Counterexamples to Hosono's conjecture on Azukawa pseudometrics and indicatrix volumes

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Abstract

We construct a bounded pseudoconvex family of hyperconvex domains in $\mathbb{C}^2$ that disproves both parts of Hosono's conjecture on variation of Azukawa pseudometrics and indicatrix volumes.

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1 Introduction

Hosono [1] proposed plurisubharmonic variation of the logarithmic Azukawa pseudometric and subharmonic variation of the negative logarithm of its indicatrix volume under pseudoconvex variation of the underlying domains. He proved both assertions when the fibers are balanced about the fixed pole [1, Theorem 1.4 in the arXiv version]. We give a single explicit family disproving both assertions. Its total space is bounded and pseudoconvex, and its fibers are bounded hyperconvex domains containing the origin. The pseudometric counterexample follows from a holomorphic retraction; the volume counterexample uses polynomial disks and rational estimates.

Let $D \subset \mathbb{C}^n$ be a bounded hyperconvex domain containing the origin. For every $Z \in D$, the negative *pluricomplex Green function* with pole at the origin is defined by

$$g_{D,0}(Z) = \sup\{u(Z) : u \in \mathrm{PSH}(D),\ u < 0,\ u(Z) \le \log\|Z\| + O(1) \text{ near } 0\}.$$

Following Hosono, for every $X \in \mathbb{C}^n$, we use the *logarithmic Azukawa pseudometric*

$$\mathcal{A}_{D,0}(X) = \limsup_{\lambda \to 0} \big(g_{D,0}(\lambda X) - \log|\lambda|\big). \tag{1.1}$$

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Thus $\mathcal{A}_{D,0}(cX) = \mathcal{A}_{D,0}(X) + \log|c|$ for $c \neq 0$, and $\mathcal{A}_{D,0}(0) = -\infty$. In this convention the *Azukawa indicatrix* is

$$I_{D,0} = \{X \in \mathbb{C}^n : \mathcal{A}_{D,0}(X) < 0\}.$$

Hosono proposed the following conjecture [1, Conjecture 1.2 in the arXiv version]. By translating the fiber coordinates, the common pole may be taken to be the origin.

CONJECTURE 1.1 (Hosono). *Let $\Delta \subset \mathbb{C}$ be a disk, and let $\Omega \subset \mathbb{C}^n \times \Delta$ be a pseudoconvex domain. For each $t \in \Delta$, suppose that the fiber*

$$\Omega_t = \{Z \in \mathbb{C}^n : (Z, t) \in \Omega\}$$

is hyperconvex and contains 0. Set $I_t = \{X \in \mathbb{C}^n : \mathcal{A}_{\Omega_t,0}(X) < 0\}$. Then:

1. *The function $(t, X) \mapsto \mathcal{A}_{\Omega_t,0}(X)$ is plurisubharmonic on $\Delta \times \mathbb{C}^n$.*
2. *The function $t \mapsto -\log \mathrm{Vol}(I_t)$ is subharmonic on Δ, where* Vol *denotes Euclidean volume.*

THEOREM 1.2. *Let $\Delta_{1/100} = \{t \in \mathbb{C} : |t| < 1/100\}$ and set*

$$\mathcal{D} = \{(z, w, t) \in \mathbb{C}^2 \times \Delta_{1/100} : |z| < 1,\ |w|^2 + |w^2 + tz|^2 < 1\}, \tag{1.2}$$

$$D_t = \{(z, w) : (z, w, t) \in \mathcal{D}\}, \qquad V(t) = \mathrm{Vol}(I_{D_t,0}). \tag{1.3}$$

The total space is bounded and pseudoconvex, and every fiber is a bounded hyperconvex domain containing 0. Both assertions of Conjecture 1.1 fail for this family:

1. *For $r = |t|$, put*

$$R_t^2 = x(r) := \frac{2r - 1 + \sqrt{5 - 4r}}{2}. \tag{1.4}$$

   Then

$$\mathcal{A}_{D_t,0}(0, 1) = -\log R_t = -\frac{1}{2}\log x(|t|). \tag{1.5}$$

   This quantity strictly decreases with $|t|$, so the logarithmic Azukawa pseudometric is not plurisubharmonic in (t, X).

2. *The function V is radial in t. For $\tau = 1/1000$,*

$$\frac{V(\tau)}{\pi^2} > \frac{123611}{200000} > \frac{15451}{25000} > \frac{V(0)}{\pi^2} = \frac{\sqrt{5} - 1}{2}. \tag{1.6}$$

   Consequently, $-\log V$ is not subharmonic on $\Delta_{1/100}$.

