\psi=\pi_1^*(2p\log|z|)+N$ be a plurisubharmonic function on M, where $N\le 0$ is a plurisubharmonic function on M and $N|_{\{0\}\times\Delta^n}\not\equiv-\infty$. Assume that there exist a holomorphic function g on Δ and a function $\tilde{\psi}_2\in Psh(M)$ such that

$$\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\log|g|)+\tilde{\psi}_2,$$

where $ord_0(g)=q$. We assume that $g=dz^qh(z)$ on Δ, where d is a constant, $h(z)$ is a holomorphic function on Δ and $h(0)=1$.

Let $\varphi_2\in Psh(Y)$. Denote $\varphi:=\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)$.

Let F be a holomorphic $(n,0)$ form on M, where

$$F=\sum_{j=k}^{\infty}\pi_1^*(z^jdz)\wedge\pi_2^*(F_j)$$

according to Lemma 3.3. Here $k\in\mathbb{N}$ and F_j is a holomorphic $(n,0)$ form on Y for any $j\ge k$.

Assume that $\int_M|F|^2e^{-\varphi}c(-\psi)<+\infty$ and $c(t)$ is increasing near $+\infty$. As $\psi=\pi_1^*(2p\log|z|)+N\le\pi_1^*(2p\log|z|)$, when t is large enough, we have

$$S:=\int_{\{\pi_1^*(2p\log|z|)<-t\}}|F|^2e^{-\varphi}c(-\pi_1^*(2p\log|z|))\le\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)<+\infty.$$

Lemma 3.6. *Let c be a positive measurable function on $(0,+\infty)$ such that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, c is increasing near $+\infty$, and $\int_0^{+\infty}c(s)e^{-s}ds<+\infty$. Assume that $k\ge q$, and*

$$S=\int_{\{\pi_1^*(2p\log|z|)<-t\}}|F|^2e^{-\varphi}c\big(-\pi_1^*(2p\log|z|)\big)<+\infty.$$

Then

$$\big(F,(0,y)\big)\in\big(\mathcal{O}(K_M)\big)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $y\in Y$.

Proof. It follows from M is a Stein manifold that there exist smooth plurisubharmonic functions $\tilde{\psi}_{2,l}$ on M such that $\tilde{\psi}_{2,l}$ are decreasingly convergent to $\tilde{\psi}_2$. Since the computation is local, we assume that $|\tilde{\psi}_{2,l}(z,w)-\tilde{\psi}_{2,l}(0,w)|\le\epsilon$ for any $(z,w)\in M=\Delta\times\Delta^n$.

We also assume that $F = z^k\tilde{h}(z,w)dz\wedge dw$ on M and $|h(z)-h(0)|<\epsilon$ for any $z\in\Delta$.

$$\begin{aligned}
S=&\int_{\{\pi_1^*(2p\log|z|)<-t\}}|F|^2e^{-\pi_1^*(2\log|g|)-\tilde{\psi}_2+\pi_1^*(2p\log|z|)+N-\pi_2^*(\varphi_2)}c(-\pi_1^*(2p\log|z|))\\
\geq&\int_{\{\pi_1^*(2p\log|z|)<-t\}}\frac{|z|^{2k+2p}|\tilde{h}(z,w)|^2}{d|z|^{2q}|h(z)|^2}e^{-\tilde{\psi}_{2,l}+N-\pi_2^*(\varphi_2)}c(-\pi_1^*(2p\log|z|))\\
\geq&\int_{w\in\Delta^n}\int_{\{2p\log|z|<-t\}}|z|^{2k+2p-2q}e^N\frac{|\tilde{h}(z,w)|^2}{d|1+\epsilon|^2}e^{-\tilde{\psi}_{2,l}(0,w)-\epsilon-\pi_2^*(\varphi_2)}c(-\pi_1^*(2p\log|z|))\\
=&\int_{w\in\Delta^n}\left(2\int_{\{2p\log|r|<-t\}}\int_0^{2\pi}|r|^{2k+2p-2q+1}e^{N(re^{i\theta},w)}|\tilde{h}(re^{i\theta},w)|^2c(-2p\log r)d\theta dr\right)\times\\
&\frac{e^{-\tilde{\psi}_{2,l}(0,w)-\epsilon-\varphi_2(w)}}{d|1+\epsilon|^2}\\
\geq&\frac{4\pi}{d|1+\epsilon|^2}e^{-\epsilon}\int_{\{2p\log|r|<-t\}}r^{2k+2p-2q+1}c(-2p\log r)dr\int_{w\in\Delta^n}|\tilde{h}(0,w)|^2e^{N(0,w)-\tilde{\psi}_{2,l}(0,w)}e^{-\varphi_2(w)}\\
=&\frac{2\pi}{pd|1+\epsilon|^2}e^{-\epsilon}\int_t^{+\infty}c(s)e^{-(\frac{k-q+1}{p}+1)s}ds\int_{w\in\Delta^n}|\tilde{h}(0,w)|^2e^{N(0,w)-\tilde{\psi}_{2,l}(0,w)}e^{-\varphi_2(w)}.
\end{aligned}\tag{3.3}$$

Since $k\geq q$ and $\int_0^{+\infty}c(s)e^{-s}ds<+\infty$, we have

$$\int_t^{+\infty}c(s)e^{-(\frac{k-q+1}{p}+1)s}ds<+\infty.$$

Letting $l\to+\infty$ in (3.3), it follows from $S<+\infty$ and Levi's Theorem that we have

$$\frac{2\pi}{pd|1+\epsilon|^2}e^{-\epsilon}\int_t^{+\infty}c(s)e^{-(\frac{k-q+1}{p}+1)s}ds\int_{w\in\Delta^n}|\tilde{h}(0,w)|^2e^{N(0,w)-\tilde{\psi}_2(0,w)}e^{-\varphi_2(w)}<+\infty. \tag{3.4}$$

It follows from inequality (3.4) that we have

$$\int_{w\in\Delta^n}|\tilde{h}(0,w)|^2e^{N(0,w)-\tilde{\psi}_2(0,w)}e^{-\varphi_2(w)}<+\infty. \tag{3.5}$$

Note that $N|_{\{0\}\times\Delta^n}\not\equiv-\infty$. It follows from (3.5) that there must exist $w_1\in Y$ such that $\tilde{\psi}_2(0,w_1)>-\infty$.

Since $k\geq q$, we know $z^k\in\mathcal{I}(2\log|g|)_0$. It follows from Lemma 3.2 that

$$\big(\pi_1^*(z^kdz)\wedge\pi_2^*(F_k),(0,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}, \tag{3.6}$$

for any $y\in Y$.

Let $0<\delta<1$ be a constant such that $w_1\in\Delta_\delta^n$. It follows from Fubini's Theorem and $c(t)e^{-t}$ is decreasing with respect to t that

$$\begin{aligned}
&\int_{\Delta_\delta\times\Delta_\delta^n}|\pi_1^*(z^kdz)\wedge\pi_2^*(F_k)|^2e^{-\varphi}c(-\pi_1^*(2p\log|z|))\\
=&\int_{\Delta_\delta}|z^k|^2e^{-\varphi_1}c(-2p\log|z|)|dz|^2\cdot\int_{\Delta_\delta^n}|F_k|^2e^{-\varphi_2}\\
\le&C\int_{\Delta_\delta}|z^k|^2e^{-\varphi_1-2p\log|z|}|dz|^2\cdot\int_{\Delta_\delta^n}|F_k|^2e^{-\varphi_2},
\end{aligned}\tag{3.7}$$

where C is a positive constant independent of F.

Consider

$$\begin{aligned}
I:=&\int_{\Delta_\delta}|z|^{2k}e^{-\varphi_1-2p\log|z|}|dz|^2\\
=&\int_{\Delta_\delta}|z|^{2k}e^{-2\log|g|-u(z)}|dz|^2\\
=&\int_{\Delta_\delta}|z|^{2k}e^{-2\log|g|+N(z,w)-\tilde\psi_2(z,w)}|dz|^2.
\end{aligned}\tag{3.8}$$

As N is a plurisubharmonic function on M, e^N has a upper bound C_1 on $\Delta_\delta\times\Delta_\delta^n$ (especially, C_1 is independent of w). Hence

$$I\le C_1\int_{\Delta_\delta}|z|^{2k}e^{-2\log|g|-\tilde\psi_2(z,w)}|dz|^2=C_1\int_{\Delta_\delta}|z|^{2k-2q}|h(z)|^2e^{-\tilde\psi_2(z,w)}|dz|^2.$$

Denote $M(w)=\int_{\Delta_\delta}|z|^{2k-2q}|h(z)|^2e^{-\tilde\psi_2(z,w)}|dz|^2$. We have $I\le M(w)$ for any $w\in\Delta_\delta^n$, especially $I\le M(w_1)$.

Next we prove $M(w_1)<+\infty$. Note that $M(w_1)=\int_{\Delta_\delta}|z|^{2k-2q}|h(z)|^2e^{-\tilde\psi_2(z,w_1)}|dz|^2$. As $e^{-\tilde\psi_2(0,w_1)}>-\infty$, $k\ge q$ and $h(z)$ is a holomorphic function on Δ_δ, by Hölder inequality, we have

$$\begin{aligned}
M(w_1)=&\int_{\Delta_\delta}|z|^{2k-2q}|h(z)|^2e^{-\tilde\psi_2(z,w_1)}|dz|^2\\
\le&(\int_{\Delta_\delta}|z|^{s(2k-2q)}|h(z)|^{2s}|dz|^2)^{\frac1s}(\int_{\Delta_\delta}e^{-\frac{s}{s-1}\tilde\psi_2(z,w_1)}|dz|^2)^{\frac{s-1}{s}}\\
<&+\infty,
\end{aligned}$$

where $s>1$ is a real number. Hence we know $I<+\infty$. Then

$$\int_{\Delta_\delta\times\Delta_\delta^n}|\pi_1^*(z^kdz)\wedge\pi_2^*(F_k)|^2e^{-\varphi}c(-\pi_1^*(2p\log|z|))<+\infty.$$

As $S=\int_{\Delta_\delta\times\Delta_\delta^n}|F|^2e^{-\varphi}c(-\pi_1^*(2p\log|z|))<+\infty$, we have

$$\int_{\Delta_\delta\times\Delta_\delta^n}|F-\pi_1^*(z^kdz)\wedge\pi_2^*(F_k)|^2e^{-\varphi}c(-\pi_1^*(2p\log|z|))<+\infty.$$

Note that

$$F-\pi_1^*(z^kdz)\wedge\pi_2^*(F_k)=\sum_{j=k+1}^{\infty}\pi_1^*(z^jdz)\wedge\pi_2^*(F_j)$$

and $k+1>q$. Using the same method as above, we can get that

$$\big(\pi_1^*(z^{k+1}dz)\wedge\pi_2^*(F_{k+1}),(0,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $y\in Y$ and

$$\int_{\Delta_\delta\times\Delta_\delta^n}\left|F-\pi_1^*(z^kdz)\wedge\pi_2^*(F_k)-\pi_1^*(z^{k+1}dz)\wedge\pi_2^*(F_{k+1})\right|^2e^{-\varphi}c(-\pi_1^*(2p\log|z|))<+\infty.$$

By induction, we know that

$$\big(\pi_1^*(z^jdz)\wedge\pi_2^*(F_j),(0,y)\big)\in\big(\mathcal{O}(K_M))\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $j\ge k$, $y\in Y$. Then it follows from Lemma 2.25 that

$$\big(F,(0,y)\big)=\big(\sum_{j=k}^{\infty}\pi_1^*(z^jdz)\wedge\pi_2^*(F_j),(0,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $y\in Y$. □

Let $\Omega=\Delta$ be the unit disk in $\mathbb{C}$, where the coordinate is z. Let $Y=\Delta^n$ be the unit polydisc in $\mathbb{C}^n$, where the coordinate is $w=(w_1,\ldots,w_n)$. Let $M=\Omega\times Y$. Let π_1, π_2 be the natural projections from M to Ω and Y.

Let $\psi_1=2p\log|z|+\psi_0$ on Ω, where $p>0$ and ψ_0 is a negative subharmonic function on Ω with $\psi_0(0)>-\infty$. Let φ_1 be a Lebesgue measurable function on Ω such that $\varphi_1+\psi$ is a subharmonic function on Ω. It follows from the Weierstrass Theorem on open Riemann surfaces (see [16]) and the Siu's Decomposition Theorem, that $\varphi_1+\psi_1=2\log|g|+2u$, where g is a holomorphic function on Ω with $ord(g)_0=q\in\mathbb{N}$, u is a subharmonic function on Ω such that $v(dd^cu,z)\in[0,1)$ for any $z\in\Omega$. Let φ_2 be a plurisubharmonic function on Y. Denote $\varphi:=\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)$ on M.

Lemma 3.7. *$\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}=\mathcal{I}\big(\pi_1^*(\psi_1)+\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)}$ for any $y\in Y$.*

Proof. It is easy to see that $\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}\supset\mathcal{I}\big(\pi_1^*(\psi_1+\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)}$ for any $y\in Y$.

Now we prove $\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}\subset\mathcal{I}\big(\pi_1^*(\psi_1+\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)}$ for any $y\in Y$.

Let $F\in\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$. Then by Lemma 3.3 (although Lemma 3.3 is stated for the holomorphic $(n,0)$ forms, since our case is local, the decomposition still holds for holomorphic functions), we know $F=\sum_{j=k}^{+\infty}z^jF_j$ on M, where the right hand side is uniformly convergent to F on M, $F_j(w)$ is a holomorphic function on Y for any $j\ge k$ and $F_k\not\equiv0$.

Since the case is local, we also assume that $F=z^kh_1(z,w)$ on M, where $h_1(z,w)$ is a holomorphic function on M satisfying $h_1(0,w)\ne0$. $F\in\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$ implies that $k\ge q$ and

$$\int_{\Delta\times\Delta^n}|z|^{2k-2q}|h_1(z,w)|e^{-\pi_2^*(\varphi(w))}<+\infty.$$

By Fubini's theorem and sub-mean value inequality of subharmonic functions, we have

$$\begin{aligned}\int_{\Delta\times\Delta^n}|z|^{2k-2q}|h_1(z,w)|e^{-\pi_2^*(\varphi_2(w))}&=\int_{w\in\Delta^n}\int_{z\in\Delta}|z|^{2k-2q}|h_1(z,w)|e^{-\pi_2^*(\varphi_2(w))}\\&\geq C\int_{w\in\Delta^n}|h_1(0,w)|e^{-\varphi_2(w)}\\&=C\int_{w\in\Delta^n}|F_k|^2e^{-\varphi_2(w)},\end{aligned}$$

where $C>0$ is a constant. Hence we have $(F_k,y)\in\mathcal{I}(\varphi_2)_y$ for any $y\in Y$.

As $v(dd^cu,0)\in[0,1)$, we have $(z^k,0)\in\mathcal{I}(2\log|g|)_0=\mathcal{I}(2\log|g|+2u)_0=\mathcal{I}(\psi_1+\varphi_1)_0$. It follows from Lemma 3.2 that we know $\big(z^kF_k,(0,y)\big)\in\mathcal{I}\big(\pi_1^*(\psi_1+\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)}$. Then we know

$$\int_{\Delta\times\Delta^n}|z^kF_k|^2e^{\pi_1^*(-2\log|g|+2u)-\pi_2^*(\varphi_2)}<+\infty,$$

which implies

$$\int_{\Delta\times\Delta^n}|z^kF_k|^2e^{\pi_1^*(-2\log|g|)-\pi_2^*(\varphi_2)}<+\infty.$$

Hence we have $\big(F-z^kF_k,(0,y)\big)\in\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$.

Denote $\tilde{F}_{k+1}=F-z^kF_k$ on $\Delta\times\Delta^n$. Note that $\tilde{F}_{k+1}=\sum_{j=k+1}^{+\infty}z^jF_j$ on M and $\big(\tilde{F}_{k+1},(0,y)\big)\in\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$.

By using similar discussion as above, we know

$$\big(z^{k+1}F_{k+1},(0,y)\big)\in\mathcal{I}\big(\pi_1^*(\psi_1+\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)},$$

and

$$\int_{\Delta\times\Delta^n}|z^{k+1}F_{k+1}|^2e^{\pi_1^*(-2\log|g|)-\pi_2^*(\varphi_2)}<+\infty.$$

Hence $\big(\tilde{F}_{k+1}-z^{k+1}F_{k+1},(0,y)\big)\in\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$. Denote $\tilde{F}_{k+2}=\tilde{F}_{k+1}-z^{k+1}F_{k+1}$ on $\Delta\times\Delta^n$. Note that $\tilde{F}_{k+2}=\sum_{j=k+2}^{+\infty}z^jF_j$ on M and $\big(\tilde{F}_{k+2},(0,y)\big)\in\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(0,y)}$.

By induction, we know that for any $j\geq k$,

$$\big(z^jF_j,(0,y)\big)\in\mathcal{I}\big(\pi_1^*(\psi_1+\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)}$$

holds. Then it follows from Lemma 2.25 that we know

$$\big(F,(0,y)\big)\in\mathcal{I}\big(\pi_1^*(\psi_1+\varphi_1)+\pi_2^*(\varphi_2)\big)_{(0,y)}$$

for any $y\in Y=\Delta^n$.

□

We recall a well known result about multiplier ideal sheaves.

Lemma 3.8 (see [2]). *Let Φ_1 and Φ_2 be plurisubharmonic functions on Δ^n satisfying $\Phi_2(o)>-\infty$, where $n\in\mathbb{Z}_{>0}$ and o is the origin in Δ^n. Then $\mathcal{I}(\Phi_1)_o=\mathcal{I}(\Phi_1+\Phi_2)_o$.*

3.2. **Optimal jet L^2 extensions.** Let Ω be an open Riemann surface with nontrivial Green functions. Let $Z_\Omega = \{z_j : j \in \mathbb{N}_+ \& j < \gamma\}$ be a subset of Ω of discrete points, where $\gamma \in \mathbb{Z}^+_{\geq 2}$ or $\gamma = +\infty$. Let Y be an $n-$dimensional weakly pseudoconvex Kähler manifold. Denote $M = \Omega \times Y$. Let π_1 and π_2 be the natural projections from M to Ω and Y respectively. Denote $Z_0 := Z_\Omega \times Y$. Denote $Z_j := \{z_j\} \times Y$.

Let $\tilde{M} \subset M$ be an $n-$dimensional weakly pseudoconvex Kähler manifold satisfying that $Z_0 \subset \tilde{M}$. Let F be a holomorphic $(n, 0)$ form on a neighborhood $U_0 \subset \tilde{M}$ of Z_0.

Let ψ be a plurisubharmonic function on $\tilde{M}$. It follows from Siu's decomposition theorem that

$$dd^c\psi = \sum_{j\geq 1} 2p_j[Z_j] + \sum_{i\geq 1} \lambda_i[A_i] + R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $(n-1)-$dimensional analytic set, and where R is a closed positive current with the property that $dimE_c(R) < n - 1$ for every $c > 0$, where $E_c(R) = \{x \in \tilde{M} : v(R, x) \geq c\}$ is the upperlevel sets of Lelong number. We assume that $p_j > 0$ for any $1 \leq j < \gamma$.

Then $N := \psi - \pi_1^*\big(\sum_{j\geq 1} 2p_jG_\Omega(z, z_j)\big)$ is a plurisubharmonic function on $\tilde{M}$. We assume that $N \leq 0$.

Let φ_1 be a Lebesgue measurable function on Ω such that $\psi+\pi_1^*(\varphi)$ is a plurisubharmonic function on $\tilde{M}$. With similar discussion as above, by Siu's decomposition theorem, we have

$$dd^c(\psi + \pi_1^*(\varphi)) = \sum_{j\geq 1} 2\tilde{q}_j[Z_j] + \sum_{i\geq 1} \tilde{\lambda}_i[\tilde{A}_i] + \tilde{R},$$

where $\tilde{q}_j \geq 0$ for any $1 \leq j < \gamma$.