Section 2 verifies the geometric hypotheses and radiality of the volume. Section 3 proves (1.5) by identifying a disk that is a holomorphic retract of each fiber. Sections 4–5 prove (1.6) without requiring an exact formula for the indicatrix when $t \neq 0$.

2 Geometry and radiality

Put

$$\rho(z, w, t) = |w|^2 + |w^2 + tz|^2 - 1.$$

This is plurisubharmonic in all variables. The total space is bounded, since $|z|, |w| < 1$, and has the plurisubharmonic exhaustion

$$-\log(1 - |z|^2) - \log(-\rho(z, w, t)) - \log(10^{-4} - |t|^2).$$

Each term is bounded below and tends to infinity at its corresponding boundary condition. For $0 \le s \le 1$, the weighted dilation $(z,w) \mapsto (s^2 z, sw)$ stays inside each fiber:

$$|sw|^2 + |(sw)^2 + t(s^2 z)|^2 = s^2|w|^2 + s^4|w^2 + tz|^2 < 1.$$

Thus the fibers are connected, and their common section $(0,0,t)$ also makes the total space connected. This proves pseudoconvexity. For fixed t, the continuous negative plurisubharmonic function

$$\max\{|z|^2 - 1,\ \rho(z,w,t)\}$$

is an exhaustion of D_t, proving hyperconvexity.

If $t = re^{i\theta}$, the unitary map $U_\theta(z,w) = (e^{-i\theta}z, w)$ maps D_r onto D_t and fixes 0. Biholomorphic invariance of the Azukawa pseudometric [2, Theorem 1.1(ii)] gives $I_{D_t,0} = U_\theta(I_{D_r,0})$. Hence $V(t) = V(r)$.

Finally, the fibers lie in $\mathbb{D}^2$ and contain a fixed Euclidean ball about 0. Domain monotonicity of the Azukawa pseudometric therefore gives $0 < V(t) < \infty$. At $t = 0$,

$$D_0 = \mathbb{D} \times \Delta_{\sqrt{s}}, \qquad s = \frac{\sqrt{5}-1}{2}, \qquad s + s^2 = 1. \tag{2.1}$$

The balanced-domain formula [2, Theorem 1.1(viii)] gives $I_{D_0,0} = D_0$ and $V(0) = \pi^2 s$.

3 The vertical Azukawa pseudometric

We compute the pseudometric in a fixed direction for every parameter. For $r = |t|$, the number $x(r)$ in (1.4) satisfies

$$x(r) + (x(r) - r)^2 = 1, \qquad x(r) > r. \tag{3.1}$$

Indeed, $x(r) - r = (\sqrt{5 - 4r} - 1)/2 > 0$ for $0 \le r < 1/100$.

Proposition 3.1. *The coordinate projection $\pi(z,w) = w$ maps D_t onto Δ_{R_t} and has a holomorphic right inverse $\sigma_t : \Delta_{R_t} \to D_t$ satisfying $\sigma_t(0) = 0$ and $\sigma_t'(0) = (0,1)$. Consequently, (1.5) holds.*

Proof. Write $v = |w|^2$ and $r = |t|$. The triangle inequality gives

$$|w^2 + tz| \ge (v - r)_+, \qquad (a)_+ := \max\{a, 0\}.$$

Thus every $(z,w) \in D_t$ satisfies $v + (v-r)_+^2 < 1$. This left side is strictly increasing in $v \ge 0$ and equals 1 at $v = x(r)$ by (3.1). It follows that $\pi(D_t) \subset \Delta_{R_t}$.

Set $\eta_t = \bar{t}/|t|$ for $t \ne 0$ and $\eta_0 = 0$, and define

$$\sigma_t(w) = \left(-\frac{\eta_t}{R_t^2} w^2, w\right), \qquad |w| < R_t. \tag{3.2}$$

Its first coordinate has modulus less than 1, and

$$|w|^2 + |w^2 + t(\sigma_t(w))_1|^2 = v + \left(1 - \frac{r}{R_t^2}\right)^2 v^2 < 1.$$

The strict inequality holds because this expression is strictly increasing in $v \ge 0$ and has value $R_t^2 + (R_t^2 - r)^2 = 1$ at $v = R_t^2$. Hence $\sigma_t(\Delta_{R_t}) \subset D_t$ and $\pi \circ \sigma_t = \mathrm{id}_{\Delta_{R_t}}$. The asserted derivatives at 0 follow directly from (3.2). Holomorphic contraction of the Azukawa pseudometric [2, Theorem 1.1(i)], applied to π and σ_t, gives

$$\mathcal{A}_{\Delta_{R_t},0}(1) \le \mathcal{A}_{D_t,0}(0,1) \le \mathcal{A}_{\Delta_{R_t},0}(1) = -\log R_t.$$