By Weierstrass theorem on open Riemann surface, there exists a holomorphic function g on Ω such that $ord_{z_j}(g) = q_j := [\tilde{q}_j]$ for any $z_j \in Z_\Omega$ and $g(z) \neq 0$ for any $z \notin Z_\Omega$, where $[q]$ equals to the integer part of the nonnegative real number q. Then we know that there exists a plurisubharmonic function $\tilde{\psi}_2 \in Psh(\tilde{M})$ such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2.$$

Let $\varphi_2 \in Psh(Y)$. Denote $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

For $1 \leq j < \gamma$, let $(V_j, \tilde{z}_j)$ be a local coordinated open neighborhood of z_j in Ω satisfying $V_j \Subset \Omega$, $\tilde{z}_j(z_j) = 0$ under the local coordinate and $V_j \cap V_k = \emptyset$ for any $j \neq k$. Denote $V_0 := \cup_{1\leq j<\gamma}V_j$. We assume that $g = d_j\tilde{z}_j^{q_j}h_j(z)$ on V_j, where d_j is a constant, $h_j(z)$ is a holomorphic function on V_j and $h(z_j) = 1$.

Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$.

We have the following lemma.

Lemma 3.9. *Let F be a holomorphic $(n, 0)$ form on U_0 such that for $1 \leq j < \gamma$, $F = \pi_1^*(\tilde{z}_j^{k_j} f_jdz_j) \wedge \pi_2^*(F_j)$ on $U_j \Subset U_0 \cap (V_j \times Y)$, where U_j is an open neighborhood of Z_j in $\tilde{M}$, k_j is a nonnegative integer, f_j is a holomorphic function on V_j satisfying $f_j(z_j) = a_j \neq 0$ and F_j is a holomorphic $(n-1, 0)$ form on Y.*

Denote $I_F := \{j : 1 \le j < \gamma \& k_j + 1 - q_j \le 0\}$. *Assume that* $k_j + 1 = q_j$ *for any* $j \in I_F$ *and* $\tilde{\psi}_2|_{Z_j}$ *is not identically* $-\infty$ *on* Z_j. *If*

$$\sum_{j\in I_F} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty \tag{3.9}$$

and

$$\int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty, \tag{3.10}$$

for any $j \notin I_F$. *Then there exists a holomorphic* $(n,0)$ *form* $\tilde{F}$ *on* $\tilde{M}$ *such that* $(\tilde{F} - F, (z_j, y)) \in (\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)))_{(z_j,y)}$ *for any* $1 \le j < \gamma$ *and* $y \in Y$ *and*

$$\int_{\tilde{M}} |\tilde{F}|^2 c(-\psi)e^{-\varphi} \le (\int_0^{+\infty} c(s)e^{-s}ds) \sum_{j\in I_F} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty. \tag{3.11}$$

Proof. Denote $G = \sum_{j\ge1} 2p_j G_\Omega(z, z_j)$. Note that $\psi \le \pi_1^*(G)$ and $c(t)e^{-t}$ is decreasing. We have

$$c(-\psi)e^{-\varphi} \le c(-\pi_1^*(G))e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)}.$$

To prove Lemma 3.9, it suffice to prove

$$\begin{aligned}&\int_{\tilde{M}} |\tilde{F}|^2 c\big(-\pi_1^*(G)\big)e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)}\\ \le&(\int_0^{+\infty} c(s)e^{-s}ds) \sum_{j\in I_F} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty.\end{aligned} \tag{3.12}$$

The following remark shows that we only need to prove formula (3.12) when Z_Ω is a finite set.

Remark 3.10. *It follows from Lemma 2.21 that there exists a sequence of open Riemann surfaces* $\{\Omega_l\}_{l\in\mathbb{Z}^+}$ *such that* $\Omega_l \Subset \Omega_{l+1} \Subset \Omega$, $\cup_{l\in\mathbb{Z}^+}\Omega_l = \Omega$ *and* $\{G_{\Omega_l}(\cdot, z_0) - G_\Omega(\cdot, z_0)\}$ *is decreasingly convergent to* 0 *on* Ω *with respect to* l *for any* $z_0 \in \Omega$.

Denote $Z_l := \Omega_l \cap Z_0$. *As* Z_Ω *is a subset of* Ω *of discrete points,* Z_l *is a set of finite points.*

Denote

$$G_l := \sum_{z_j\in Z_l} 2p_j G_\Omega(z, z_j)$$

and

$$\varphi_{1,l} := \varphi_1 + G - G_l.$$

Then we have $N + \pi_1^*(G_l) + \pi_1^*(\varphi_{1,l}) = N + \pi_1^*(G) + \pi_1^*(\varphi_1) = \psi + \pi_1^*(\varphi_1)$.

Let $I_l := I_F \cap \{j : z_j \in Z_l\}$. *Denote* $\tilde{M}_l := (\Omega_l \times Y) \cap \tilde{M}$. *We note that* $\tilde{M}_l$ *is weakly pseudoconvex Kähler. Now we assume that the formula* (3.12) *holds on* $\tilde{M}_l$, *i.e. we have*

$$\begin{aligned}&\int_{\tilde{M}_l} |\tilde{F}_l|^2 c\big(-\pi_1^*(G_l)\big)e^{-N-\pi_1^*(\varphi_{1,l})-\pi_2^*(\varphi_2)}\\ \le&(\int_0^{+\infty} c(s)e^{-s}ds) \sum_{j\in I_l} \frac{2\pi|a_j|^2}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} < +\infty,\end{aligned}$$

where F_l is a holomorphic $(n,0)$ form on M_l satisfying $(F_l-F,(z_j,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g_0|)+\pi_2^*(\varphi_2)))_{(z_j,y)}$ for any l, $z_j\in Z_l$, and $y\in Y$.

As $G\le G_l$ and $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, we have

$$\begin{aligned}&\int_{\tilde{M}_l}|\tilde{F}_l|^2c\big(-\pi_1^*(G)\big)e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)}\\ \le&\int_{\tilde{M}_l}|\tilde{F}_l|^2c\big(-\pi_1^*(G_l)\big)e^{-N-\pi_1^*(\varphi_{1,l})-\pi_2^*(\varphi_2)}\\ \le&(\int_0^{+\infty}c(s)e^{-s}ds)\sum_{j\in I_l}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}<+\infty.\end{aligned}\tag{3.13}$$

Note that $\pi_1^*(G)$ is continuous on $\tilde{M}\backslash Z_0$, where Z_0 is a closed analytic subset of $\tilde{M}$ and $N+\pi_1^*(G)+\pi_1^*(\varphi_1)=\psi+\pi_1^*(\varphi_1)$ is a plurisubharmonic function on M. For any compact subset K of $\tilde{M}\backslash Z_0$, there exist $\hat{l}$ (depending on the choice of K) such that $K\subset\subset\tilde{M}_{\hat{l}}$ and $C_K>0$ such that $\frac{e^{N+\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)}}{c(-\pi_1^*(G))}=\frac{e^{N+\pi_1^*(\varphi_1)+\pi_2^*(\varphi_2)+\pi_1^*(G)}}{c(-\pi_1^*(G))e^{\pi_1^*(G)}}\le C_K$ on K. It follows from Lemma 2.24 and diagonal method that there exists a subsequence of $\{F_l\}$ (also denoted by $\{F_l\}$), which is compactly convergent to a holomorphic $(n,0)$ form $\tilde{F}$ on $\tilde{M}$. Combining formula (3.13) *and Fatou's lemma, we have*

$$\begin{aligned}&\int_{\tilde{M}_l}|\tilde{F}|^2c(-\pi_1^*(G))e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)}\\ \le&\liminf_{l\to+\infty}\int_{\tilde{M}_l}|\tilde{F}_l|^2c(-\pi_1^*(G))e^{-N-\pi_1^*(\varphi_1)-\pi_2^*(\varphi_2)}\\ \le&\liminf_{l\to+\infty}\int_{\tilde{M}_l}|\tilde{F}_l|^2c(-\pi_1^*(G_l))e^{-N-\pi_1^*(\varphi_{1,l})-\pi_2^*(\varphi_2)}\\ \le&\liminf_{l\to+\infty}(\int_0^{+\infty}c(s)e^{-s}ds)\sum_{j\in I_l}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}\\ \le&(\int_0^{+\infty}c(s)e^{-s}ds)\sum_{j\in I_F}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}<+\infty.\end{aligned}$$

As $\{F_l\}$ is compactly convergent to $\tilde{F}$ on $\tilde{M}$ and $(F_l-F,(z_j,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g_0|)+\pi_2^*(\varphi_2)))_{(z_j,y)}$ for any l, $z_j\in Z_l$, and $y\in Y$. It follows from Lemma 2.25 that $(\tilde{F}-F,(z_j,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g_0|)+\pi_2^*(\varphi_2)))_{(z_j,y)}$ for any $z_j\in Z_\Omega$, and $y\in Y$.

We continue to prove Lemma 3.9. Now we assume that $\gamma=m+1$ i.e. $Z_\Omega=\{z_j:1\le j\le m\}$ and $I_F=\{1,2,\ldots,m_1\}$, where $m_1\le m$.

Denote $\tilde{\psi}_{2,l}=\max\{\tilde{\psi}_2,-l\}$, where l is a positive integer. As $\tilde{\psi}_2$ is plurisubharmonic, we know $\{\tilde{\psi}_{2,l}\}_{l=1}^{+\infty}$ is a sequence of plurisubharmonic functions on $\tilde{M}$ decreasingly convergent to $\tilde{\psi}_2$. We also note that every $\tilde{\psi}_{2,l}$ is lower bounded.

When t_0 is large enough, we know that $\{G<-t\}\Subset V_0$, for any $t>t_0$. As $\tilde{M}$ is a weakly pseudoconvex Kähler manifold, there exists a sequence of weakly pseudoconvex Kähler manifolds $\tilde{M}_s$ satisfying $\tilde{M}_1\Subset\tilde{M}_2\cdots\Subset\tilde{M}_s\Subset\cdots\tilde{M}$ and $\cup_{s\in\mathbb{N}^+}\tilde{M}_s=\tilde{M}$.

It is easy to verify that $\int_{\{\pi_1^*(G)<-t\}\cap\tilde{M}_s}|F|^2<+\infty$, and it follows from formula (3.10) and Fubini's theorem that

$$\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}<+\infty.$$

It follows from Lemma 2.23 that there exists a holomorphic $(n,0)$ form $F_{t,l,s}$ on M such that

$$\begin{aligned}&\int_{\tilde{M}_s}|F_{t,l,s}-(1-b_{t,1}(\pi_1^*(G)))F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+v_{t,1}(\pi_1^*(G))}c(-v_{t,1}(\pi_1^*(G)))\\ \leq&(\int_0^{t+1}c(s)e^{-s}ds)\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}<+\infty,\end{aligned}\tag{3.14}$$

for any $t\geq t_0$.

As $b_{t,1}(\hat{t})=0$ for $\hat{t}$ largely enough, then we know $(F_{t,l,s}-F,(z_0,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g_0|)+\pi_2^*(\varphi_2)))_{(z_j,y)}$ for any $(z_j,y)\in Z_0\cap\tilde{M}_s$.

For any $\epsilon>0$, there exists $t_1>t_0$ such that

(1) for any $j\in\{1,2,\ldots,m\}$

$$\sup_{\tilde{z}_j\in\{G<-t_1\}\cap V_j}|g_1(\tilde{z})-g_1(z_j)|<\epsilon,$$

where $g_1(z)$ is a smooth function on V_0 satisfying $g_1(\tilde{z})|_{V_j}:=G-2p_j\log|\tilde{z}_j|$.

(2) for any $j\in\{1,2,\ldots,m\}$

$$\sup_{\tilde{z}_j\in\{G<-t_1\}\cap V_j}|f_j(\tilde{z}_j)-a_j|<\epsilon.$$

(3) for any $j\in\{1,2,\ldots,m\}$

$$\sup_{\tilde{z}_j\in\{G<-t_1\}\cap V_j}|h_j(\tilde{z}_j)-1|<\epsilon.$$

For any $(z_j,y)\in Z_0\cap\tilde{M}_s$, letting (U_y,w) be a small local coordinated open neighborhood of y and shrinking V_j if necessary, we have $V_j\times U_y\Subset U_j$ for any j. Recall that $V_0:=\cup_{j\geq 1}V_j$. Assume that $F_j=\tilde{h}_j(w)dw$ on U_y, where $\tilde{h}_j$ is a holomorphic function on U_y and $dw=dw_1\wedge dw_2\wedge\ldots\wedge dw_n$. There exists $t_2>t_1$ such that when $t>t_2$, $(\{G<-t\}\times U_y)\cap(V_j\times U_y)\Subset V_j\times U_y$, for any j.

When $t>t_2$, direct calculation shows

$$\begin{aligned}&\int_{V_0\times U_y}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}\\ =&\sum_{j=1}^m\int_{\{-t-1<G<-t\}\times U_y}\frac{|\tilde{z}_j|^{2k_j}|f_j|^2|F_j|^2}{|d_j|^2|\tilde{z}|^{2q_j}|h_j|^2}e^{-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}|d\tilde{z}_j|^2|dw|^2\\ \leq&\sum_{j=1}^m\int_{y\in U_y}\left(\int_{\{-t-1-\epsilon-g_1(z_0)<2p_j\log|\tilde{z}|<-t+\epsilon-g_1(z_0)\}}|\tilde{z}_j|^{2k_j-2q_j}\frac{|f_j|^2}{|d_j|^2|h_j|^2}e^{-\tilde{\psi}_{2,l}(z,w)}|d\tilde{z}_j|^2\right)\times\\ &|\tilde{h}_j|^2e^{-\varphi_2}|dw|^2.\end{aligned}$$

Note that $2k_j-2q_j=-2$ for any $1\leq j\leq m_1$ and $2k_j-2q_j\geq 0$ for $m_1<j\leq m$. When t is laege enough, for any j, the integral

$$\int_{\{-t-1-\epsilon-g_1(z_0)<2p_j\log|\tilde{z}|<-t+\epsilon-g_1(z_0)\}}|\tilde{z}_j|^{2k_j-2q_j}\frac{|f_j|^2}{|d_j|^2|h_j|^2}e^{-\tilde{\psi}_{2,l}(z,w)}|d\tilde{z}_j|^2$$

is uniformly bounded with respect to t. It follows from (3.10) that $\int_{U_y}|\tilde{h}_j|^2e^{-\varphi_2}|dw|^2<+\infty$ for any j. Then, by Fatou's lemma and Lemma 2.29 (we use Lemma 2.29 for the case $n=1$), we have

$$\begin{aligned}
&\limsup_{t\to+\infty}\int_{V_0\times U_y}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}\\
\le&\limsup_{t\to+\infty}\sum_{j=1}^{m}\int_{y\in U_y}(\int_{\{-t-1-\epsilon-g_1(z_0)<2p_j\log|\tilde{z}|<-t+\epsilon-g_1(z_0)\}}|\tilde{z}_j|^{2k_j-2q_j}\times\\
&\frac{|f_j|^2}{|d_j|^2|h_j|^2}e^{-\tilde{\psi}_{2,l}(z,w)}|d\tilde{z}_j|^2)|\tilde{h}_j|^2e^{-\varphi_2}|dw|^2\\
\le&\sum_{j=1}^{m}\int_{y\in U_y}\limsup_{t\to+\infty}(\int_{\{-t-1-\epsilon-g_1(z_0)<2p_j\log|\tilde{z}|<-t+\epsilon-g_1(z_0)\}}|\tilde{z}_j|^{2k_j-2q_j}\times\\
&\frac{|f_j|^2}{|d_j|^2|h_j|^2}e^{-\tilde{\psi}_{2,l}(z,w)}|d\tilde{z}_j|^2)|\tilde{h}_j|^2e^{-\varphi_2}|dw|^2\\
\le&\sum_{j=1}^{m_1}\int_{y\in U_y}\frac{2\pi(1+2\epsilon)|a_j|^2}{p_j|d_j|^2}e^{-\tilde{\psi}_{2,l}(z_0,w)}|\tilde{h}_j|^2e^{-\varphi_2}|dw|^2.
\end{aligned}$$

Let $\epsilon\to+\infty$, we have

$$\begin{aligned}
&\limsup_{t\to+\infty}\int_{V_0\times U_y}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}\\
\le&\sum_{j=1}^{m_1}\int_{y\in U_y}\frac{2\pi|a_j|^2}{p_j|d_j|^2}e^{-\tilde{\psi}_{2,l}(z_0,w)}|\tilde{h}_j|^2e^{-\varphi_2}|dw|^2.
\end{aligned}$$

As y and U_y are arbitrarily chosen, we have

$$\begin{aligned}
&\limsup_{t\to+\infty}\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}\\
&\qquad\le\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\tilde{\psi}_{2,l}(z_0,w)-\varphi_2}.
\end{aligned}\tag{3.15}$$

Since $v_{t,1}(\pi_1^*(G))\ge\pi_1^*(G)$ and $c(t)e^{-t}$ is decreasing with respect to t, it follows from (3.14) and (3.15) that

$$\begin{aligned}
&\limsup_{t\to+\infty}\int_{\tilde{M}_s}|F_{t,l,s}-(1-b_{t,1}(\pi_1^*(G)))F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\
\le&\limsup_{t\to+\infty}\int_{\tilde{M}_s}|F_{t,l,s}-(1-b_{t,1}(\pi_1^*(G)))F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+v_{t,1}(\pi_1^*(G))}c(-v_{t,1}(\pi_1^*(G)))\\
\le&\limsup_{t\to+\infty}(\int_0^{t+1}c(t_1)e^{-t_1}dt_1)\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<\pi_1^*(G)<-t\}}|F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}\\
\le&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\tilde{\psi}_{2,l}(z_0,w)-\varphi_2}.
\end{aligned}\tag{3.16}$$

Note that $k_j-q_j=-1$ for $1\le j\le m_1$, $k_j-q_j\ge 0$ for $m_1<j\le m$, when t is large enough, we have

$$\begin{aligned}&\int_{(\{\pi_1^*(G)<-t\}\times Y)\cap\tilde{M}_s}|F|^2e^{\pi_1^*(-2\log|g|+G)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)}c(-\pi_1^*(G))\\ \le&\sum_{j=1}^m C_1\int_{\{G<-t\}}|z_j|^{2k_j}|f_j|^2e^{-2\log|g_0|}e^{-G}c(-G)\int_{\{z_j\}\times Y}|F_j|^2e^{-\varphi_2}\\ \le&C_2\sum_{j=1}^m\int_t^{+\infty}c(t_1)e^{-t_1}dt_1\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\varphi_2}<+\infty,\end{aligned}$$

where C_1 and C_2 are constants. Hence we have

$$\limsup_{t\to+\infty}\int_{\tilde{M}_s}|(1-b_{t,1}(\pi_1^*(G)))F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))<+\infty.$$

Combining with (3.16), we know

$$\limsup_{t\to+\infty}\int_{\tilde{M}_s}|F_{t,l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))<+\infty.$$

By Lemma 2.24, we know there exists a subsequence of $\{F_{t,l,s}\}_{t\to+\infty}$ (still denoted by $\{F_{t,l,s}\}_{t\to+\infty}$) compactly convergent to a holomorphic $(n,0)$ form $F_{l,s}$ on M_s. It follows from (3.16) and Fatou's lemma that

$$\begin{aligned}&\int_{\tilde{M}_s}|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \le&\liminf_{t\to+\infty}\int_{\tilde{M}_s}|F_{t,l,s}-(1-b_{t,1}(\pi_1^*(G)))F|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \le&\left(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1\right)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\tilde{\psi}_{2,l}(z_0,w)-\varphi_2}<+\infty.\end{aligned}\tag{3.17}$$

As $\tilde{\psi}_{2,l}$ is decreasingly convergent to $\tilde{\psi}_2$, when $l\to+\infty$, and $\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\pi_2^*(\varphi_2)}<+\infty$ for any j,

$$\begin{aligned}&\int_{\tilde{M}_s}|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \le&\left(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1\right)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}<+\infty.\end{aligned}\tag{3.18}$$

As $\tilde{\psi}_{2,l}$ is decreasingly convergent to $\tilde{\psi}_2$, for any compact subset $K\subset\tilde{M}_s$, we have

$$\inf_l\inf_K e^{-\tilde{\psi}_{2,l}}\ge\inf_K e^{-\tilde{\psi}_{2,1}}>0,$$

then it follows from (3.18) that

$$\sup_l\int_K|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))<+\infty.$$

Hence it follows from Lemma 2.24 and diagonal method that there exists a subsequence of $\{F_{l,s}\}_{l\to+\infty}$ (still denoted by $\{F_{l,s}\}_{l\to+\infty}$) compactly convergent to a holomorphic $(n,0)$ form F_s on $\tilde{M}_s$. It follows from (3.10), (3.18) and Fatou's lemma that

$$\begin{aligned}&\int_{\tilde{M}_s}|F_s|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_2-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \leq&\liminf_{l\to+\infty}\int_{\tilde{M}_s}|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \leq&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_{(\{z_j\}\times Y)\cap\tilde{M}_s}|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}\\ \leq&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}<+\infty.\end{aligned}\tag{3.19}$$

Again using Lemma 2.24 and diagonal method, we know that there exists a subsequence of $\{F_s\}_{s\to+\infty}$ (still denoted by $\{F_s\}_{s\to+\infty}$) compactly convergent to a holomorphic $(n,0)$ form $\tilde{F}$ on $\tilde{M}$. It follows from (3.19) and Fatou's lemma that

$$\begin{aligned}&\int_{\tilde{M}}|\tilde{F}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_2-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \leq&\liminf_{s\to+\infty}\int_{M_s}|F_{l,s}|^2e^{\pi_1^*(-2\log|g|)-\tilde{\psi}_{2,l}-\pi_2^*(\varphi_2)+\pi_1^*(G)}c(-\pi_1^*(G))\\ \leq&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}<+\infty.\end{aligned}\tag{3.20}$$

Note that $N+\pi_1^*(G)+\pi_1^*(\varphi_1)=2\log|g|+\tilde{\psi}_2$. We have

$$\begin{aligned}&\int_{\tilde{M}}|\tilde{F}|^2e^{-\pi_1^*(\varphi_1)-N-\pi_2^*(\varphi_2)}c(-\pi_1^*(G))\\ \leq&(\int_0^{+\infty}c(t_1)e^{-t_1}dt_1)\sum_{j=1}^{m_1}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\tilde{\psi}_2(z_j,w)-\varphi_2}<+\infty.\end{aligned}\tag{3.21}$$

It follows from $(F_{t,l,s}-F,(z_j,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)))_{(z_j,y)}$ for any $(z_j,y)\in Z_0\cap\tilde{M}_s$, Lemma 2.25 and the compactly convergence of all the sequences, we know that

$$(\tilde{F}-F,(z_j,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)))_{(z_j,y)}$$

for any $y\in Y$.