This proves (1.5). ∎

Since

$$x'(r) = 1 - \frac{1}{\sqrt{5-4r}} > 0 \qquad (0 \le r < 1/100),$$

the function $-\frac{1}{2}\log x(r)$ is strictly decreasing. Therefore, for every $0 < a < 1/100$,

$$\mathcal{A}_{D_0,0}(0,1) > \frac{1}{2\pi}\int_0^{2\pi} \mathcal{A}_{D_{ae^{i\theta}},0}(0,1)\, d\theta. \tag{3.3}$$

A plurisubharmonic function of (t, X) would restrict to a subharmonic function on the complex line $X = (0,1)$. The strict reverse submean inequality (3.3) therefore disproves part (1) of Conjecture 1.1.

4 Polynomial disks and a volume bound

We first record how analytic disks provide vectors strictly inside the indicatrix.

Lemma 4.1. *Let D be a bounded hyperconvex domain containing 0, and let f be holomorphic on a neighborhood of $\overline{\mathbb{D}}$, with $f(0) = 0$ and $f(\overline{\mathbb{D}}) \subset D$. Then $f'(0) \in I_{D,0}$.*

Proof. Compactness gives $R > 1$ such that $\zeta \mapsto f(R\zeta)$ maps $\mathbb{D}$ into D. Holomorphic contraction and logarithmic homogeneity of the Azukawa pseudometric [2, Theorem 1.1(i), (iii)] imply

$$\mathcal{A}_{D,0}(f'(0)) + \log R = \mathcal{A}_{D,0}(Rf'(0)) \le \mathcal{A}_{\mathbb{D},0}(1) = 0.$$

Thus $\mathcal{A}_{D,0}(f'(0)) < 0$. ∎

For the remainder of the proof, fix

$$\tau = \frac{1}{1000}, \qquad \beta = \frac{1}{4}.$$

For $b \in \{0, \beta\}$, let x_b be the positive solution of

$$x_b + (x_b - \tau b)^2 = 1 - 16\tau^2, \qquad x_b = \frac{2\tau b - 1 + \sqrt{5 - 4\tau b - 64\tau^2}}{2}. \tag{4.1}$$

Write $r_b = \sqrt{x_b}$. Direct comparison in (4.1) gives

$$\frac{3}{5} < x_0 < x_\beta < \frac{7}{10}, \qquad \frac{1}{2} < r_0 < r_\beta < 1. \tag{4.2}$$

Indeed, the left side of (4.1), as a function of x, is strictly increasing for $x \ge 3/5$; its values at $3/5$ and $7/10$ straddle $1 - 16\tau^2$. At $x = x_0$ the replacement of $b = 0$ by $b = \beta$ strictly decreases that left side, so $x_\beta > x_0$.

Proposition 4.2. *The indicatrix at parameter τ contains*

$$\big(\mathbb{D} \times \Delta_{r_0}\big) \;\cup\; \big(\Delta_{1/2} \times \Delta_{r_\beta}\big). \tag{4.3}$$

Proof. We construct disks with derivative (a, y) for the two ranges

$$b = 0, \qquad |a| < 1, \qquad 0 \le y \le r_0, \tag{4.4}$$

$$b = \beta, \qquad |a| < 1/2, \qquad r_0 \le y \le r_\beta. \tag{4.5}$$

For $y > 0$, set

$$c = y^2 - \tau b, \qquad h = -\frac{c\bar{a}}{y(1+2c)}, \qquad f(\zeta) = \big(a\zeta - b\zeta^2,\; y\zeta + \tau h\zeta^2\big). \tag{4.6}$$

When $b = y = 0$, put $h = 0$ and use the same polynomial formula. In (4.4), $c = y^2$ and $|h| = y|a|/(1+2y^2) \le 1$. In (4.5), (4.2) gives $c > 0$ and $y > 1/2$, so

$$|h| \le \frac{c}{y(1+2c)} < \frac{1}{2y} < 1.$$

In both ranges, $0 \le c \le 1$, $y < 1$, and

$$y^2 + c^2 \le 1 - 16\tau^2. \tag{4.7}$$

For (4.4), this follows from the monotonicity of $x + x^2$ for $x \ge 0$. For (4.5), use the monotonicity of $x + (x - \tau\beta)^2$ on $[x_0, x_\beta]$.