Lemma 3.9 is proved. □

Let Ω_j be an open Riemann surface, which admits a nontrivial Green function G_{Ω_j} for any $1\leq j\leq n_1$. Let Y be an n_2-dimensional weakly pseudoconvex Kähler manifold, and let K_Y be the canonical (holomorphic) line bundle on Y. Let $M=\left(\prod_{1\leq j\leq n_1}\Omega_j\right)\times Y$ be an $n-$dimensional complex manifold, where $n=n_1+n_2$, and let K_M be the canonical (holomorphic) line bundle on M. Let π_1, $\pi_{1,j}$ and π_2 be the natural projections from M to $\prod_{1\leq j\leq n_1}\Omega_j$, Ω_j and Y respectively.

Let $\tilde{M}\subset M$ be an $n-$dimensional weakly pseudoconvex Kähler manifold satisfying that $Z_0\subset\tilde{M}$. Let F be a holomorphic $(n,0)$ form on a neighborhood $U_0\subset\tilde{M}$ of Z_0.

Let $Z_j = \{z_{j,k} : 1 \le k < \tilde{m}_j\}$ be a discrete subset of Ω_j for any $j \in \{1, \ldots, n_1\}$, where $\tilde{m}_j \in \mathbb{Z}_{\ge 2} \cup \{+\infty\}$. Denote that $Z_0 := \left(\prod_{1\le j\le n_1} Z_j\right) \times Y$.

Let $p_{j,k}$ be a positive number for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$, which satisfies that $\sum_{1\le k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \le j \le n_1$. Denote that

$$G := \max_{1\le j\le n_1} \left\{ 2 \sum_{1\le k<\tilde{m}_j} p_{j,k} \pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k})) \right\}.$$

Let $N \le 0$ be a plurisubharmonic function on $\tilde{M}$. Denote $\psi := G + N$.

Let φ_X be a Lebesgue measurable function on $\prod_{1\le j\le n_1} \Omega_j$. Assume that $\pi_1^*(\varphi_X) + N$ is a plurisubharmonic function on $\tilde{M}$ and $(\pi_1^*(\varphi_X) + N)|_{Z_0} \not\equiv -\infty$. Denote $\Phi = \pi_1^*(\varphi_X) + N$. Let φ_Y be a plurisubharmonic function on Y, and

$$\varphi := \pi_1^*(\varphi_X) + \pi_2^*(\varphi_Y)$$

on M.

Let $w_{j,k}$ be a local coordinate on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ for any $1 \le j \le n_1$ and $1 \le k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \ne k'$.

Denote that $\tilde{I}_1 := \{\beta = (\beta_1, \ldots, \beta_{n_1}) : 1 \le \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \ldots, n_1\}\}$, $V_\beta := \prod_{1\le j\le n_1} V_{z_{j,\beta_j}}$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n_1,\beta_{n_1}})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n_1,\beta_{n_1}}) \in \prod_{1\le j\le n_1} \Omega_j$ for any $\beta = (\beta_1, \ldots, \beta_{n_1}) \in \tilde{I}_1$. Denote that $E_\beta := \left\{\alpha = (\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ and $\tilde{E}_\beta := \left\{\alpha = (\alpha_1, \ldots, \alpha_{n_1}) : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} \ge 1 \,\&\, \alpha_j \in \mathbb{Z}_{\ge 0}\right\}$ for any $\beta \in \tilde{I}_1$. Let U_0 be an open neighborhood of Z_0 in $\tilde{M}$. Let f be a holomorphic $(n, 0)$ form on U_0 such that

$$f = \sum_{\alpha\in\tilde{E}_\beta} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(f_{\alpha,\beta})$$

on $U_0 \cap (V_\beta \times Y)$, where $f_{\alpha,\beta}$ is a holomorphic $(n_2, 0)$ form on Y for any $\alpha \in \tilde{E}_\beta$ and $\beta \in \tilde{I}_1$. Denote that

$$c_{j,k} := \exp \lim_{z\to z_{j,k}} \left(\frac{\sum_{1\le k_1<\tilde{m}_j} p_{j,k_1} G_{\Omega_j}(z, z_{j,k_1})}{p_{j,k}} - \log |w_{j,k}(z)| \right)$$

for any $j \in \{1, \ldots, n\}$ and $1 \le k < \tilde{m}_j$ (following from Lemma 2.18 and Lemma 2.19, we get that the above limit exists).

Lemma 3.11. *Let c be a positive function on $(0, +\infty)$ such that $\int_0^{+\infty} c(t)e^{-t}dt < +\infty$ and $c(t)e^{-t}$ is decreasing on $(0, +\infty)$. Assume that $f_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)_y$ for any $y \in Y$, where $\alpha \in \tilde{E}_\beta \backslash E_\beta$ and $\beta \in \tilde{I}_1$, and*

$$\sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - (N+\pi_1^*(\varphi_X))(z_\beta, w')}}{\prod_{1\le j\le n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j+2}} < +\infty. \tag{3.22}$$

Then there exists a holomorphic $(n,0)$ *form* F *on* $\tilde{M}$ *satisfying that* $\big(F-f,(z,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y))\big)_{(z,y)}$ *for any* $(z,y)\in Z_0$ *and*

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Remark 3.12. *If we don't assume that* $f_{\alpha,\beta}\in\mathcal{I}(\varphi_Y)_y$ *for any* $y\in Y$*, where* $\alpha\in\tilde{E}_\beta\backslash E_\beta$ *and* $\beta\in\tilde{I}_1$*, we can still find a holomorphic* $(n,0)$ *form* F *on* $\tilde{M}$ *satisfying that* $\big(F-f,(z,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G)\big)_{(z,y)}$ *for any* $(z,y)\in Z_0$ *and*

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$$

as long as (3.22) *holds.*

Proof of Lemma 3.11. As $\psi=G+N\le G$, we know that $c(-\psi)e^{-\varphi}\le c(-G)e^{-\varphi-N}$. To prove Lemma 3.11, it suffice to prove that there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,z)\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y)))_z$ for any $z\in Z_0$ and

$$\int_{\tilde{M}}|F|^2e^{-\varphi-N}c(-G)\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

The following Remark shows that it suffices to prove Proposition 3.11 for the case $\tilde{m}_j<+\infty$ for any $j\in\{1,\dots,n_1\}$.

Remark 3.13. *Assume that Proposition 3.11 holds for the case* $\tilde{m}_j<+\infty$ *for any* $j\in\{1,\dots,n_1\}$*. For any* $j\in\{1,\dots,n_1\}$*, it follows from Lemma 2.21 that there exists a sequence of Riemann surfaces* $\{\Omega_{j,l}\}_{l\in\mathbb{Z}_{\ge1}}$*, which satisfies that* $\Omega_{j,l}\Subset\Omega_{j,l+1}\Subset\Omega_j$ *for any* l*,* $\cup_{l\in\mathbb{Z}_{\ge1}}\Omega_{j,l}=\Omega_j$ *and* $\{G_{\Omega_{j,l}}(\cdot,z)-G_{\Omega_j}(\cdot,z)\}_{l\in\mathbb{Z}_{\ge1}}$ *is decreasingly convergent to* 0 *with respect to* l *for any* $z\in\Omega_j$*. As* Z_j *is a discrete subset of* Ω_j*,* $Z_{j,l}:=\Omega_{j,l}\cap Z_j$ *is a set of finite points. Denote that* $\tilde{M}_l:=\Big((\prod_{1\le j\le n_1}\Omega_{j,l})\times Y\Big)\cap\tilde{M}$ *and* $G_l:=\max_{1\le j\le n_1}\Big\{\pi_{1,j}^*\Big(\sum_{z_{j,k}\in Z_{j,l}}2p_{j,k}G_{\Omega_{j,l}}(\cdot,z_{j,k})\Big)\Big\}$ *on* $\tilde{M}_l$*. Note that* $\tilde{M}_l$ *is weakly pseudoconvex Kähler manifold. Denote that*

$$c_{j,k,l}=\exp\lim_{z\to z_{j,k}}\left(\frac{\sum_{z_{j,k_1}\in Z_{j,l}}p_{j,k_1}G_{\Omega_{j,l}}(z,z_{j,k_1})}{p_{j,k}}-\log|w_{j,k}(z)|\right)$$

for any $1\le j\le n_1$*,* $l\in\mathbb{Z}_{\ge1}$ *and* $1\le k<\tilde{m}_j$ *satisfying* $z_{j,k}\in Z_{j,l}$*. Hence* $c_{j,k,l}$ *is decreasingly convergent to* $c_{j,k}$ *with respect to* l*,* G_l *is decreasingly convergent to* G *with respect to* l *and* $\cup_{l\in\mathbb{Z}_{\ge1}}\tilde{M}_l=\tilde{M}$*.*

Then there exists a holomorphic $(n,0)$ *form* F_l *on* $\tilde{M}_l$ *such that* $(F_l-f,(z_\beta,y))\in(\mathcal{O}(K_{\tilde{M}_l})\otimes(\mathcal{I}(G_l)+\pi_2^*(\varphi_Y))_{(z_\beta,y)}=(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y)))_{(z_\beta,y)}$ *for any* $\beta\in$

$\{\tilde\beta\in\tilde I_1 : z_{\tilde\beta}\in\prod_{1\le j\le n_1}\Omega_{j,l}\}$ *and* $y\in Y$, *and* F_l *satisfies*

$$\begin{aligned}&\int_{\tilde M_l}|F_l|^2e^{-\varphi-N}c(-G_l)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\{\tilde\beta\in\tilde I_1:z_{\tilde\beta}\in\prod_{1\le j\le n_1}\Omega_{j,l}\}}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j,l}^{2\alpha_j+2}}\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\\ <&+\infty.\end{aligned}$$

Since $\psi\le\psi_l$ *and* $c(t)e^{-t}$ *is decreasing on* $(0,+\infty)$, *we have*

$$\begin{aligned}&\int_{\tilde M_l}|F_l|^2e^{-\varphi-N-G_l+G}c(-G)\\ \le&\int_{\tilde M_l}|F_l|^2e^{-\varphi-N}c(-G_l)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}\tag{3.23}$$

Note that ψ *is continuous on* $M\backslash Z_0$, ψ_l *is continuous on* $M_l\backslash Z_0$ *and* Z_0 *is a closed complex submanifold of* M. *For any compact subset* K *of* $M\backslash Z_0$, *there exist* $l_K>0$ *such that* $K\Subset M_{l_K}$ *and* $C_K>0$ *such that* $\frac{e^{\varphi+N+G_l-G}}{c(-G)}\le C_K$ *for any* $l\ge l_K$. *It follows from Lemma 2.24 and the diagonal method that there exists a subsequence of* $\{F_l\}$, *denoted still by* $\{F_l\}$, *which is uniformly convergent to a holomorphic* $(n,0)$ *form* F *on any compact subset of* M. *It follows from Fatou's Lemma and inequality* (3.23) *that*

$$\begin{aligned}\int_{\tilde M}|F|^2e^{-\varphi-N}c(-G)=&\int_{\tilde M}\lim_{l\to+\infty}|F_l|^2e^{-\varphi-N-G_l+G}c(-G)\\ \le&\liminf_{l\to+\infty}\int_{\tilde M_l}|F_l|^2e^{-\varphi-N-G_l+G}c(-G)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}$$

Since $\{F_l\}$ *is uniformly convergent to* F *on any compact subset of* $\tilde M$ *and* $(F_l-f,(z_\beta,y))\in(\mathcal O(K_{\tilde M})\otimes\mathcal I(G+\pi_2^*(\varphi_Y)))_{(z_\beta,y)}$ *for any* $\beta\in\left\{\tilde\beta\in\tilde I_1:z_{\tilde\beta}\in\prod_{1\le j\le n_1}\Omega_{j,l}\right\}$ *and* $y\in Y$, *it follows from Lemma 2.25 that* $(F-f,(z_\beta,y))\in(\mathcal O(K_{\tilde M})\otimes\mathcal I(G+\pi_2^*(\varphi_Y)))_{(z_\beta,y)}$ *for any* $\beta\in\tilde I_1$ *and* $y\in Y$.

In the following, we assume that $\tilde m_j<+\infty$ for any $1\le j\le n_1$. Denote that $m_j=\tilde m_j-1$.

As $\tilde M$ is a weakly pseudoconvex Kähler manifold, there exists a sequence of weakly pseudoconvex Kähler manifolds $\tilde M_s$ satisfying $\tilde M_1\Subset\tilde M_2\cdots\Subset\tilde M_s\Subset\cdots\tilde M$ and $\cup_{s\in\mathbb N^+}\tilde M_s=\tilde M$.

Recall that $\Phi=\pi_1^*(\varphi_X)+N\in Psh(M)$. Denote $\Phi_l=\max\{\Phi,-l\}$, where l is a positive integer. We note that Φ_l is a bounded plurisubharmonic function on $\tilde M$.

It follows from Lemma 2.18 and Lemma 2.19 that there exists a local coordinate $\tilde{w}_{j,k}$ on a neighborhood $\tilde{V}_{z_{j,k}} \Subset V_{z_{j,k}}$ of $z_{j,k}$ satisfying $\tilde{w}_{j,k}(z_{j,k}) = 0$ and

$$|\tilde{w}_{j,k}| = \exp\left(\frac{\sum_{1\le k_1\le m_j} p_{j,k_1} G_{\Omega_j}(\cdot, z_{j,k_1})}{p_{j,k}}\right)$$

on $\tilde{V}_{z_{j,k}}$.

Denote that $\tilde{V}_\beta := \prod_{1\le j\le n_1} \tilde{V}_{j,\beta_j}$ for any $\beta \in \tilde{I}_1$. Let $\tilde{f}$ be a holomorphic $(n,0)$ form on $(\cup_{\beta\in\tilde{I}_1}\tilde{V}_\beta \times Y)\cap U_0$ satisfying

$$\tilde{f} = \sum_{\alpha\in E_\beta} c_{\alpha,\beta}\pi_1^*(\tilde{w}_\beta^\alpha d\tilde{w}_{1,\beta_1}\wedge d\tilde{w}_{2,\beta_2}\wedge\ldots\wedge d\tilde{w}_{n_1,\beta_{n_1}})\wedge\pi_2^*(f_{\alpha,\beta})$$

on $(\tilde{V}_\beta\times Y)\cap U_0$, where $c_{\alpha,\beta} = \prod_{1\le j\le n_1}\left(\lim_{z\to z_{j,\beta_j}}\frac{w_{j,\beta_j}(z)}{\tilde{w}_{j,\beta_j}(z)}\right)^{\alpha_j+1}$. It follows from $f_{\alpha,\beta}\in\mathcal{I}(\varphi_Y)_y$ for any $y\in Y$, where $\alpha\in\tilde{E}_\beta$ and $\beta\in\tilde{I}_1$, and Lemma 3.5 that

$$(f-\tilde{f},(z,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y)))_{(z,y)} \tag{3.24}$$

for any $(z,y)\in Z_0$.

Denote that $G_1 := \max_{1\le j\le n_1}\left\{\tilde{\pi}_j^*\left(2\sum_{1\le k<\tilde{m}_j} p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$ on $\prod_{1\le j\le n_1}\Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\le j\le n_1}\Omega_j$ to Ω_j and $G=\pi_1^*(G_1)$. It follows from Lemma 2.19 and Lemma 2.20 that there exists $t_0>0$ such that $\{G_1<-t_0\}\Subset\cup_{\beta\in\tilde{I}_1}\tilde{V}_\beta$, which implies that $\int_{\{G<-t\}\cap\tilde{M}_s}|\tilde{f}|^2<+\infty$. It follows from (3.22), Φ_l is bounded and Fubini's theorem that we know

$$\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<G<-t\}}|\tilde{f}|^2e^{-G-\pi_2^*(\varphi_Y)-\Phi_l}<+\infty.$$

Using Lemma 2.23, there exists a holomorphic $(n,0)$ form $F_{l,s,t}$ on $\tilde{M}_s$ such that

$$\begin{aligned}&\int_{\tilde{M}_s}|F_{l,s,t}-(1-b_{t,1}(\psi))\tilde{f}|^2e^{-G-\pi_2^*(\varphi_Y)-\Phi_l+v_{t,1}(G)}c(-v_{t,1}(G))\\ \le&\left(\int_0^{t+1}c(t)e^{-t}dt\right)\int_{\tilde{M}_s}\mathbb{I}_{\{-t-1<G<-t\}}|\tilde{f}|^2e^{-G-\pi_2^*(\varphi_Y)-\Phi_l},\end{aligned} \tag{3.25}$$

where $t\ge t_0$. Note that $b_{t,1}(t_1)=0$ for large enough t_1, then $(F_{l,s,t}-\tilde{f},(z,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y)))_{(z,y)}$ for any $(z,y)\in Z_0\cap\tilde{M}_s$.

For any $(z_\beta,y)\in Z_0\cap\tilde{M}_s$, letting (U_y,w) be a small local coordinated open neighborhood of y and shrinking $\tilde{V}_\beta$ if necessary, we have $\tilde{V}_\beta\times U_y\Subset U_0\cap(V_\beta\times Y)$ for any $\beta\in\tilde{I}_1$. Assume that $F_j=\tilde{h}_j(w)dw$ on U_y, where $\tilde{h}_j$ is a holomorphic function on U_y and $dw=dw_1\wedge dw_2\wedge\ldots\wedge dw_n$. There exists $t_2>t_1$ such that when $t>t_2$, $(\{G_1<-t\}\times U_y)\cap(\tilde{V}_\beta\times U_y)\Subset(V_\beta\times Y)\cap\tilde{M}$, for any $\beta\in\tilde{I}_1$.

Now we consider

$$\begin{aligned}&\int_{(\cup_{\beta\in\tilde{I}_1}\tilde{V}_\beta)\times U_y}\mathbb{I}_{\{-t-1<G<-t\}}|\tilde{f}|^2e^{-G-\pi_2^*(\varphi_Y)-\Phi_l}\\ =&\sum_{\beta\in\tilde{I}_1}\int_{w'\in U_y}(\int_{\{-t-1<G_1<-t\}}|f(\tilde{w},w')|^2\frac{e^{-\Phi_l(\tilde{w},w')}}{\max_{1\le j\le n}\{|\tilde{w}_{j,\beta_j}|^{2p_{j,\beta_j}}\}})e^{-\varphi_Y}.\end{aligned} \tag{3.26}$$

Note that $|c_{\alpha,\beta}| = \frac{1}{\prod_{1\le j\le n_1} c_{j,\beta_j}^{\alpha_j+1}}$ and $\int_{w'\in Y_s} |f_{\alpha,\beta}|^2 e^{-\varphi_Y} < +\infty$. It follows from (3.26), Fatou's Lemma and Lemma 2.30 that we have

$$\begin{aligned}
&\limsup_{t\to+\infty} \int_{\cup_{\beta\in\tilde I_1}\tilde V_\beta\times U_y} \mathbb{I}_{\{-t-1<G<-t\}}|\tilde f|^2 e^{-G-\pi_2^*(\varphi_Y)-\Phi_l}\\
=&\limsup_{t\to+\infty} \sum_{\beta\in\tilde I_1}\int_{w'\in U_y}\Big(\int_{\{-t-1<G_1<-t\}}|f(\tilde w,w')|^2\frac{e^{-\Phi_l(\tilde w,w')}}{\max_{1\le j\le n}\{|\tilde w_{j,\beta_j}|^{2p_{j,\beta_j}}\}}\Big)e^{-\varphi_Y}\\
\le&\sum_{\beta\in\tilde I_1}\int_{w'\in U_y}\limsup_{t\to+\infty}\Big(\int_{\{-t-1<G_1<-t\}}|f(\tilde w,w')|^2\frac{e^{-\Phi_l(\tilde w,w')}}{\max_{1\le j\le n}\{|\tilde w_{j,\beta_j}|^{2p_{j,\beta_j}}\}}\Big)e^{-\varphi_Y}\\
\le&\sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{U_y}|f_{\alpha,\beta}|^2e^{-\varphi_Y-\Phi_l(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}\tag{3.27}$$

Note that y and U_y are arbitrarily chosen, $v_{t,1}(\psi)\ge\psi$ and $c(t)e^{-t}$ is decreasing. Combining inequalities (3.25) and (3.27), then we have

$$\begin{aligned}
&\int_{\tilde M_s}|F_{l,s,t}-(1-b_{t,1}(\psi))\tilde f|^2e^{-\pi_2^*(\varphi_Y)-\Phi_l}c(-G)\\
\le&\int_{\tilde M_s}|F_{l,s,t}-(1-b_{t,1}(\psi))\tilde f|^2e^{-G-\pi_2^*(\varphi_Y)-\Phi_l+v_{t,1}(G)}c(-v_{t,1}(G))\\
\le&\left(\int_0^{t+1}c(t_1)e^{-t_1}dt_1\right)\int_{\tilde M_s}\mathbb{I}_{\{-t-1<G<-t\}}|\tilde f|^2e^{-G-\pi_2^*(\varphi_Y)-\Phi_l}\\
\le&\left(\int_0^{t+1}c(t_1)e^{-t_1}dt_1\right)\sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde M_s}|f_{\alpha,\beta}|^2e^{-\varphi_Y-\Phi_l(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}\tag{3.28}$$

Note that G is continuous on $\tilde M\backslash Z_0$. For any open set $K\Subset\tilde M_s\backslash Z_0$, as $b_{t,1}(t_1)=1$ for any t_1 large enough and $c(t_2)e^{-t_2}$ is decreasing with respect to t_2, we get that there exists a constant $C_K>0$ such that

$$\int_K|(1-b_{t,1}(\psi))\tilde f|^2e^{-\pi_2^*(\varphi_Y)-\Phi_l}c(-G)\le C_K\int_{\{G<-t_1\}\cap K}|\tilde f|^2<+\infty$$

for any $t>t_1$, which implies that

$$\limsup_{t\to+\infty}\int_K|F_{l,s,t}|^2e^{-\pi_2^*(\varphi_Y)-\Phi_l}c(-G)<+\infty.$$

Using Lemma 2.24 and the diagonal method, we obtain that there exists a subsequence of $\{F_{l,s,t}\}_{t\to+\infty}$ denoted by $\{F_{l,s,t_m}\}_{m\to+\infty}$ uniformly convergent on any compact subset of $\tilde M_s\backslash Z_0$. As Z_0 is a closed complex submanifold of $\tilde M$, we obtain that $\{F_{l,s,t_m}\}_{m\to+\infty}$ is uniformly convergent to a holomorphic $(n,0)$ form $F_{l,s}$ on $\tilde M_s$ on any compact subset of $\tilde M_s$. Then it follows from inequality (3.28) and

Fatou's Lemma that

$$\begin{aligned}
&\int_{\tilde{M}_s} |F_{l,s}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l} c(-G) \\
=&\int_{\tilde{M}_s} \liminf_{m\to+\infty} |F_{l,s,t_m} - (1-b_{t_m,1}(\psi))\tilde{f}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l} c(-G) \\
\le&\liminf_{m\to+\infty} \int_{\tilde{M}_s} |F_{l,s,t_m} - (1-b_{t_m,1}(\psi))\tilde{f}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l} c(-G) \\
\le&\left(\int_0^{+\infty} c(t_1)e^{-t_1}dt_1\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s} |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\Phi_l(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \\
<&+\infty.
\end{aligned}$$

As $\Phi_l(z_\beta,w')$ is decreasingly convergent to $\Phi(z_\beta,w')$ for any $\beta\in\tilde{I}_1$, then we have

$$\begin{aligned}
&\limsup_{l\to+\infty} \int_{\tilde{M}_s} |F_{l,s}|^2 e^{-\pi_2^*(\varphi_Y)-\Phi_l} c(-G) \\
\le&\left(\int_0^{+\infty} c(t_1)e^{-t_1}dt_1\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s} |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\Phi(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \\
=&\left(\int_0^{+\infty} c(t_1)e^{-t_1}dt_1\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s} |f_{\alpha,\beta}|^2 e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \\
<&+\infty.
\end{aligned} \tag{3.29}$$

Note that G is continuous on $\tilde{M}\backslash Z_0$ and Z_0 is a closed complex submanifold of $\tilde{M}$. Using Lemma 2.24, we obtain that there exists a subsequence of $\{F_{l,s}\}_{l\to+\infty}$ (also denoted by $\{F_{l,s}\}_{l\to+\infty}$) uniformly convergent to a holomorphic $(n,0)$ form F_s on $\tilde{M}_s$ on any compact subset of $\tilde{M}_s$, which satisfies that

$$\begin{aligned}
&\int_{\tilde{M}_s} |F_s|^2 e^{-\pi_2^*(\varphi_Y)-N-\varphi_X} c(-G) \\
\le&\left(\int_0^{+\infty} c(t_1)e^{-t_1}dt_1\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s} |f_{\alpha,\beta}|^2 e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.
\end{aligned}$$

As $\cup_{s\in\mathbb{Z}_{\ge1}}\tilde{M}_s=\tilde{M}$, we have

$$\begin{aligned}
&\limsup_{s\to+\infty} \int_{\tilde{M}_s} |F_s|^2 e^{-\varphi-N} c(-\psi) \\
\le&\lim_{s\to+\infty} \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_{(\{z_\beta\}\times Y)\cap\tilde{M}_s} |f_{\alpha,\beta}|^2 e^{-\varphi_Y-(N+\varphi_X)(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \\
=&\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_{Y} |f_{\alpha,\beta}|^2 e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}} \\
<&+\infty.
\end{aligned} \tag{3.30}$$

Note that ψ is continuous on $\tilde{M}\backslash Z_0$, Z_0 is a closed complex submanifold of $\tilde{M}$ and $\cup_{s\in\mathbb{Z}_{\ge1}}\tilde{M}_s=\tilde{M}$. Using Lemma 2.24 and the diagonal method, we get that there exists a subsequence of $\{F_s\}$ (also denoted by $\{F_s\}$) uniformly convergent to a holomorphic $(n,0)$ form F on $\tilde{M}$ on any compact subset of $\tilde{M}$. Then it follows from inequality (3.30) and Fatou's Lemma that

$$\begin{aligned}\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)&=\int_{\tilde{M}}\liminf_{s\to+\infty}\mathbb{I}_{\tilde{M}_s}|F_{l'}|^2e^{-\varphi}c(-\psi)\\&\le\liminf_{s\to+\infty}\int_{\tilde{M}_s}|F_{l'}|^2e^{-\varphi}c(-\psi)\\&\le\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-(N+\pi_1^*(\varphi_X))(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}$$

Following from Lemma 2.25, we have $(F-f,(z,y))\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y)))_{(z,y)}$ for any $(z,y)\in Z_0$.

Thus, Proposition 3.11 holds. □

Proof of Remark 3.12. If we don't assume that $f_{\alpha,\beta}\in\mathcal{I}(\varphi_Y)_y$ for any $y\in Y$, where $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in\tilde{I}_1$, it follows from Lemma 3.5 that for any $(z,y)\in Z_0$, we will have

$$(f-\tilde{f},(z,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(G))_{(z,y)}. \tag{3.31}$$

Replace the formula (3.24) by $(f-\tilde{f},(z,y))\in(\mathcal{O}(K_M)\otimes\mathcal{I}(G))_{(z,y)}$. Then if (3.22) holds, by the same proof as Proposition 3.11, we know Remark 3.12 holds. □

3.3. **Other Calculations.** Let Ω be an open Riemann surface with nontrivial Green functions. Let $Z_\Omega=\{z_j:j\in\mathbb{N}_+\&j<\gamma\}$ be a subset of Ω of discrete points. Let Y be an $(n-1)-$dimensional weakly pseudoconvex Kähler manifold. Denote $M=\Omega\times Y$. Let π_1 and π_2 be the natural projections from M to Ω and Y respectively. Denote $Z_0:=Z_\Omega\times Y$. Denote $Z_j:=\{z_j\}\times Y$.

Let ψ be a plurisubharmonic function on M. It follows from Siu's decomposition theorem that

$$dd^c\psi=\sum_{j\ge1}2p_j[Z_j]+\sum_{i\ge1}\lambda_i[A_i]+R,$$

where $[Z_j]$ and $[A_i]$ are the currents of integration over an irreducible $(n-1)-$dimensional analytic set, and where R is a closed positive current with the property that $dimE_c(R)<n-1$ for every $c>0$. We assume that $p_j>0$ for any $1\le j<\gamma$.

Then $N:=\psi-\pi_1^*\big(\sum\limits_{j\ge1}2p_jG_\Omega(z,z_j)\big)$ is a plurisubharmonic function on M. We assume that $N\le0$ and $N|_{Z_j}$ is not identically $-\infty$ for any j.

Let φ_1 be a Lebesgue measurable function on Ω such that $\psi+\pi_1^*(\varphi)$ is a plurisubharmonic function on M. With Similar discussion as above, by Siu's decomposition theorem, we have

$$dd^c(\psi+\pi_1^*(\varphi))=\sum_{j\ge1}2\tilde{q}_j[Z_j]+\sum_{i\ge1}\tilde{\lambda}_i[\tilde{A}_i]+\tilde{R},$$

where $\tilde{q}_j\ge0$ for any $1\le j<\gamma$.

By Weierstrass theorem on open Riemann surfaces, there exists a holomorphic function g on Ω such that $ord_{z_j}(g)=q_j:=[\tilde{q}_j]$ for any $z_j\in Z_\Omega$ and $g(z)\neq0$ for any $z\notin Z_\Omega$, where $[q]$ equals to the integer part of the nonnegative real number q.

Then we know that there exists a plurisubharmonic function $\tilde{\psi}_2 \in Psh(M)$ such that

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2.$$

Let $\varphi_2 \in Psh(Y)$. Denote $\varphi := \pi_1^*(\varphi_1) + \pi_2^*(\varphi_2)$.

For $1 \le j < \gamma$, let $(V_j, \tilde{z}_j)$ be a local coordinated open neighborhood of z_j in Ω satisfying $V_j \Subset \Omega$, $\tilde{z}_j(z_j) = 0$ under the local coordinate and $V_j \cap V_k = \emptyset$ for any $j \ne k$. Denote $V_0 := \cup_{1\le j<\gamma} V_j$. We assume that $g = d_j \tilde{z}_j^{q_j} h_j(z)$ on V_j, where d_j is a constant, $h_j(z)$ is a holomorphic function on V_j and $h_j(z_j) = 1$.

Let $c(t)$ be a positive measurable function on $(0, +\infty)$ satisfying that $c(t)e^{-t}$ is decreasing and $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$.

Lemma 3.14. *Assume that $c(t)$ is increasing near $+\infty$. Let F be a holomorphic $(n,0)$ form on M such that $F = \sum_{l=k_j}^{+\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$ on $V_j \times Y$ $(1 \le j < \gamma)$, where k_j is a nonnegative integer, $F_{j,l}$ is a holomorphic $(n-1,0)$ form on Y, for any l, j, and $F_j := F_{j,k_j} \not\equiv 0$ on Y. Denote that*

$$I_F := \{j : k_j + 1 - q_j \le 0 \& 1 \le j < \gamma\}.$$

Assume that

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} < +\infty.$$

Then $k_j + 1 - q_j = 0$ for any $j \in I_F$, and

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \ge \sum_{j\in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \tag{3.32}$$

Proof. As $\tilde{\psi}_2$ and N are upper semi-continuous functions on M, there exists continuous functions $\tilde{\psi}_{2,\alpha}$ and N_β on M decreasingly convergent to $\tilde{\psi}_2$ and N respectively.

There exists some $t_1 \ge 0$ such that $c(t)$ is increasing on $[t_1, +\infty)$. Recall that $\psi = \pi_1^*(\sum\limits_{j\ge1} 2p_j \pi_1^*(G_\Omega(z, z_j))) + N$ and denote $G = \sum\limits_{j\ge1} 2p_j \pi_1^*(G_\Omega(z, z_j))$. For any $t > t_1$,

$$\begin{aligned}
&\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) \\
=& \int_{\{G+N<-t\}} |F|^2 e^{-2\pi_1^*(\log|g|) - \tilde{\psi}_2 + \pi_1^*(G) + N - \pi_2^*(\varphi_2)} c(-(G+N)) \\
\ge& \int_{\{G+N_\beta<-t\}} |F|^2 e^{-2\pi_1^*(\log|g|) - \tilde{\psi}_{2,\alpha} + \pi_1^*(G) + N - \pi_2^*(\varphi_2)} c(-(G+N_\beta)).
\end{aligned} \tag{3.33}$$

For any $y \in Y$, let (U_y, w) be a local coordinated open neighborhood of y in Y satisfying $U_y \Subset Y$. We assume that $F = \tilde{z}_j^{k_j} \tilde{h}_j(\tilde{z}_j, w) d\tilde{z}_j \wedge dw$ on $V_j \times U_y$.

Let $m \in \mathbb{N}$. For any $0 < \epsilon < \frac{1}{2}$, let s_0 be large enough and shrink U_y if necessary such that,

(1) for any $j \in \{1, 2, \ldots, m\}$,

$$\tilde{V}_j := \{|\tilde{z}_j| < s_0, z \in V_j\} \Subset V_j,$$

(2) for any $j \in \{1, 2, \ldots, m\}$ and any $w \in U_y$, we have

$$\sup_{\tilde{z}\in\tilde{V}_j} |\tilde{\psi}_{2,\alpha}(\tilde{z}, w) - \tilde{\psi}_{2,\alpha}(z_0, w)| < \epsilon.$$

(3) Denote $H_j(\tilde{z}_j, w) := G - 2p_j \log|\tilde{z}_j| + N_\beta(\tilde{z}_j, w) + \epsilon$ on $\tilde{V}_j \times U_y$. Then for any

$w \in U_y$, we have

$$\sup_{\tilde{z} \in \tilde{V}_j} |H(\tilde{z}_j, w) - H(z_j, w)| < \epsilon.$$

(4) Recall that $g = d_j \tilde{z}^{q_j} h_j(\tilde{z}_j)$ on V_j, where d_j is a constant, $h_j(\tilde{z})$ is a holomorphic function on V_j and $h(z_j) = 1$. We assume that

$$\sup_{\tilde{z} \in \tilde{V}_j} |h_j(\tilde{z}_j) - 1| < \epsilon.$$

(5) Denote that $2g_j(\tilde{z}_j) = G - 2p_j \log |\tilde{z}_j|$ on $\tilde{V}_j$. Note that g_j is a harmonic function on $\tilde{V}_j$.

It follows from Lemma 2.20 that there exists $t' > t_1$ such that

$$\Big(\{G + N_\beta < -t\} \cap (\tilde{V}_j \times U_y)\Big) \Subset \tilde{V}_j \times U_y,$$

for any $t > t'$. We also have $G + N_\beta \le 2p_j \log |\tilde{z}_j| + H_j(z_j, w)$ on $V_j \times U_y$.