Write $f = (z, w)$. On $|\zeta| = 1$, define

$$Q(\zeta) = \tau a\zeta + 2\tau y h\zeta^3 + \tau^2 h^2 \zeta^4.$$

Then $w^2 + \tau z = c\zeta^2 + Q(\zeta)$. The choice of h cancels the first-order nonconstant terms because $y(1+2c)h + c\bar{a} = 0$. Expanding the two squares therefore gives the exact identity

$$|w|^2 + |w^2 + \tau z|^2 = y^2 + c^2 + \tau^2|h|^2 + 2c\tau^2 \operatorname{Re}(h^2\zeta^2) + |Q(\zeta)|^2. \tag{4.8}$$

Since $|a| \le 1$, $y < 1$, and $|h| \le 1$,

$$|Q(\zeta)| \le \tau(3+\tau), \qquad 3 + (3+\tau)^2 < 13.$$

Equations (4.7)–(4.8) yield

$$|w|^2 + |w^2 + \tau z|^2 < 1 - 16\tau^2 + 13\tau^2 = 1 - 3\tau^2 < 1 \quad \text{on } \partial\mathbb{D}. \tag{4.9}$$

The first coordinate satisfies $|z| \le |a| + b < 1$ on $\overline{\mathbb{D}}$ in both ranges (4.4)–(4.5). The maximum principle applied to the subharmonic function $|w|^2 + |w^2 + \tau z|^2$ extends (4.9) to $\overline{\mathbb{D}}$. Consequently $f(\overline{\mathbb{D}}) \subset D_\tau$. Lemma 4.1 gives $(a, y) = f'(0) \in I_{D_\tau, 0}$.

For any real φ, the weighted rotation

$$(z, w) \longmapsto (e^{2i\varphi}z, e^{i\varphi}w)$$

is an automorphism of D_τ fixing 0. For a prescribed (X, Y), write $Y = ye^{i\varphi}$ and use the disk above with $a = e^{-2i\varphi}X$. This gives all phases of Y without changing the permitted bound on $|X|$. The first range covers $\mathbb{D} \times \Delta_{r_0}$; the second adds the annular part of $\Delta_{1/2} \times \Delta_{r_\beta}$ above r_0. This proves (4.3). ∎

5 Exact comparison and failure of subharmonicity

The volume of the union in (4.3) gives

$$\frac{V(\tau)}{\pi^2} \ge \frac{3}{4}x_0 + \frac{1}{4}x_\beta. \tag{5.1}$$

The following bounds involve only rational numbers:

$$q_0 = \frac{30901}{50000} = 0.61802, \qquad q_\beta = \frac{7727}{12500} = 0.61816, \qquad q_* = \frac{15451}{25000} = 0.61804.$$

They satisfy

$$\begin{aligned} q_0 + q_0^2 &= 0.9999687204 < 0.999984 = 1 - 16\tau^2, \\ q_\beta + (q_\beta - \tau\beta)^2 &= 0.9999727681 < 0.999984, \\ q_* + q_*^2 &= 1.0000134416 > 1. \end{aligned}$$

All displayed decimals are exact finite decimals. Monotonicity in (4.1) and (2.1) implies

$$x_0 > q_0, \qquad x_\beta > q_\beta, \qquad s < q_*.$$

Hence (5.1) gives

$$\frac{V(\tau)}{\pi^2} > \frac{3}{4}q_0 + \frac{1}{4}q_\beta = \frac{123611}{200000} > q_* > \frac{V(0)}{\pi^2},$$

which proves (1.6). In particular, $V(\tau) - V(0) > 3\pi^2/200000$.

To finish the proof of Theorem 1.2, set $u(t) = -\log V(t)$. Radiality and (1.6) give

$$u(0) > u(\tau) = \frac{1}{2\pi}\int_0^{2\pi} u(\tau e^{i\theta})\, d\theta.$$

This contradicts the submean inequality for a subharmonic function. The parameter circle lies strictly inside $\Delta_{1/100}$.

AI declaration and verification status

The research question and proposed strategy were supplied by Shijie Bao. OpenAI Codex assisted with the construction, calculations, and exposition. This version has not undergone independent mathematical review.

References

[1] G. Hosono. Subharmonic variation of Azukawa pseudometrics for balanced domains. *Internat. J. Math.*, 32(3):2150015, 9 pp., 2021. doi:10.1142/S0129167X21500154. Preprint: arXiv:1811.07154.

[2] W. Zwonek. Regularity properties of the Azukawa metric. *J. Math. Soc. Japan*, 52(4):899–914, 2000. doi:10.2969/jmsj/05240899.