Following from (3.33), for any $t > t'$, direct calculation shows that

$$\begin{aligned}
&\int_{\{\psi<-t\}\cap(V_0\times U_y)} |F|^2 e^{-\varphi} c(-\psi) \\
\ge &\int_{\{G+N_\beta<-t\}\cap(V_0\times U_y)} |F|^2 e^{-2\pi_1^*(\log|g|)-\tilde{\psi}_{2,\alpha}+\pi_1^*(G)+N-\pi_2^*(\varphi_2)} c(-(G+N_\beta)) \\
\ge &\sum_{j=1}^m \int_{\{2p_j\log|\tilde{z}_j|+H_j(z_j,w)<-t\}\cap(\tilde{V}_j\times U_y)} |\tilde{z}_j|^{2k_j+2p_j-2q_j} \frac{|\tilde{h}_j(\tilde{z}_j,w)|^2}{|d_j|^2|h_j(\tilde{z}_j)|^2} \times \\
&e^{-\tilde{\psi}_{2,l,\alpha}(z_j,w)-\epsilon+2g_j(\tilde{z}_j)+N-\pi_2^*(\varphi_2)} c(-2p_j\log|\tilde{z}_j| - H_j(z_j,w)) \\
\ge &\sum_{j=1}^m \int_{w\in U_y} \Big(\int_{\{2p_j\log|\tilde{z}_j|+H_j(z_j,w)<-t\}} |\tilde{h}_j(\tilde{z}_j,w)|^2 |\tilde{z}_j|^{2k_j+2p_j-2q_j} e^{2g_j(\tilde{z}_j)+N} \times \\
&c(-2p_j\log|\tilde{z}_j| - H_j(z_j,w))|d\tilde{z}_j|^2\Big) \frac{e^{-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)}}{|d|^2|1+\epsilon|^2} |dw|^2 \\
= &\sum_{j=1}^m \int_{w\in U_y} \Big(2\int_0^{e^{\frac{-t-H_j(z_j,w)}{2p_j}}} \int_0^{2\pi} |\tilde{h}_j(re^{-\theta},w)|^2 r^{2k_j+2p_j-2q_j+1} e^{2g_j(re^{i\theta})+N(re^{i\theta},w)} \times \\
&c(-2p_j\log r - H_j(z_j,w))d\theta dr\Big) \frac{e^{-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)}}{|d|^2|1+\epsilon|^2} |dw|^2 \\
\ge &\sum_{j=1}^m \int_{w\in U_y} 4\pi\Big(\int_0^{e^{\frac{-t-H_j(z_j,w)}{2p_j}}} r^{2k_j+2p_j-2q_j+1} c(-2p_j\log r - H_j(z_j,w))\Big) \times \\
&|\tilde{h}_j(z_j,w)|^2 e^{2g_j(z_j)+N(z_j,w)} \frac{e^{-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)}}{|d|^2|1+\epsilon|^2} |dw|^2 \\
= &\sum_{j=1}^m \frac{2\pi}{p_j|d_j|^2|1+\epsilon|^2} \left(\int_t^{+\infty} c(s) e^{-(\frac{k_j+1-q_j}{p_j}+1)s} ds\right) \times \\
&\int_{w\in U_y} |\tilde{h}_j(z_j,w)|^2 e^{-(\frac{k_j+1-q_j}{p_j}+1)H_j(z_j,w)} e^{2g_j(z_j)+N(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)} |dw|^2.
\end{aligned} \tag{3.34}$$

As $\int_0^{+\infty} c(s)e^{-s}ds < +\infty$, hence we have

$$
\begin{aligned}
&\frac{\int_{\{\psi<-t\}\cap(V_0\times U_y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \\
\geq &\sum_{j=1}^{m} \frac{2\pi}{p_j|d_j|^2|1+\epsilon|^2} \left(\frac{\int_t^{+\infty} c(s)e^{-(\frac{k+1-q_2}{p}+1)s}ds}{\int_t^{+\infty} c(s)e^{-s}ds} \right) \times \\
&\int_{w\in U_y} |\tilde{h}_j(z_j,w)|^2 e^{-(\frac{k_j+1-q_j}{p_j}+1)H_j(z_j,w)} e^{2g_j(z_j)+N(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\epsilon-\varphi_2(w)}|dw|^2.
\end{aligned} \tag{3.35}
$$

Denote

$$I_m := \{1 \leq j \leq m : k_j + 1 - q_j \leq 0\}.$$

Note that

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} < +\infty,$$

$N(z_j,w) \not\equiv -\infty$ and $|\tilde{h}(z_j,w)|^2 \not\equiv 0$. It follows from Lemma 2.31 and (3.35) that we know $k_j + 1 - q_j = 0$ for any $j \in I_m$ and

$$
\begin{aligned}
&\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}\cap(V_0\times U_y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \\
\geq &\sum_{j\in I_m} \frac{2\pi}{p_j|d_j|^2} \int_{w\in U_y} |\tilde{h}_j(z_j,w)|^2 e^{-H(z_j,w)} e^{2g_j(z_j)+N(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\varphi_2(w)}|dw|^2 \\
= &\sum_{j\in I_m} \frac{2\pi}{p_j|d_j|^2} \int_{w\in U_y} |\tilde{h}_j(z_j,w)|^2 e^{-2g_j(z_j,0)+N_\beta(z_j,w)} e^{2g_j(z_j)+N(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\varphi_2(w)}|dw|^2 \\
= &\sum_{j\in I_m} \frac{2\pi}{p_j|d_j|^2} \int_{w\in U_y} |\tilde{h}_j(z_j,w)|^2 e^{N(z_j,w)-N_\beta(z_j,w)-\tilde{\psi}_{2,\alpha}(z_j,w)-\varphi_2(w)}|dw|^2.
\end{aligned}
$$

By Monotone convergence theorem, letting $\alpha \to +\infty$ and $\beta \to +\infty$, we have

$$
\begin{aligned}
&\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}\cap(V_0\times U_y)} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \\
\geq &\sum_{j\in I_m} \frac{2\pi}{p_j|d_j|^2} \int_{w\in U_y} |\tilde{h}_j(z_j,w)|^2 e^{-\tilde{\psi}_2(z_j,w)-\varphi_2(w)}|dw|^2.
\end{aligned} \tag{3.36}
$$

As y and U_y are arbitrarily chosen, it follows from Y is a weakly pseudoconvex Kähler manifold that we have

$$
\begin{aligned}
&\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \\
\geq &\sum_{j\in I_m} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j,w)-\varphi_2(w)}.
\end{aligned} \tag{3.37}
$$

Let $m \to +\infty$ and we have

$$\liminf_{t \to +\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s) e^{-s} ds} \geq \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j,w)-\varphi_2(w)}. \tag{3.38}$$

Especially, for any $j \in I_F$, we have

$$\int_Y |F_j|^2 e^{-\tilde{\psi}_2(z_j,w)-\varphi_2(w)} < +\infty.$$

□

Denote $M = \prod_{1 \leq j \leq n_1} \Omega_j \times Y$. Let $p_{j,k}$ be a positive number for any $1 \leq j \leq n_1$ and $1 \leq k < \tilde{m}_j$, which satisfies that $\sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \not\equiv -\infty$ for any $1 \leq j \leq n_1$. Recall that

$$G := \max_{1 \leq j \leq n_1} \left\{ 2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} \pi_{1,j}^* (G_{\Omega_j}(\cdot, z_{j,k})) \right\}.$$

Let $N \leq 0$ be a plurisubharmonic function on M satisfying that $N|_{Z_0} \not\equiv -\infty$. Denote $\psi := G + N$.

Let $\varphi_X := \sum_{1 \leq j \leq n_1} \varphi_j(z)$, where each φ_j is an upper semi-continuous function on Ω_j satisfying $\varphi_j(z_j) \neq -\infty$ for any $z_j \in \Omega_j$. We assume that $\pi_1^*(\varphi_X) + N$ is a plurisubharmonic function on M. Let φ_Y be a plurisubharmonic function on Y. Denote $\varphi := \pi_1^*(\varphi_X) + \pi_2^*(\varphi_Y)$.

It follows from Lemma 2.18 and Lemma 2.19 that there exists a local coordinate $w_{j,k}$ on a neighborhood $V_{z_{j,k}} \Subset \Omega_j$ of $z_{j,k} \in \Omega_j$ satisfying $w_{j,k}(z_{j,k}) = 0$ and

$$\log |w_{j,k}| = \frac{1}{p_{j,k}} \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})$$

for any $j \in \{1, \ldots, n_1\}$ and $1 \leq k < \tilde{m}_j$, where $V_{z_{j,k}} \cap V_{z_{j,k'}} = \emptyset$ for any j and $k \neq k'$.

Denote that $\tilde{I}_1 := \{(\beta_1, \ldots, \beta_{n_1}) : 1 \leq \beta_j < \tilde{m}_j \text{ for any } j \in \{1, \ldots, n_1\}\}$, $V_\beta := \prod_{1 \leq j \leq n_1} V_{z_{j,\beta_j}}$ for any $\beta = (\beta_1, \ldots, \beta_n) \in \tilde{I}_1$ and $w_\beta := (w_{1,\beta_1}, \ldots, w_{n,\beta_n})$ is a local coordinate on V_β of $z_\beta := (z_{1,\beta_1}, \ldots, z_{n,\beta_n}) \in M$.

Let

$$G_1 = \max_{1 \leq j \leq n_1} \left\{ \tilde{\pi}_j^* \left(2 \sum_{1 \leq k < \tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k}) \right) \right\}$$

on $\prod_{1 \leq j \leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1 \leq j \leq n_1} \Omega_j$ to Ω_j. Note that $G = \pi_1^*(G_1)$.

Let F be a holomorphic $(n, 0)$ form on $\{\psi < -t_0\} \subset M$ for some $t_0 > 0$ satisfying $\int_{\{\psi<-t_0\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$. For any $\beta \in \tilde{I}_1$, it follows from Lemma 3.3 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on $V_\beta \times Y$.

Denote that $E_\beta := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} = 1\right\}$, $E_{1,\beta} := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} < 1\right\}$ and $E_{2,\beta} := \left\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1\leq j\leq n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} > 1\right\}$.

Assume that $c(t)$ is increasing near $+\infty$.

Lemma 3.15. *If* $\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} < +\infty$, *we have* $F_{\alpha,\beta} \equiv 0$ *for any* $\alpha \in E_{1,\beta}$ *and* $\beta \in \tilde{I}_1$, *and*

$$\liminf_{t\to+\infty} \frac{\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\leq j\leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta,w')}.$$

Proof. As φ_X is an upper semi-continuous functions on $X := \prod_{1\leq j\leq n_1} \Omega_j$ and N is an upper semi-continuous function on M, there exist continuous functions $\varphi_{X,l}$ and N_γ on X and M decreasingly convergent to φ_X and N respectively.

When t is large enough, $c(t)$ is increasing, then we have

$$\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) \geq \int_{\{G+N_\gamma<-t\}} |F|^2 e^{-\pi_1^*(\varphi_{X,l}) - \pi_2^*(\varphi_Y)} c(-G-N_\gamma).$$

As Y is a weakly pseudoconvex Kähler manifold, there exist open weakly pseudoconvex Kähler manifolds $Y_1 \Subset \ldots \Subset Y_s \Subset Y_{s+1} \Subset \ldots$ such that $\cup_{s\in\mathbb{Z}_{\geq 1}} Y_s = Y$.

Fix Y_s. For any $\beta \in \tilde{I}_1$, there exists $t_\beta > t_0$ such that $\{G + N_\gamma < -t_\beta\} \cap (V_\beta \times Y_s) \Subset (V_\beta \times Y_{s+1})$.

On $V_\beta \times Y_s$, for any $\epsilon > 0$, there exists t_ϵ large enough such that when $t > t_\epsilon$,
(1) for any $(w, w') \in \{G + N_\gamma < -t\}$,
$$|\varphi_{X,l}(w) - \varphi_{X,l}(z_\beta)| < \epsilon.$$
(2) Denote $H(w, w') := N_\gamma(w, w') + \epsilon$. For any $(w, w') \in \{G + N_\gamma < -t\}$,
$$|H(w, w') - H(z_\beta, w')| < \epsilon.$$
Then we have $G + H(z_\beta, w') \geq G + N_s(w, w')$, for any $(w, w') \in \{G + N_\gamma < -t_\beta\} \cap (V_\beta \times Y_s)$.

Let

$$G_1 = \max_{1\leq j\leq n_1} \left\{\tilde{\pi}_j^* \left(2 \sum_{1\leq k<\tilde{m}_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$$

on $\prod_{1\leq j\leq n_1} \Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\leq j\leq n_1} \Omega_j$ to Ω_j. Note that $G = \pi_1^*(G_1)$.

For any $t \geq \max\{t_\beta, t_\epsilon\}$, note that $\{G_1 < -t\} = \prod_{1\leq j\leq n_1} \left\{|w_{j,\beta_j}| < e^{-\frac{t}{2p_{j,\beta_j}}}\right\}$ and $F = \sum_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$ on $\{G_1 < -t\} \times Y$, then we have

$$\begin{aligned}
&\int_{\{\psi<-t\}\cap(V_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)\\
\geq&\int_{\{G+N_\gamma<-t\}\cap(V_\beta\times Y_s)}|F|^2e^{-\pi_1^*(\varphi_{X,l})-\pi_2^*(\varphi_Y)}c(-G-N_\gamma)\\
\geq&e^{-\epsilon}\int_{\{G+H(z_\beta,w')<-t\}\cap(V_\beta\times Y_s)}|F|^2e^{-\pi_1^*(\varphi_{X,l}(z_\beta,w'))-\pi_2^*(\varphi_Y)}c(-G-H(z_\beta,w'))\\
=&e^{-\epsilon}\sum_{\alpha\in\mathbb{Z}_{\geq0}^{n_1}}\int_{w'\in Y_s}\left(\int_{\{G_1+H(z_\beta,w')<-t\}}|w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}}|^2c(-G-H(z_\beta,w'))\right)\times\\
&e^{-\varphi_{X,l}(z_\beta)}|F_{\alpha,\beta}|^2e^{-\varphi_Y}.
\end{aligned}\tag{3.39}$$

Denote that $q_\alpha:=\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}-1$. It follows from Lemma 2.28 and inequality (3.39) that

$$\begin{aligned}
&\int_{\{\psi<-t\}\cap(V_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)\\
\geq&e^{-\varphi_{X,l}(z_\beta)-\epsilon}\sum_{\alpha\in\mathbb{Z}_{\geq0}^{n_1}}\left(\int_t^{+\infty}c(s)e^{-(q_\alpha+1)s}ds\right)\frac{(q_\alpha+1)(2\pi)^{n_1}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)}\int_{Y_s}|F_{\alpha,\beta}|^2e^{-\varphi_Y-(q_\alpha+1)H(z_\beta,w')}.
\end{aligned}$$

It follows from $\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}<+\infty$ and Lemma 2.31 that

$$F_{\alpha,\beta}\equiv0$$

for any α satisfying $q_\alpha<0$ and

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap(V_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\geq e^{-\varphi_{X,l}(z_\beta)-\epsilon}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{Y_s}|F_{\alpha,\beta}|^2e^{-\varphi_Y-H(z_\beta,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)}.$$

Note that $H(w,w'):=N_\gamma(w,w')+\epsilon$. Letting $\epsilon\to0$, $\gamma\to+\infty$ and $s\to+\infty$, we have

$$\begin{aligned}
&\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap(V_\beta\times Y)}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\\
\geq&\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}.
\end{aligned}\tag{3.40}$$

Note that $V_\beta\cap V_{\tilde\beta}=\emptyset$ for any $\beta\neq\tilde\beta$ and $\{\psi_1<-t_\beta\}\cap V_\beta\Subset V_\beta$ for any $\beta\in\tilde I_1$. It follows from inequality (3.40) that

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\geq\sum_{\beta\in\tilde I_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}.$$

Thus, Lemma 3.15 holds. □

Let $\tilde M$ be an open complex submanifold of M satisfying that $Z_0=\{z_\beta:\beta\in\tilde I_1\}\times Y\subset\tilde M$, and let $K_{\tilde M}$ be the canonical (holomorphic) line bundle on M_1. Let F_1 be a holomorphic $(n,0)$ form on $\{\psi<-t_0\}\cap\tilde M$ for $t_0>0$ satisfying that

$\int_{\{\psi<-t_0\}\cap\tilde{M}}|F_1|^2e^{-\varphi}c(-\psi)<+\infty$. For any $\beta\in\tilde{I}_1$, it follows from Lemma 3.4 that there exist a sequence of holomorphic $(n_2,0)$ forms $\{F_{\alpha,\beta}\}_{\alpha\in\mathbb{Z}_{\geq0}^{n_1}}$ on Y and an open subset U_β of $\{\psi<-t_0\}\cap\tilde{M}\cap(V_\beta\times Y)$ such that

$$F_1=\sum_{\alpha\in\mathbb{Z}_{\geq0}^{n_1}}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(F_{\alpha,\beta})$$

on U_β and

$$\int_K|F_{\alpha,\beta}|^2e^{-\varphi_Y}<+\infty$$

for any $\alpha\in\mathbb{Z}_{\geq0}^{n_1}$ and compact subset K of Y. Using the similar method in Lemma 3.15, we have the following Remark.

Remark 3.16. *If* $\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}<+\infty$, *we have* $F_{\alpha,\beta}\equiv0$ *for any* $\alpha\in E_{1,\beta}$ *and* $\beta\in\tilde{I}_1$, *and*

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\geq\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N}.$$

Proof. As φ_X is an upper semi-continuous functions on $X:=\prod_{1\leq j\leq n_1}\Omega_j$ and N is an upper semi-continuous function on M, there exist continuous functions $\varphi_{X,l}$ and N_γ on X and M decreasingly convergent to φ_X and N respectively.

When t is large enough, $c(t)$ is increasing, then we have

$$\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\geq\int_{\{G+N_\gamma<-t\}\cap\tilde{M}}|F|^2e^{-\pi_1^*(\varphi_{X,l})-\pi_2^*(\varphi_Y)}c(-G-N_\gamma).$$

As Y is a weakly pseudoconvex Kähler manifold, there exist open weakly pseudoconvex Kähler manifolds $Y_1\Subset\ldots\Subset Y_s\Subset Y_{s+1}\Subset\ldots$ such that $\cup_{s\in\mathbb{Z}_{\geq1}}Y_s=Y$.

For any $\beta\in\tilde{I}_1$ and any Y_s , there exists open subset $\hat{V}_\beta\Subset V_\beta$ and $t_{\beta,s}>t_0$ such that $\{G+N_\gamma<-t_{\beta,s}\}\cap(\hat{V}_\beta\times Y_s)\Subset(\hat{V}_\beta\times Y_{s+1})\Subset U_\beta$.

On $\hat{V}_\beta\times Y_s$, for any $\epsilon>0$, there exists t_ϵ large enough such that when $t>t_\epsilon$,
(1) for any $(w,w')\in\{G+N_\gamma<-t\}$,

$$|\varphi_{X,l}(w)-\varphi_{X,l}(z_\beta)|<\epsilon.$$

(2) Denote $H(w,w'):=N_\gamma(w,w')+\epsilon$. For any $(w,w')\in\{G+N_\gamma<-t\}$,

$$|H(w,w')-H(z_\beta,w')|<\epsilon.$$

Then we have $G+H(z_\beta,w')\geq G+N_s(w,w')$, for any $(w,w')\in\{G+N_\gamma<-t_\beta\}\cap(\hat{V}_\beta\times Y_s)$.

Recall that $G_1=\max_{1\leq j\leq n_1}\left\{\tilde{\pi}_j^*\left(2\sum_{1\leq k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$ on $\prod_{1\leq j\leq n_1}\Omega_j$, where $\tilde{\pi}_j$ is the natural projection from $\prod_{1\leq j\leq n_1}\Omega_j$ to Ω_j. Note that $G=\pi_1^*(G_1)$. For any $t\geq\max\{t_{\beta,s},t_\epsilon\}$, note that $\{G_1<-t\}=\prod_{1\leq j\leq n_1}\left\{|w_{j,\beta_j}|<e^{-\frac{t}{2p_{j,\beta_j}}}\right\}$

and $F=\sum_{\alpha\in\mathbb{Z}_{\ge0}^{n_1}}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(F_{\alpha,\beta})$ on U_β, then we have

$$\begin{aligned}&\int_{\{\psi<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)\\ \ge&\int_{\{G+N_\gamma<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\pi_1^*(\varphi_{X,l})-\pi_2^*(\varphi_Y)}c(-G-N_\gamma)\\ \ge&e^{-\epsilon}\int_{\{G+H(z_\beta,w')<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\pi_1^*(\varphi_{X,l}(z_\beta,w'))-\pi_2^*(\varphi_Y)}c(-G-H(z_\beta,w'))\\ =&e^{-\epsilon}\sum_{\alpha\in\mathbb{Z}_{\ge0}^{n_1}}\int_{w'\in Y_s}\left(\int_{\{G_1+H(z_\beta,w')<-t\}}|w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}}|^2c(-G-H(z_\beta,w'))\right)\times\\ &e^{-\varphi_{X,l}(z_\beta)}|F_{\alpha,\beta}|^2e^{-\varphi_Y}.\end{aligned}\tag{3.41}$$

Denote that $q_\alpha:=\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}-1$. It follows from Lemma 2.28 and inequality (3.41) that

$$\begin{aligned}&\int_{\{\psi<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)\\ \ge&e^{-\varphi_{X,l}(z_\beta)-\epsilon}\sum_{\alpha\in\mathbb{Z}_{\ge0}^{n_1}}\left(\int_t^{+\infty}c(s)e^{-(q_\alpha+1)s}ds\right)\frac{(q_\alpha+1)(2\pi)^{n_1}}{\prod_{1\le j\le n_1}(\alpha_j+1)}\int_{Y_s}|F_{\alpha,\beta}|^2e^{-\varphi_Y-(q_\alpha+1)H(z_\beta,w')}.\end{aligned}$$

It follows from $\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}<+\infty$ and Lemma 2.31 that

$$F_{\alpha,\beta}\equiv0$$

for any α satisfying $q_\alpha<0$ and

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\ge e^{-\varphi_{X,l}(z_\beta)-\epsilon}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_{Y_s}|F_{\alpha,\beta}|^2e^{-\varphi_Y-H(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)}.$$

Note that $H(w,w'):=N_\gamma(w,w')+\epsilon$. Letting $\epsilon\to0$, $\gamma\to+\infty$ and $s\to+\infty$, we have

$$\begin{aligned}&\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap(\hat{V}_\beta\times Y_s)}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\\ \ge&\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\le j\le n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\le j\le n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}.\end{aligned}\tag{3.42}$$

Note that $V_\beta\cap V_{\tilde{\beta}}=\emptyset$ implies that $\hat{V}_\beta\cap\hat{V}_{\tilde{\beta}}=\emptyset$ for any $\beta\neq\tilde{\beta}$. It follows from inequality (3.42) that

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\ge\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\le j\le n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\le j\le n_1}(\alpha_j+1)}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}.$$

Thus, Remark 3.16 holds.

□

4. Proofs of Theorem 1.2, Proposition 1.4 and Proposition 1.6

4.1. **Proof of Theorem 1.2.**

Proof. We firstly give the proof of the sufficiency in Theorem 1.2.

As $\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1$ is a subharmonic function on Ω. By the construction of $g\in\mathcal{O}_\Omega$, we have $2u(z):=2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1-2\log|g|$ is a subharmonic function on Ω which satisfies $v(dd^cu,z)\in[0,1)$ for any $z\in Z_\Omega$. Then it follows from Lemma 3.7 that we know $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. It follows from Theorem 2.5 that we know the sufficiency part of Theorem 1.2 holds.

Now we prove the necessity part of Theorem 1.2.

Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$. Then according to Lemma 2.3, there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f)\in H^0(Z_0,\Big(\mathcal{O}(K_M)\otimes\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)\Big)|_{Z_0})$, and $G(t;c)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$ for any $t\geq0$. Then according to Lemma 2.3, Remark 2.4 and Lemma 2.32, we can assume that c is increasing near $+\infty$.

Following from Lemma 3.3, for any $j\in\{1,2,\dots,m\}$, we assume that

$$F=\sum_{l=k_j}^{\infty}\pi_1^*(\tilde{z}_j^l d\tilde{z}_j)\wedge\pi_2^*(F_{j,l})$$

on $V_j\times Y$, where $k_j\in\mathbb{N}$, $F_{j,l}$ is a holomorphic $(n-1,0)$ form on Y for any $l\geq k_j$, and $\tilde{F}_j:=F_{j,k_j}\not\equiv0$.

We firstly recall the construction of ψ and φ.

Recall that $\psi=\pi_1^*\big(\sum\limits_{j=1}^m 2p_jG_\Omega(z,z_j)\big)+N$ is a plurisubharmonic function on M. We assume that $N\leq0$ is a plurisubharmonic function on M and $N|_{Z_j}$ is not identically $-\infty$ for any j. φ_1 is a Lebesgue measurable function on Ω such that $\psi+\pi_1^*(\varphi)$ is a plurisubharmonic function on M. We also note that, by Siu's decomposition theorem and Weierstrass theorem on open Riemann surfaces, we have

$$\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\log|g|)+\tilde{\psi}_2,$$

where g is a holomorphic function on Ω and $\tilde{\psi}_2\in Psh(M)$. Denote $ord_{z_j}(g)=p_j$ and $d_j:=\lim_{z\to z_j}(g/\tilde{z}_j^{q_j})(z)$.

Now we prove that $\psi=\pi_1^*\big(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)\big)$. As $G(h^{-1}(r))$ is linear and c is increasing near $+\infty$, using Lemma 3.14, we can get that $k_j-q_j+1=0$ for any $j\in I_F$ and

$$\frac{\int_M|F|^2e^{-\varphi}c(-\psi)}{\int_0^{+\infty}c(s)e^{-s}ds}\geq\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)},\tag{4.1}$$

where $I_F:=\{j:k_j-q_j+1\leq0\}$. Especially, $\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}<+\infty$ and $\tilde{\psi}_2$ is not identically $-\infty$ on Z_j for any $j\in I_F$.

Let $V_j\times U_y$ be an open neighborhood of (z_j,y). Assume that $F=\tilde{z}_j^{k_j}h(\tilde{z}_j,w)d\tilde{z}_j\wedge dw$ on $V_j\times U_y$, where $h(z_j,w)dw=\tilde{F}_j$. Note that $h(z_j,w)$ is not identically zero on $\{z_j\}\times U_y$, there must exist $\hat{w}\in U_y$ such that $h(z_j,\hat{w})\neq0$. Then we know $|h(z_j,w)|^2$

has a positive lower bound on $\tilde{V}_j\times U_{\hat{w}}$, where $\tilde{V}_j\times U_{\hat{w}}$ is a small open neighborhood of $(z_j,\hat{w})$. Then, according to Lemma 3.1, $\psi\le\pi_1^*\big(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)\big)$ and $c(t)$ is increasing near $+\infty$, we know that for any $w\in U_{\hat{w}}$, we have $\int_{\tilde{V}_j}|\tilde{z}_j^{k_j}d\tilde{z}_j|^2e^{-\varphi_1}c(-\sum_{j=1}^m 2p_jG_\Omega(\cdot,z_j))<+\infty$. It follows from Lemma 3.2 that we have

$$\int_{\tilde{V}_j\times U_{\hat{w}}}|\pi_1^*(\tilde{z}_j^{k_j}d\tilde{z}_j)\wedge\pi_2^*(\tilde{F}_j)|^2e^{-\varphi}c\Big(-\pi_1^*(\sum_{j=1}^m 2p_jG_\Omega(\cdot,z_j))\Big)<+\infty.$$

Then by Lemma 3.6, we have

$$\big(F-\pi_1^*(\tilde{z}_j^{k_j}d\tilde{z}_j)\wedge\pi_2^*(\tilde{F}_j),(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j\in I_F$, $y\in Y$. And according to Lemma 3.6 we also have

$$\big(F,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j\in\{1,2,\ldots,m\}\setminus I_F$, $y\in Y$.

Denote that $\tilde{\psi}:=\pi_1^*\big(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)\big)$, $\tilde{\varphi}_1:=\pi_1^*(\varphi_1)+\psi-\tilde{\psi}$, and $\tilde{\varphi}:=\tilde{\varphi}_1+\pi_2^*(\varphi_2)$. Then according to Lemma 3.9 $\big(F=\pi_1^*(\tilde{z}_j^{k_j}dw_j)\wedge\pi_2^*(\tilde{F}_j)$ on $V_j\times Y$ for $j\in I_F$ and $F\equiv 0$ on $V_j\times Y$ for $j\notin I_F$ in Lemma 3.9$\big)$, there exists a holomorphic $(n,0)$ form $\tilde{F}$ on M such that $\big(\tilde{F}-\pi_1^*(\tilde{z}_j^{k_j}dw_j)\wedge\pi_2^*(\tilde{F}_j),(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j\in I_F$, $y\in Y$, $\big(\tilde{F},(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j\in\{1,2,\ldots,m\}\setminus I_F$, $y\in Y$, and

$$\int_M|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi})\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\tag{4.2}$$

Then $\big(\tilde{F}-F,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Combining inequality (4.1) with inequality (4.2), we have that

$$\int_M|\tilde{F}|^2e^{-\varphi}c(-\psi)\ge\int_M|F|^2e^{-\varphi}c(-\psi)\ge\int_M|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi}).\tag{4.3}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi\le\tilde{\psi}$, we have $\int_M|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi})\ge\int_M|\tilde{F}|^2e^{-\varphi}c(-\psi)$. Then all "$\ge$" in (4.3) should be "$=$". It follows from Lemma 2.22 that $N=0$ and $\psi=\pi_1^*\big(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)\big)$.

As $\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1$ is a subharmonic function on Ω. By the construction of $g\in\mathcal{O}_\Omega$, we have $2u(z):=2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1-2\log|g|$ is a subharmonic function on Ω which satisfies $v(dd^cu,z)\in[0,1)$ for any $z\in Z_0$. Then it follows from Lemma 3.7 that we know $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$.

Then it follows from Theorem 2.5 that we know Theorem 1.2 holds. □

4.2. **Proof of Proposition 1.4.**

Proof of Proposition 1.4. Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$. Then according to Lemma 2.3, there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f)\in H^0(Z_0,\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)|_{Z_0})$, and $G(t;c)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$ for any $t\ge 0$. Then according to Lemma 2.3, Remark 2.4 and Lemma 2.32, we can assume that c is increasing near $+\infty$.

Following from Lemma 3.3, for any $j \geq 1$, we assume that

$$F = \sum_{l=k_j}^{\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$$

on $V_j \times Y$, where $k_j \in \mathbb{N}$, $F_{j,l}$ is a holomorphic $(n-1,0)$ form on Y for any $l \geq k_j$, and $\tilde{F}_j := F_{j,k_j} \not\equiv 0$.

We firstly recall the construction of ψ and φ.

Recall that $\psi = \pi_1^*\big(\sum_{j\geq 1} 2p_j G_\Omega(z, z_j)\big) + N$ is a plurisubharmonic function on M. We assume that $N \leq 0$ is a plurisubharmonic function on M and $N|_{Z_j}$ is not identically $-\infty$ for any j. φ_1 is a Lebesgue measurable function on Ω such that $\psi + \pi_1^*(\varphi)$ is a plurisubharmonic function on M. We also note that, by Siu's decomposition theorem and Weierstrass theorem on open Riemann surface, we have

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2,$$

where g is a holomorphic function on Ω and $\tilde{\psi}_2 \in Psh(M)$. Denote $ord_{z_j}(g) = p_j$ and $d_j := \lim_{z\to z_j}(g/\tilde{z}_j^{q_j})(z)$.

Now we prove that $\psi_1 = 2\sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j)$. As $G(h^{-1}(r))$ is linear and c is increasing near $+\infty$, using Lemma 3.14, we can get that $k_j - q_j + 1 = 0$ for any $j \in I_F$ and

$$\frac{\int_M |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j\in I_F} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)}, \tag{4.4}$$

where $I_F := \{j : k_j - q_j + 1 \leq 0\}$. Especially, $\sum_{j\in I_F} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)} < +\infty$ and $\tilde{\psi}_2$ is not identically $-\infty$ on Z_j for any $j \in I_F$.

Let $V_j \times U_y$ be an open neighborhood of (z_j, y). Assume that $F = \tilde{z}_j^{k_j} h(\tilde{z}_j, w) d\tilde{z}_j \wedge dw$ on $V_j \times U_y$, where $h(z_j, w)dw = \tilde{F}_j$. Note that $h(z_j, w)$ is not identically zero on $\{z_j\} \times U_y$, there must exist $\hat{w} \in U_y$ such that $h(z_j, \hat{w}) \neq 0$. Then we know $|h(z_j, w)|^2$ has a positive lower bound on $\tilde{V}_j \times U_{\hat{w}}$, where $\tilde{V}_j \times U_{\hat{w}}$ is a small open neighborhood of $(z_j, \hat{w})$. Then, according to Lemma 3.1, $\psi \leq \pi_1^*(\sum_{j=1}^{\gamma} 2p_j G_\Omega(\cdot, z_j))$ and $c(t)$ is increasing near $+\infty$, we know that for any $w \in U_{\hat{w}}$, we have $\int_{\tilde{V}_j} |\tilde{z}_j^{k_j} d\tilde{z}_j|^2 e^{-\varphi_1} c(-\sum_{j=1}^m 2p_j G_\Omega(\cdot, z_j)) < +\infty$. It follows from Lemma 3.2 that we have

$$\int_{\tilde{V}_j \times U_{\hat{w}}} |\pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j)|^2 e^{-\varphi} c(-\pi_1^*(\sum_{j=1}^m 2p_j G_\Omega(\cdot, z_j))) < +\infty.$$

Then by Lemma 3.6, we have

$$\big(F - \pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j \in I_F$, $y \in Y$. And according to Lemma 3.6 we also have

$$\big(F, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j \in \{1, 2, \ldots, m\} \setminus I_F$, $y \in Y$.

Denote that $\tilde{\psi} := \pi_1^*\big(2\sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j)\big)$, $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$, and $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$. Then according to Lemma 3.9 $\big(F = \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j)$ on $V_j \times$

Y for $j \in I_F$ and $F \equiv 0$ on $V_j \times Y$ for $j \notin I_F$ in Lemma 3.9$\big)$, there exists a holomorphic $(n,0)$ form $\tilde{F}$ on M such that $\big(\tilde{F} - \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j \in I_F$, $y \in Y$, $\big(\tilde{F}, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j \in \{1, 2, \ldots, m\} \setminus I_F$, $y \in Y$, and

$$\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \leq \left(\int_0^{+\infty} c(s) e^{-s} ds\right) \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j, w)}. \quad (4.5)$$

Then $\big(\tilde{F} - F, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Combining inequality (4.4) with inequality (4.5), we have that

$$\int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi) \geq \int_M |F|^2 e^{-\varphi} c(-\psi) \geq \int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}). \quad (4.6)$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi \leq \tilde{\psi}$, we have $\int_M |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \geq \int_M |\tilde{F}|^2 e^{-\varphi} c(-\psi)$. Then all "$\geq$" in (4.6) should be "$=$". It follows from Lemma 2.22 that $N = 0$ and $\psi = \pi_1^*\big(2\sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j)\big)$.

As $\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j) + \varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j) + \varphi_1$ is a subharmonic function on Ω. By the construction of $g \in \mathcal{O}_\Omega$, we have $2u(z) := 2\sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j) + \varphi_1 - 2\log|g|$ is a subharmonic function on Ω which satisfies $v(dd^c u, z) \in [0,1)$ for any $z \in Z_0$. Then it follows from Lemma 3.7 that we know $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}\big(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$.

Then it follows from Proposition 2.6 that we know Proposition 1.4 holds. □

4.3. **Proof of Proposition 1.6.**

Proof. Assume that $\tilde{G}\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s} ds]$. Then according to Lemma 2.3, there exists a unique holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying $(F - f) \in H^0(Z_0, \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)|_{Z_0})$, and $\tilde{G}(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq 0$. Then according to Lemma 2.3, Remark 2.4 and Lemma 2.32, we can assume that c is increasing near $+\infty$.

We firstly recall the construction of ψ and φ.

Recall that $\psi = \pi_1^*\big(\sum_{j \geq 1} 2p_j G_\Omega(z, z_j)\big) + N$ is a plurisubharmonic function on $\tilde{M}$. We assume that $N \leq 0$ is a plurisubharmonic function on $\tilde{M}$ and $N|_{Z_j}$ is not identically $-\infty$ for any j. φ_1 is a Lebesgue measurable function on Ω such that $\psi + \pi_1^*(\varphi)$ is a plurisubharmonic function on $\tilde{M}$. We also note that, by Siu's decomposition theorem and Weierstrass theorem on open Riemann surface, we have

$$\psi + \pi_1^*(\varphi_1) = \pi_1^*(2\log|g|) + \tilde{\psi}_2,$$

where g is a holomorphic function on Ω and $\tilde{\psi}_2 \in Psh(\tilde{M})$. Denote $ord_{z_j}(g) = p_j$ and $d_j := \lim_{z \to z_j} (g/\tilde{z}_j^{q_j})(z)$.

Following from Lemma 3.4, for any $j \in \{1, 2, \ldots, m\}$, we assume that

$$F = \sum_{l=k_j}^{\infty} \pi_1^*(\tilde{z}_j^l d\tilde{z}_j) \wedge \pi_2^*(F_{j,l})$$

on $U_j \Subset (V_j \times Y) \cap \tilde{M}$ is a neighborhood of $Z_j := \{z_j\} \times Y$ in $\tilde{M}$, and $k_j \in \mathbb{N}$, $F_{j,l}$ is a holomorphic $(n-1,0)$ form on Y for any $l \geq k_j$, and $\tilde{F}_j := F_{j,k_j} \not\equiv 0$.

Let W be an open subset of Y such that $W \Subset Y$. Then for any j, $1 \leq j < \gamma$, there exists $r_{j,W} > 0$ such that $V_{j,W} \times W \subset \tilde{M}$, where $V_{j,W} := \{z \in \Omega : |\tilde{z}_j(z)| < r_{j,W}\}$.

As $\tilde{G}(h^{-1}(r))$ is linear and c is increasing near $+\infty$, using Lemma 3.14, we can get that $k_j - q_j + 1 = 0$ for any $j \in I_F$ and

$$\frac{\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_W |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)},$$

where $I_F := \{j : 1 \leq j < \gamma \ \& \ \tilde{k}_j + 1 - k_j \leq 0\}$.

By the arbitrariness of W, we have $\int_Y |\tilde{F}_j|^2 e^{-\varphi_2} < +\infty$ for any $j \in I_F$, and

$$\frac{\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)}{\int_0^{+\infty} c(s)e^{-s}ds} \geq \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)}. \tag{4.7}$$

Especially, we know $\sum_{j \in I_F} \frac{2\pi}{p_j|d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)} < +\infty$ and $\tilde{\psi}_2$ is not identically $-\infty$ on Z_j for any $j \in I_F$.

Let $V_j \times U_y$ be an open neighborhood of (z_j, y) in $\tilde{M}$. Assume that $F = \tilde{z}_j^{k_j} h(\tilde{z}_j, w) d\tilde{z}_j \wedge dw$ on $V_j \times U_y$, where $h(z_j, w)dw = \tilde{F}_j$. Note that $h(z_j, w)$ is not identically zero on $\{z_j\} \times U_y$, there must exist $\hat{w} \in U_y$ such that $h(z_j, \hat{w}) \neq 0$. Then we know $|h(z_j, w)|^2$ has a positive lower bound on $\tilde{V}_j \times U_{\hat{w}}$, where $\tilde{V}_j \times U_{\hat{w}} \Subset V_j \times U_y$ is a small open neighborhood of $(z_j, \hat{w})$. Then, according to Lemma 3.1, $\psi \leq \sum_{j=1}^{\gamma} 2p_j G_\Omega(\cdot, z_j)$ and $c(t)$ is increasing near $+\infty$, we know that for any $w \in U_{\hat{w}}$, we have $\int_{\tilde{V}_j} |\tilde{z}_j^{k_j} d\tilde{z}_j|^2 e^{-\varphi_1} c(-\sum_{j=1}^{\gamma} 2p_j G_\Omega(\cdot, z_j)) < +\infty$. It follows from Lemma 3.2 that we have

$$\int_{\tilde{V}_j \times U_{\hat{w}}} |\pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j)|^2 e^{-\varphi} c\big(-\pi_1^*(\sum_{j=1}^{\gamma} 2p_j G_\Omega(\cdot, z_j))\big) < +\infty.$$

Then by Lemma 3.6, we have

$$\big(F - \pi_1^*(\tilde{z}_j^{k_j} d\tilde{z}_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j \in I_F$, $y \in Y$. And according to Lemma 3.6 we also have

$$\big(F, (z_j, y)\big) \in \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$$

for any $j \in \{1, 2, \ldots, m\} \setminus I_F$, $y \in Y$.

Now we prove that $N \equiv 0$. Denote that $\tilde{\psi} := \pi_1^*(2\sum_{j=1}^{\gamma} p_j G_\Omega(\cdot, z_j))$, $\tilde{\varphi}_1 := \pi_1^*(\varphi_1) + \psi - \tilde{\psi}$, and $\tilde{\varphi} := \tilde{\varphi}_1 + \pi_2^*(\varphi_2)$. Then according to Lemma 3.9 $\big(F = \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j)$ on U_j for $j \in I_F$ and $F \equiv 0$ on U_j for $j \notin I_F$ in Lemma 3.9$\big)$, there exists a holomorphic $(n,0)$ form $\tilde{F}$ on $\tilde{M}$ such that $\big(\tilde{F} - \pi_1^*(\tilde{z}_j^{k_j} dw_j) \wedge \pi_2^*(\tilde{F}_j), (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j \in I_F$, $y \in Y$, $\big(\tilde{F}, (z_j, y)\big) \in \big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\pi_1^*(2\log|g|) + \pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j \geq 1 \& j \notin I_F$ and any $y \in Y$, and

$$\int_{\tilde{M}} |\tilde{F}|^2 e^{-\tilde{\varphi}} c(-\tilde{\psi}) \leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{j \in I_F} \frac{2\pi}{p_j |d_j|^2} \int_Y |F_j|^2 e^{-\varphi_2 - \tilde{\psi}_2(z_j,w)}. \tag{4.8}$$

Note that we have $\big(\tilde{F}-F,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$.

It follows from (4.7) and (4.10) that, we have

$$\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi)\geq\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\geq\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi}). \tag{4.9}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi\leq\tilde{\psi}$, we have $\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\tilde{\psi})\geq\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi)$. Then all "$\geq$" in (4.9) should be "$=$". It follows from Lemma 2.22 that $N=0$ and $\psi=\pi_1^*\big(2\sum_{j=1}^{\gamma}p_jG_\Omega(\cdot,z_j)\big)$.

Using Lemma 3.9 $\big(\tilde{M}\sim M$ and $F=\pi_1^*(\tilde{z}_j^{k_j}dw_j)\wedge\pi_2^*(\tilde{F}_j)$ for $j\in I_F$ and $F\equiv0$ for $j\notin I_F$ in Lemma 3.9$\big)$, there exists a holomorphic $(n,0)$ form $\hat{F}$ on M such that $\big(\hat{F}-\pi_1^*(\tilde{z}_j^{k_j}dw_j)\wedge\pi_2^*(\tilde{F}_j),(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j\in I_F$, $y\in Y$, $\big(\hat{F},(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $j\geq1\&j\notin I_F$ and any $y\in Y$, and

$$\int_M|\hat{F}|^2e^{-\varphi}c(-\psi)\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}. \tag{4.10}$$

Then $\big(\hat{F}-F,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$.

Now According to the choice of F, we have that

$$\begin{aligned}\tilde{G}(0)=&\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\leq\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \leq&\int_M|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \leq&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j\in I_F}\frac{2\pi}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned} \tag{4.11}$$

Combining inequality (4.7) with inequality (4.11), we get that

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)=\int_M|\hat{F}|^2e^{-\varphi}c(-\psi).$$

As $\tilde{F}\not\equiv0$, the above equality implies that $\tilde{M}=M$.

□

5. Proofs of Theorem 1.8, Theorem 1.10, Theorem 1.12 and Proposition 1.14

In this section, we prove Theorem 1.8, Theorem 1.10, Theorem 1.12 and Proposition 1.14.

5.1. **Proof of Theorem 1.8.**

Proof. We firstly give the proof of the sufficiency of Theorem 1.8. It follows from Theorem 2.9 that $G\big(h^{-1}(r);\mathcal{I}(\varphi+\psi)\big)$ is linear with respect to r.

When $N\equiv0$, then φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\leq j\leq n_1$, it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$.

Hence $G\big(h^{-1}(r);\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big)$ is linear with respect to r. The sufficiency of Theorem 1.8 is proved.

We prove the necessity part of Theorem 1.8.

Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$. Then according to Lemma 2.3, there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F-f)\in H^0(Z_0,\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big)|_{Z_0})$, and $G(t;c)=\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)$ for any $t\geq 0$. Then according to Lemma 2.3, Remark 2.4 and Lemma 2.32, we can assume that c is increasing near $+\infty$.

It follows from Lemma 3.3 that there exists a sequence of holomorphic $(n_2,0)$ forms $\{F_\alpha\}_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that

$$F=\sum_{\alpha\in\mathbb{Z}_{\geq 0}^{n_1}}\pi_1^*(w^\alpha dw_1\wedge\ldots\wedge dw_{n_1})\wedge\pi_2^*(F_\alpha)$$

on $V_0\times Y$.

Denote that $E_0:=\Big\{\alpha\in\mathbb{Z}_{\geq0}^{n_1}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_j}=1\Big\}$, $E_1:=\Big\{\alpha\in\mathbb{Z}_{\geq0}^{n_1}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_j}<1\Big\}$ and $E_2:=\Big\{\alpha\in\mathbb{Z}_{\geq0}^{n_1}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_j}>1\Big\}$.

Now we prove $N\equiv 0$. It follows from Lemma 3.15 that we have $F_\alpha\equiv 0$ for any $\alpha\in E_1$, and

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\geq\sum_{\alpha\in E_0}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_j)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_0,w')}. \tag{5.1}$$

Note that $\psi=\hat{G}+N\leq\hat{G}$, $c(t)$ is increasing near $+\infty$, φ_X is upper semi-continuous on $\prod_{j=1}^{n_1}\Omega_j$. When t is large enough, we have

$$\begin{aligned}&\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)\geq\int_{\{\hat{G}<-t\}}|F|^2e^{-\pi_1^*\left(-\sum_{1\leq j\leq n_1}\varphi_j(z_j)\right)-\pi_2^*(\varphi_Y)}c(-\hat{G})\\ \geq&C_0\int_{\{\hat{G}<-t\}}|F|^2e^{-\pi_2^*(\varphi_Y)}c(-\hat{G}),\end{aligned}$$

where $C_0>0$ is a constant, then it follows from $\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)<+\infty$ and Lemma 3.3 that for any $\alpha\in E_2:=\Big\{\alpha\in\mathbb{Z}_{\geq0}^{n_1}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_j}>1\Big\}$, we know $F_\alpha\in\mathcal{I}(\varphi_Y)$ for any $y\in Y$.

Recall that

$$\hat{G}=\max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*\big(G_{\Omega_j}(\cdot,z_j)\big)\right\}.$$

Denote $\tilde{\varphi}:=\varphi+\psi-\hat{G}$. It follows from Lemma 3.11 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on M satisfying that $(\hat{F}-F,z)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big)_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_M|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\\ \leq&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E_0}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_j)}\int_Y|f_\alpha|^2e^{-\varphi_Y-N(z_0,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}.\end{aligned}\tag{5.2}$$

It follows from (5.1) and (5.2) that we have

$$\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \leq \int_M |F|^2 e^{-\varphi} c(-\psi) \leq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi) \tag{5.3}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi \leq \hat{G}$, we have $\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \geq \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi)$. Then all "$\geq$" in (5.3) should be "$=$". It follows from Lemma 2.22 that we know $N \equiv 0$ and $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^* \big(G_{\Omega_j}(\cdot, z_j)\big)\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \leq j \leq n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1 \leq j \leq n_1} \{2p_j \pi_{1,j}^* \big(G_{\Omega_j}(\cdot, z_j)\big)\}$. It follows from Theorem 2.9 that the necessity part of Theorem 1.8 holds. □

5.2. **Proof of Theorem 1.10.**

Proof. We firstly give the proof of the sufficiency of Theorem 1.10. It follows from Theorem 2.10 that $G\big(h^{-1}(r); \mathcal{I}(\varphi + \psi)\big)$ is linear with respect to r.

When $N \equiv 0$, then φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \leq j \leq n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$.

Hence $G\big(h^{-1}(r); \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\big)$ is linear with respect to r. The sufficiency of Theorem 1.10 is proved.

We prove the necessity part of Theorem 1.10.

Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2.3, there exists a unique holomorphic $(n, 0)$ form F on M satisfying $(F - f) \in H^0(Z_0, \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y)\big)|_{Z_0})$, and $G(t; c) = \int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \geq 0$. Then according to Lemma 2.3, Remark 2.4 and Lemma 2.32, we can assume that c is increasing near $+\infty$.

It follows from Lemma 3.3 that there exists a sequence of holomorphic $(n_2, 0)$ forms $\{F_{\alpha,\beta}\}_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}}$ on Y such that

$$F = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{n_1}} \pi_1^*(w_\beta^\alpha dw_{1,\beta_1} \wedge \ldots \wedge dw_{n_1,\beta_{n_1}}) \wedge \pi_2^*(F_{\alpha,\beta})$$

on $V_\beta \times Y$, for any $\beta \in \tilde{I}_1$. Denote that $E_\beta := \Big\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} = 1\Big\}$, $E_{\beta,1} := \Big\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} < 1\Big\}$ and $E_{\beta,2} := \Big\{\alpha \in \mathbb{Z}_{\geq 0}^{n_1} : \sum_{1 \leq j \leq n_1} \frac{\alpha_j + 1}{p_{j,\beta_j}} > 1\Big\}$.

Now we prove $N \equiv 0$. It follows from Lemma 3.15 that we have we have $F_\alpha \equiv 0$ for any $\alpha \in E_{\beta,1}$, and

$$\liminf_{t \to +\infty} \frac{\int_{\{\psi < -t\}} |F|^2 e^{-\varphi} c(-\psi)}{\int_t^{+\infty} c(s)e^{-s}ds} \geq \sum_{\beta \in \tilde{I}_1} \sum_{\alpha \in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1 \leq j \leq n_1} \varphi_j(z_{j,\beta_j})}}{\prod_{1 \leq j \leq n_1} (\alpha_j + 1) c_{j,\beta_j}^{2\alpha_j + 2}} \int_Y |F_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}. \tag{5.4}$$

Note that $\psi = \hat{G} + N \le \hat{G}$, $c(t)$ is increasing near $+\infty$, φ_X is upper semi-continuous on $\prod_{j=1}^{n_1} \Omega_j$. When t is large enough, we have

$$\begin{aligned}&\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) \ge \int_{\{\hat{G}<-t\}} |F|^2 e^{-\pi_1^*\left(\sum_{1\le j\le n_1} \pi_{1,j}^*(\varphi_j)\right)-\pi_2^*(\varphi_Y)} c(-\hat{G})\\ \ge & C_0 \int_{\{\hat{G}<-t\}} |F|^2 e^{-\pi_2^*(\varphi_Y)} c(-\hat{G}),\end{aligned}$$

then it follows from $\int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi) < +\infty$ and Lemma 3.3 that for any $\alpha \in E_{\beta,2} := \left\{\alpha \in \mathbb{Z}_{\ge 0}^{n_1} : \sum_{1\le j\le n_1} \frac{\alpha_j+1}{p_{j,\beta_j}} > 1\right\}$ and any $\beta \in \mathcal{I}_1$, we know $F_{\alpha,\beta} \in \mathcal{I}(\varphi_Y)$ for any $y \in Y$.

Recall that

$$\hat{G} := \max_{1\le j\le n_1} \left\{\pi_{1,j}^*\left(2\sum_{1\le k\le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\},$$

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Lemma 3.11 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on M satisfying that $(\hat{F} - F, z) \in \left(\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))\right)_z$ for any $z \in Z_0$ and

$$\begin{aligned}&\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G})\\ \le & \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1} \sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1} e^{-\sum_{1\le j\le n_1} \varphi_j(z_{j,\beta_j})} \int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y - N(z_\beta, w')}}{\prod_{1\le j\le n_1} (\alpha_j+1) c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned} \tag{5.5}$$

It follows from (5.4) and (5.5) that we have

$$\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \le \int_M |F|^2 e^{-\varphi} c(-\psi) \le \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi). \tag{5.6}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi \le \hat{G}$, we have $\int_M |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G}) \ge \int_M |\hat{F}|^2 e^{-\varphi} c(-\psi)$. Then all "$\ge$" in (5.6) should be "$=$". It follows from Lemma 2.22 that we know $N \equiv 0$ and $\psi = \max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k\le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k\le m_j} p_{j,k} G_{\Omega_j}(\cdot, z_{j,k})\right)\right\}$. It follows from Theorem 2.10 that the necessity part of Theorem 1.10 holds. □

5.3. **Proof of Theorem 1.12.**

Proof. We prove Theorem 1.12 by contradiction.

Assume that $G\big(h^{-1}(r)\big)$ is linear with respect to $r \in (0, \int_0^{+\infty} c(s)e^{-s}ds]$. Then according to Lemma 2.3, there exists a unique holomorphic $(n,0)$ form F on M satisfying $(F - f) \in H^0(Z_0, \big(\mathcal{O}(K_M) \otimes \mathcal{I}(\hat{G} + \pi_2^*(\varphi_Y))|_{Z_0})$, and $G(t;c) = \int_{\{\psi<-t\}} |F|^2 e^{-\varphi} c(-\psi)$ for any $t \ge 0$. Then according to Lemma 2.3, Remark 2.4 and Lemma 2.32, we can assume that c is increasing near $+\infty$.

It follows from Lemma 3.3 that there exists a sequence of holomorphic $(n_2,0)$ forms $\{F_{\alpha,\beta}\}_{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}}$ on Y such that

$$F=\sum_{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(F_{\alpha,\beta})$$

on $V_\beta\times Y$ for any $\beta\in\tilde{I}_1$. Denote that $E_\beta:=\left\{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}=1\right\}$, $E_{\beta,1}:=\left\{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}<1\right\}$ and $E_{\beta,2}:=\left\{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}>1\right\}$.

Now we prove $N\equiv 0$. It follows from Lemma 3.15 that we have we have $F_\alpha\equiv 0$ for any $\alpha\in E_{\beta,1}$, and

$$\liminf_{t\to+\infty}\frac{\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\geq\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}. \tag{5.7}$$

Note that $\psi=\hat{G}+N\leq\hat{G}$, $c(t)$ is increasing near $+\infty$, $\sum_{1\leq j\leq n_1}\pi_{1,j}^*(\varphi_j)$ is upper semi-continuous on $\prod_{j=1}^{n_1}\Omega_j$. When t is large enough, we have

$$\begin{aligned}&\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)\geq\int_{\{G<-t\}}|F|^2e^{-\sum_{1\leq j\leq n_1}\pi_{1,j}^*(\varphi_j)-\pi_2^*(\varphi_Y)}c(-\hat{G})\\ \geq&C_0\int_{\{\hat{G}<-t\}}|F|^2e^{-\pi_2^*(\varphi_Y)}c(-\hat{G}),\end{aligned}$$

where C_0 is a constant. Then it follows from $\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)<+\infty$ and Lemma 3.3 that for any $\alpha\in E_{\beta,2}:=\left\{\alpha\in\mathbb{Z}^{n_1}_{\geq 0}:\sum_{1\leq j\leq n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}>1\right\}$ and any $\beta\in\mathcal{I}_1$, we know $F_{\alpha,\beta}\in\mathcal{I}(\varphi_Y)$ for any $y\in Y$.

Recall that

$$\hat{G}:=\max_{1\leq j\leq n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\leq k\leq m_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}.$$

Denote $\tilde{\varphi}:=\varphi+\psi-\hat{G}$. It follows from Lemma 3.11 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on M satisfying that $(\hat{F}-F,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\right)_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_M|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\\ \leq&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\leq j\leq n_1}\varphi_j(z_{j,\beta_j})}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned} \tag{5.8}$$

It follows from (5.7) and (5.8) that we have

$$\int_M|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\leq\int_M|F|^2e^{-\varphi}c(-\psi)\leq\int_M|\hat{F}|^2e^{-\varphi}c(-\psi). \tag{5.9}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi\leq\hat{G}$, we have $\int_M|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\geq\int_M|\hat{F}|^2e^{-\varphi}c(-\psi)$. Then all "$\geq$" in (5.9) should be "$=$". It follows from Lemma 2.22 that we know $N\equiv 0$ and $\psi=\max_{1\leq j\leq n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\leq k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$.

As $N\equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\le j\le n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Note that $\psi=\max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k<\tilde{m}_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$. It follows from Theorem 2.11 that we get a contradiction. □

5.4. **Proof of proposition 1.14.**

Proof. As $\tilde{G}(h^{-1}(r))$ is linear with respect to $r\in(0,\int_0^{+\infty}c(s)e^{-s}ds]$, then according to Lemma 2.3, there exists a unique holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying $(F-f)\in H^0(Z_0,\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big|_{Z_0})$, and $G(t)=\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$ for any $t\ge 0$. Then according to Lemma 2.3, Remark 2.4 and Lemma 2.32, we can assume that c is increasing near $+\infty$.

It follows from Lemma 3.4 that there exists a sequence of holomorphic $(n_2,0)$ forms $\{F_{\alpha,\beta}\}_{\alpha\in\mathbb{Z}_{\ge0}^{n_1}}$ on Y such that

$$F=\sum_{\alpha\in\mathbb{Z}_{\ge0}^{n_1}}\pi_1^*(w_\beta^\alpha dw_{1,\beta_1}\wedge\ldots\wedge dw_{n_1,\beta_{n_1}})\wedge\pi_2^*(F_{\alpha,\beta})$$

on a open neighborhood $U_\beta\subset(V_\beta\times Y)\cap\tilde{M}$ of $\{z_\beta\}\times Y$ for any $\beta\in\tilde{I}_1$. Denote that $E_\beta:=\left\{\alpha\in\mathbb{Z}_{\ge0}^{n_1}:\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}=1\right\}$, $E_{\beta,1}:=\left\{\alpha\in\mathbb{Z}_{\ge0}^{n_1}:\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}<1\right\}$ and $E_{\beta,2}:=\left\{\alpha\in\mathbb{Z}_{\ge0}^{n_1}:\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}>1\right\}$.

It follows from Remark 3.16 that we have we have $F_{\alpha,\beta}\equiv 0$ for any $\alpha\in E_{\beta,1}$, and

$$\frac{G(0)}{\int_t^{+\infty}c(s)e^{-s}ds}\ge\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\le j\le n_1}\varphi_j(z_{j,\beta_j})}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}\int_Y|F_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}. \tag{5.10}$$

Note that $\psi=\hat{G}+N\le\hat{G}$, $c(t)$ is increasing near $+\infty$, $\sum_{1\le j\le n_1}\pi_{1,j}^*(\varphi_j)$ is upper semi-continuous on $\prod_{j=1}^{n_1}\Omega_j$. When t is large enough, we have

$$\begin{aligned}&\int_{\{\psi<-t\}\cap\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\ge\int_{\{\hat{G}<-t\}\cap\tilde{M}}|F|^2e^{-\sum_{1\le j\le n_1}\pi_{1,j}^*(\varphi_j)-\pi_2^*(\varphi_Y)}c(-\hat{G})\\ \ge&C_0\int_{\{\hat{G}<-t\}\cap\tilde{M}}|F|^2e^{-\pi_2^*(\varphi_Y)}c(-\hat{G}),\end{aligned}$$

then it follows from $\int_{\{\psi<-t\}}|F|^2e^{-\varphi}c(-\psi)<+\infty$ and Lemma 3.4 that for any $\alpha\in E_{\beta,2}:=\left\{\alpha\in\mathbb{Z}_{\ge0}^{n_1}:\sum_{1\le j\le n_1}\frac{\alpha_j+1}{p_{j,\beta_j}}>1\right\}$ and any $\beta\in\mathcal{I}_1$, we know $F_{\alpha,\beta}\in\mathcal{I}(\varphi_Y)$ for any $y\in Y$.

Recall that $\hat{G}:=\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}$ and $\tilde{\varphi}=\varphi+N$. It follows from Lemma 3.11 that there exists a holomorphic $(n,0)$ form $\tilde{F}$ on $\tilde{M}$

satisfying that $(\tilde{F}-F,z)\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y)))_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\le j\le n_1}\varphi_j(z_{j,\beta_j})}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}\tag{5.11}$$

Combining with (5.10) and (5.11), we have

$$\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi)\ge\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\ge\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\hat{G}).\tag{5.12}$$

As $c(t)e^{-t}$ is decreasing with respect to t and $\psi\le\hat{G}$, we have $\int_{\tilde{M}}|\tilde{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\ge\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi)$. Then all "$\ge$" in (5.12) should be "$=$". It follows from Lemma 2.22 that $N=0$ and $\psi=\max_{1\le j\le n_1}\left\{\pi_{1,j}^*\left(2\sum_{1\le k\le m_j}p_{j,k}G_{\Omega_j}(\cdot,z_{j,k})\right)\right\}$.

It follows from Lemma 3.11 ($\tilde{M}\sim M$) that there exists a holomorphic $(n,0)$ form $\hat{F}$ on M satisfying that $(\hat{F}-F,z)\in\left(\mathcal{O}(K_M)\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\right)_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_{M}|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}e^{-\sum_{1\le j\le n_1}\varphi_j(z_{j,\beta_j})}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-N(z_\beta,w')}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}\tag{5.13}$$

We also have

$$\frac{\tilde{G}(0)}{\int_t^{+\infty}c(s)e^{-s}ds}=\frac{\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}\le\frac{\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)}{\int_t^{+\infty}c(s)e^{-s}ds}.\tag{5.14}$$

Combining (5.10), (5.13) and (5.14), we have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)=\int_M|\hat{F}|^2e^{-\varphi}c(-\psi).$$

As $\hat{F}\ne0$, the above equality shows that $M=\tilde{M}$.

□

6. Proofs of Theorem 1.17 and Theorem 1.19

In this section, we prove Theorem 1.17 and Theorem 1.19.

6.1. **Proof of Theorem 1.17.**

Proof. It follows from Lemma 3.9 that we know there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ such that $(F-f,(z_j,y))\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\right)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$ and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\tag{6.1}$$

In the following, we prove the characterization of the holding of the equality $\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}=\inf\big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):$ $\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $\big(\tilde{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0\big\}$.

We firstly give the proof of the sufficiency in Theorem 1.17.

As $\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1$ is a subharmonic function on Ω. By the construction of $g\in\mathcal{O}_\Omega$, we have $2u(z):=2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1-2\log|g|$ is a subharmonic function on Ω which satisfies $v(dd^cu,z)\in[0,1)$ for any $z\in Z_\Omega$. Then it follows from Lemma 3.7 that we know $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. It follows from Theorem 2.7 that we know the sufficiency part of Theorem 1.17 holds.

Next we prove the necessity part of Theorem 1.17.

Denote that $\tilde{\psi}:=\pi_1^*\big(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)\big)$, $\tilde{\varphi}_1:=\pi_1^*(\varphi_1)+\psi-\tilde{\psi}$, and $\tilde{\varphi}:=\tilde{\varphi}_1+\pi_2^*(\varphi_2)$. It follows from Lemma 3.9 that there exists a holomorphic $(n,0)$ form $\tilde{F}_1\neq0$ on $\tilde{M}$ such that $\big(\tilde{F}_1-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\tilde{\varphi}}c\left(-\tilde{\psi}\right)\\&\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned}\tag{6.2}$$

As $\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}=\inf\big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):$ $\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $\big(\tilde{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0\big\}$ holds, we know

$$\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\varphi}c\left(-\psi\right)\geq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, $\psi\leq\tilde{\psi}$. Hence we must have

$$\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\varphi}c\left(-\psi\right)=\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\tilde{\varphi}}c\left(-\tilde{\psi}\right).$$

It follows from Lemma 2.22 that we have $N\equiv0$ and $\psi=\pi_1^*\big(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)\big)$.

It follows from Lemma 3.9 ($\tilde{M}\sim M$) that there exists a holomorphic $(n,0)$ form $\hat{F}\neq0$ on M such that $\big(\hat{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$ and

$$\begin{aligned}&\int_M|\hat{F}|^2e^{-\varphi}c\left(-\psi\right)\\&\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned}\tag{6.3}$$

As $\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^m\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}=\inf\big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):$ $\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $\big(\tilde{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes$

$\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)))_{(z_j,y)}$ for any $(z_j,y)\in Z_0\big\}$ holds, we know

$$\begin{aligned}&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{m}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}\\ \le&\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \le&\int_{M}|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{m}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned}$$

As $\hat{F}\neq 0$, we have $\tilde{M}=M$.

Now $\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1)$ is a plurisubharmonic function on M. By definition, we know that $2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1$ is a subharmonic function on Ω. By the construction of $g\in\mathcal{O}_\Omega$, we have $2u(z):=2\sum_{j=1}^m p_jG_\Omega(\cdot,z_j)+\varphi_1-2\log|g|$ is a subharmonic function on Ω and satisfies $v(dd^cu,z)\in[0,1)$ for any $z\in Z_\Omega$. Then it follows from Lemma 3.7 that we know $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$.

Then it follows from the necessity part of Theorem 2.7 that we know Theorem 1.17 holds.

□

6.2. **Proof of Theorem 1.19.**

Proof. It follows from Lemma 3.9 that we know there exists a holomorphic $(n,0)$ form F on M such that $\big(F-f,(z_j,y)\big)\in(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$ and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\le\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{+\infty}\frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.$$

Now it suffices to show that the equality $\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{+\infty}\frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}=\inf\big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $\big(\tilde{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0\big\}$ can not hold. We assume that the equality holds to get a contradiction.

Denote that $\tilde{\psi}:=\pi_1^*\big(2\sum_{j=1}^m(k_j+1)G_\Omega(\cdot,z_j)\big)$, $\tilde{\varphi}_1:=\pi_1^*(\varphi_1)+\psi-\tilde{\psi}$, and $\tilde{\varphi}:=\tilde{\varphi}_1+\pi_2^*(\varphi_2)$. It follows from Lemma 3.9 that there exists a holomorphic $(n,0)$ form $\tilde{F}_1\neq 0$ on $\tilde{M}$ such that $\big(\tilde{F}_1-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\tilde{\varphi}}c\left(-\tilde{\psi}\right)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{+\infty}\frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned}\tag{6.4}$$

As $\left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^{+\infty}\frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y |F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} = \inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):$ $\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $\big(\tilde{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0\Big\}$, we know

$$\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\varphi}c(-\psi)\geq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{+\infty}\frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$ and $\psi\leq\tilde{\psi}$. Hence we must have

$$\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\varphi}c(-\psi)=\int_{\tilde{M}}|\tilde{F}_1|^2e^{-\tilde{\varphi}}c\left(-\tilde{\psi}\right).$$

Hence, by Lemma 2.22, we have $N\equiv 0$ and $\psi=\pi_1^*\big(2\sum_{j=1}^{+\infty}(k_j+1)G_\Omega(\cdot,z_j)\big)$.

It follows from Lemma 3.9 ($\tilde{M}\sim M$) that there exists a holomorphic $(n,0)$ form $\hat{F}\neq 0$ on M such that $\big(\hat{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_M)\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$ and

$$\begin{aligned}&\int_M|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \leq&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{m}\frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned}\tag{6.5}$$

As $\left(\int_0^{+\infty} c(s)e^{-s}ds\right)\sum_{j=1}^{m}\frac{2\pi|a_j|^2}{p_j|d_j|^2}\int_Y |F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)} = \inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):$ $\tilde{F}$ is a holomorphic $(n,0)$ form on $\tilde{M}$ such that $\big(\tilde{F}-f,(z_j,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2))\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0\Big\}$ holds, we know

$$\begin{aligned}&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{m}\frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}\\ \leq&\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \leq&\int_M|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \leq&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{j=1}^{m}\frac{2\pi|a_j|^2}{(k_j+1)|d_j|^2}\int_Y|F_j|^2e^{-\varphi_2-\tilde{\psi}_2(z_j,w)}.\end{aligned}$$

As $\hat{F}\neq 0$, we have $\tilde{M}=M$.

Now $\psi+\pi_1^*(\varphi_1)=\pi_1^*(2\sum_{j=1}^{+\infty}(k_j+1)G_\Omega(\cdot,z_j)+\varphi_1)$ is a plurisubharmonic function on M, by definition, we know that $2\sum_{j=1}^{m}(k_j+1)G_\Omega(\cdot,z_j)+\varphi_1$ is a subharmonic function on Ω. By the construction of $g\in\mathcal{O}_\Omega$, we have $2u(z):=2\sum_{j=1}^{+\infty}(k_j+1)G_\Omega(\cdot,z_j)+\varphi_1-2\log|g|$ is a subharmonic function on Ω and satisfies $v(dd^cu,z)\in[0,1)$ for any $z\in Z_\Omega$. Then it follows from Lemma 3.7 that we know $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\pi_1^*(2\log|g|)+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$.

Then it follows from Theorem 2.8 that we know Theorem 1.19 holds.

$\square$

7. Proofs of Theorem 1.21 and Theorem 1.23 and Theorem 1.25

In this section, we prove Theorem 1.21 and Theorem 1.23 and Theorem 1.25.

7.1. **Proofs of Theorem 1.21 and Remark 1.22.**

Proof of Theorem 1.21. It follows from Remark 3.12 that there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)\right)_z$ for any $z \in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\alpha\in E} \frac{(2\pi)^{n_1}\int_Y |f_\alpha|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}.\end{aligned}$$

In the following, we prove the characterization of the holding of the equality. It follows from Theorem 2.12 that we only need to show the necessity.

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Remark 3.12 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F}-F,z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G})\right)_z$ for any $z \in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G})\\ &\le \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\alpha\in E} \frac{(2\pi)^{n_1}\int_Y |f_\alpha|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}.\end{aligned} \tag{7.1}$$

When the equality $\inf\left\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi) : \tilde{F} \in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F}-f,z) \in \left(\mathcal{O}\left(K_{\tilde{M}}\right)\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}\right)\right)_z \text{ for any } z\in Z_0\right\} = \left(\int_0^{+\infty}c(s)e^{-s}ds\right)\times \sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}$ holds, we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \ge \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\alpha\in E} \frac{(2\pi)^{n_1}\int_Y |f_\alpha|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, $\psi \le \hat{G}$. Hence we must have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c\left(-\psi\right) = \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c\left(-\hat{G}\right).$$

It follows from Lemma 2.22 that we have $N \equiv 0$ and $\psi = \max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \le j \le n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j,y) \in Z_0$. Note that $\psi = \max_{1\le j\le n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}$. Then It follows from the necessity part of Theorem 2.12 that Theorem 1.21 holds. □

Proof of Remark 1.22. As $(f_\alpha,y) \in \left(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y)\right)_y$ for any $y \in Y$ and $\alpha \in \tilde{E}\backslash E$, it follows from Lemma 3.11 that there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,(z,y)) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\hat{G}+\pi_2^*(\varphi_Y)\right)\right)_{(z,y)}$ for any $(z,y) \in Z_0$

and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$$
$$\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}.$$

In the following, we prove the characterization of the holding of the equality.

When $N\equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\leq j\leq n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Note that $\psi=\max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}$. It follows from Remark 2.13 that we have the sufficiency of Remark 1.22.

Next we show the necessity of Remark 1.22.

Denote $\tilde{\varphi}:=\varphi+\psi-\hat{G}$. It follows from Remark 3.12 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F}-F,z)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big)_z$ for any $z\in Z_0$ and

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})$$
$$\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}. \tag{7.2}$$

Now we know the equality $\inf\big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}\in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,\big(\tilde{F}-f,(z,y)\big)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big)_{(z,y)}$ for any $(z,y)\in Z_0\big\}=\Big(\int_0^{+\infty}c(s)e^{-s}ds\Big)\times$ $\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_j(z_j)^{2\alpha_j+2}}$ holds, then we have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)$$
$$\geq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\alpha\in E}\frac{(2\pi)^{n_1}\int_Y|f_\alpha|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_0,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_j^{2\alpha_j+2}(z_j)}. \tag{7.3}$$

It follows from (7.2) and (7.3) that we have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c\left(-\psi\right)=\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c\left(-\hat{G}\right).$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, $\psi\leq\hat{G}$. It follows from Lemma 2.22 that we have $N\equiv 0$ and $\psi=\max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}$.

As $N\equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\leq j\leq n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Note that $\psi=\max_{1\leq j\leq n_1}\left\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\right\}$. Then It follows from Remark 2.13 that Remark 1.22 holds. $\square$

7.2. **Proofs of Theorem 1.23 and Remark 1.24.**

Proof of Theorem 1.23. It follows from Remark 3.12 that there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,z)\in\Big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\Big(\max_{1\leq j\leq n_1}\Big\{2\sum_{1\leq k\leq m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\Big\}\Big)\Big)_z$

for any $z \in Z_0$ and

$$\int_{\tilde{M}} |F|^2 e^{-\varphi} c(-\psi)$$
$$\leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

In the following, we prove the characterization of the holding of the equality. It follows from Theorem 2.14 that we only need to show the necessity.

Denote $\tilde{\varphi} := \varphi + \psi - \hat{G}$. It follows from Remark 3.12 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F} - F, z) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}(\hat{G})\right)_z$ for any $z \in Z_0$ and

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c(-\hat{G})$$
$$\leq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}. \tag{7.4}$$

When the equality $\inf\Bigg\{ \int_{\tilde{M}} |\tilde{F}|^2 e^{-\varphi} c(-\psi) : \tilde{F} \in H^0(\tilde{M}, \mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F} - f, z) \in \Big(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\Big(\max_{1\leq j\leq n_1}\Big\{2\sum_{1\leq k\leq m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\Big\}\Big)\Big)_z$ for any $z \in Z_0\Bigg\} = \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ holds, we have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) \geq \left(\int_0^{+\infty} c(s)e^{-s}ds\right) \sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta} \frac{(2\pi)^{n_1}\int_Y |f_{\alpha,\beta}|^2 e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\leq j\leq n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\leq j\leq n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0, +\infty)$, $\psi \leq \hat{G}$. Hence we must have

$$\int_{\tilde{M}} |\hat{F}|^2 e^{-\varphi} c(-\psi) = \int_{\tilde{M}} |\hat{F}|^2 e^{-\tilde{\varphi}} c\left(-\hat{G}\right).$$

It follows from Lemma 2.22 that we have $N \equiv 0$ and $\psi = \max_{1\leq j\leq n_1}\left\{2\sum_{1\leq k\leq m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}$.

As $N \equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j) > -\infty$ for any $1 \leq j \leq n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi + \psi)_{(z_j,y)} = \mathcal{I}\big(\hat{G} + \pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j, y) \in Z_0$. Note that $\psi = \max_{1\leq j\leq n_1}\left\{2\sum_{1\leq k\leq m_j} p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot, z_{j,k}))\right\}$. Then It follows from the necessity part of Theorem 2.14 that Theorem 1.23 holds.

□

Proof of Remark 1.24. As $(f_{\alpha,\beta}, y) \in \left(\mathcal{O}(K_Y) \otimes \mathcal{I}(\varphi_Y)\right)_y$ holds for any $y \in Y$, $\alpha \in \tilde{E}_\beta \backslash E_\beta$ and $\beta \in \tilde{I}_1$, it follows from Lemma 3.11 that there exists a holomorphic $(n,0)$ form F on M satisfying that $\left(F - f, (z,y)\right) \in \left(\mathcal{O}(K_{\tilde{M}}) \otimes \mathcal{I}\left(\hat{G} + \pi_2^*(\varphi_Y)\right)\right)_{(z,y)}$ for any $(z,y)\in Z_0$ and

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)$$
$$\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

In the following, we prove the characterization of the holding of the equality.

When $N\equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\le j\le n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Note that $\psi=\max_{1\le j\le n_1}\{2p_j\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_j))\}$. It follows from Remark 2.15 that we have the sufficiency of Remark 1.24.

Next we show the necessity of Remark 1.24.

Denote $\tilde{\varphi}:=\varphi+\psi-\hat{G}$. It follows from Remark 3.12 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F}-F,z)\in\big(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\big)_{(z,y)}$ for any $(z,y)\in Z_0$ and

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})$$
$$\leq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\tag{7.5}$$

Now we know the equality $\inf\big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}\in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F}-f,(z,y)\in(\mathcal{O}\left(K_{\tilde{M}}\right)\otimes\mathcal{I}\left(\hat{G}+\pi_2^*(\varphi_Y)\right)\right)_{(z,y)}$ for any $(z,y)\in Z_0\big\}=\left(\int_0^{+\infty}c(s)e^{-s}ds\right)$ $\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ holds, then we have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)$$
$$\geq\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\tag{7.6}$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, $\psi\le\hat{G}$. It follows from (7.5) and (7.6) that we have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c\left(-\psi\right)=\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi-N}c\left(-\hat{G}\right).$$

It follows from Lemma 2.22 that we have $N\equiv 0$ and $\psi=\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}$.

As $N\equiv 0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\le j\le n_1$, it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Note that $\psi=\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}$. Then It follows from Remark 2.15 that Remark 1.24 holds. □

7.3. **Proofs of Theorem 1.25 and Remark 1.26.**

Proof of Theorem 1.25. It follows from Remark 3.12 that there exists a holomorphic $(n,0)$ form F on $\tilde{M}$ satisfying that $(F-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}$$

In the following, we prove the characterization of the holding of the equality. It follows from Theorem 2.14 that we only need to show the necessity.

We show the equality $\inf\Big\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi)\,:\,\tilde{F}\in H^0(\tilde{M},\mathcal{O}(K_{\tilde{M}}))\,\&\,(\tilde{F}-f,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\max_{1\le j\le n_1}\left\{2\sum_{1\le k\le m_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}\right)\right)_z$ for any $z\in Z_0\Big\}=\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ can not hold. We assume that the equality holds to get a contradiction.

Denote $\tilde{\varphi}:=\varphi+\psi-\hat{G}$. It follows from Remark 3.12 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $(\hat{F}-F,z)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G})\right)_z$ for any $z\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}\tag{7.7}$$

As the equality holds, we have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)\ge\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, $\psi\le\hat{G}$. Hence we must have

$$\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c\,(-\psi)=\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c\,(-G)\,.$$

It follows from Lemma 2.22 that we have $\psi=\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}$.

As $N\equiv0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\le j\le n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(G+\pi_2^*(\varphi_Y)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Note that $\psi=\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}$. Then It follows from Theorem 2.16 that we get the contradiction and Theorem 1.25 is proved. □

Proof of Remark 1.26. As $(f_{\alpha,\beta},y)\in\left(\mathcal{O}(K_Y)\otimes\mathcal{I}(\varphi_Y)\right)_y$ holds for any $y\in Y$, $\alpha\in\tilde{E}_\beta\backslash E_\beta$ and $\beta\in\tilde{I}_1$, it follows Lemma 3.11 that there exists a holomorphic $(n,0)$

form F on $\tilde{M}$ satisfying that $\left(F-f,(z,y)\right)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}\left(\hat{G}+\pi_2^*(\varphi_Y)\right)\right)_{(z,y)}$ for any $(z,y)\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|F|^2e^{-\varphi}c(-\psi)\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}$$

In the following, we assume that the equality $\inf\left\{\int_{\tilde{M}}|\tilde{F}|^2e^{-\varphi}c(-\psi):\tilde{F}\in H^0\left(\tilde{M},\mathcal{O}(K_{\tilde{M}})\right)\&\left(\tilde{F}-f,(z,y)\right)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(G+\pi_2^*(\varphi_Y))\right)_{(z,y)}\text{ for any }(z,y)\in Z_0\right\}=\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}$ holds to get a contradiction.

Denote $\tilde{\varphi}:=\varphi+\psi-\hat{G}$. It follows from Remark 3.12 that there exists a holomorphic $(n,0)$ form $\hat{F}$ on $\tilde{M}$ satisfying that $\left(\hat{F}-F,z\right)\in\left(\mathcal{O}(K_{\tilde{M}})\otimes\mathcal{I}(\hat{G}+\pi_2^*(\varphi_Y))\right)_{(z,y)}$ for any $(z,y)\in Z_0$ and

$$\begin{aligned}&\int_{\tilde{M}}|\hat{F}|^2e^{-\tilde{\varphi}}c(-\hat{G})\\ \le&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}\tag{7.8}$$

Since the equality holds, we have

$$\begin{aligned}&\int_{\tilde{M}}|\hat{F}|^2e^{-\varphi}c(-\psi)\\ \ge&\left(\int_0^{+\infty}c(s)e^{-s}ds\right)\sum_{\beta\in\tilde{I}_1}\sum_{\alpha\in E_\beta}\frac{(2\pi)^{n_1}\int_Y|f_{\alpha,\beta}|^2e^{-\varphi_Y-\left(N+\pi_{1,j}^*(\sum_{1\le j\le n_1}\varphi_j)\right)(z_\beta,w)}}{\prod_{1\le j\le n_1}(\alpha_j+1)c_{j,\beta_j}^{2\alpha_j+2}}.\end{aligned}\tag{7.9}$$

Note that $c(t)e^{-t}$ is decreasing on $(0,+\infty)$, $\psi\le\hat{G}$. It follows from (7.8) and (7.9) that we have

$$\int_{\tilde{M}}|F|^2e^{-\varphi}c\left(-\psi\right)=\int_{\tilde{M}}|F|^2e^{-\varphi-N}c\left(-\hat{G}\right).$$

It follows from Lemma 2.22 that we have $N\equiv0$ and $\psi=\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}$.

As $N\equiv0$, we know φ_j is a subharmonic function on Ω_j satisfying $\varphi_j(z_j)>-\infty$ for any $1\le j\le n_1$ and it follows from Lemma 3.8 that $\mathcal{I}(\varphi+\psi)_{(z_j,y)}=\mathcal{I}\big(\hat{G}+\pi_2^*(\varphi_2)\big)_{(z_j,y)}$ for any $(z_j,y)\in Z_0$. Note that $\psi=\max_{1\le j\le n_1}\left\{2\sum_{1\le k<\tilde{m}_j}p_{j,k}\pi_{1,j}^*(G_{\Omega_j}(\cdot,z_{j,k}))\right\}$. Then It follows from Remark 2.17 that Remark 1.26 holds. □

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