# CRITICAL DIVISORIAL DETECTORS AND EQUISINGULAR ANALYTIC APPROXIMATION

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Abstract. We characterize decreasing equisingular analytic approximation for log canonical psh germs in arbitrary dimension and for psh germs with finite-colength multiplier ideals on smooth surfaces. The criteria require exact divisorial detection of the non-klt components and critical functions, respectively. For toric weights, we construct a toric analytic weight between the original weight and any analytic upper bound. This removes the toric-approximant restriction from the An–Seo criterion and recovers Guan's diagonal theorem.

## 1. Introduction

Demailly's Bergman regularization [4] approximates psh singularities by analytic ones. Requiring decreasing convergence together with equality of the multiplier ideal is more restrictive. Guan first disproved general existence in every dimension at least two [7], and later gave a complete criterion for diagonal weights [8]. An–Seo [1] characterize toric weights under the additional requirement that the approximants are toric. We give divisorial criteria for lc weights and surface germs, and remove the toric-approximant restriction by an analytic sandwich construction.

Let $u$ be psh near $0 \in \mathbb{C}^n$, $R = \mathcal{O}_{\mathbb{C}^n,0}$, and $\mathfrak{m} \subset R$ the maximal ideal. Set

$$\mathcal{I}(u)_0 = \{f \in R : |f|^2 e^{-2u} \in L^1_{\mathrm{loc}}\}, \qquad c_0(u) = \sup\{t > 0 : \mathcal{I}(tu)_0 = R\}.$$

Analytic singularities mean $c \log|\mathfrak{a}| + O(1)$, with $c > 0$ and bounded remainder. Here $\log|\mathfrak{a}| = \frac{1}{2}\log\sum_j |g_j|^2$. For a prime divisor $E$ on a smooth birational model $\pi : Y \to X$, where $X$ is a neighborhood of the origin, write $c_X(E)$ for its center and set $v_E = \mathrm{ord}_E$ and $A_E = 1 + \mathrm{ord}_E K_{Y/X}$. Let $\nu_E(u)$ denote the generic Lelong number of $u \circ \pi$ along $E$. We ask whether there is a decreasing sequence of psh weights satisfying

$$u_j \in \mathrm{PSH}, \quad u_j \downarrow u, \quad u_j \text{ has analytic singularities}, \quad \mathcal{I}(u_j) = \mathcal{I}(u). \tag{E}$$

By Lemma 2.1, this is equivalent to the existence of an analytic weight $\psi = c\log|\mathfrak{a}|$ such that

$$u \le \psi + O(1), \qquad \mathcal{I}(\psi) = \mathcal{I}(u). \tag{U}$$

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All ideal equalities hold on a common sufficiently small neighborhood. A weight is log canonical (lc) there if $\mathcal{I}(tu) = \mathcal{O}$ for every $0 < t < 1$.

**Theorem A** (Log canonical weights)**.** *Let $u$ be lc near $0 \in \mathbb{C}^n$, and let $Z_1, \dots, Z_s$ be the irreducible components of $V(\mathcal{I}(u))$. Then (E) holds if and only if, for every component $Z_i$, there is a prime divisor $E_i$ over $X$ whose center is $Z_i$ and which satisfies*

$$\nu_{E_i}(u) = A_{E_i}. \tag{1.1}$$

*If every $Z_i$ has codimension at least two, choose such divisors $E_i$ and put*

$$\mathfrak{b}_k = \bigcap_i \{f : v_{E_i}(f) \geq kA_{E_i}\}.$$

*Then there is an integer $r > 0$ for which $\psi = r^{-1} \log |\mathfrak{b}_r|$ satisfies (U).*

Kim–Kollár [10, Theorems 1.1, 1.3, 5.2] prove same-ideal analytic realization, quasimonomial attainment, and divisorial approximation of lc places, using an analytic-singularity convention that allows sums with unrelated real coefficients. Theorem A characterizes the existence of an analytic realization satisfying the upper bound in (U); Example 7.1 illustrates the distinction.

**Corollary 1.1** (The maximal ideal)**.** *Let $n \geq 2$ and suppose $\mathcal{I}(u)_0 = \mathfrak{m}$. For a prime divisor $E$ centered at the origin, set $\mathfrak{a}_k(E) = \{f \in R : v_E(f) \geq k\}$.*

(1) *Property (E) holds if and only if there is a prime divisor $E$ centered at the origin such that*

$$\nu_E(u) \geq A_E. \tag{1.2}$$

(2) *Suppose in addition that $c_0(u) = 1$, and fix any prime divisor $E$ centered at the origin satisfying (1.2). For this divisor, $\nu_E(u) = A_E$ and the graded $R$-algebra $\bigoplus_{k\geq 0} \mathfrak{a}_k(E)$ is finitely generated. Moreover, for this same $E$ and every sufficiently divisible positive integer $r$, the weight*

$$\psi = \frac{A_E}{r} \log |\mathfrak{a}_r(E)| \tag{1.3}$$

*satisfies (U).*

Part (1) follows from Theorem A when $c_0(u) = 1$, and from Lemma 2.3 when $c_0(u) < 1$. Part (2) is proved in Section 3.1.

**Definition 1.2** (Critical divisorial detector)**.** For a psh germ $u$, set $J = \mathcal{I}(u)_0$ and

$$J_- = \bigcap_{0<t<1} \mathcal{I}(tu)_0, \qquad \mathcal{C}(u) = J_- \setminus J. \tag{1.4}$$

A germ $f \in \mathcal{C}(u)$ is called critical. A *critical divisorial detector* for $f$ is a prime divisor $E$ over $X$, with $0 \in c_X(E)$, such that

$$\nu_E(u) = v_E(f) + A_E. \tag{1.5}$$

Critical germs are integrable at every scale below one and fail to be integrable at scale one. If $J$ is $\mathfrak{m}$-primary, the descending family $\mathcal{I}(tu)_0/J \subset R/J$ stabilizes as $t \uparrow 1$, so $J_- = \mathcal{I}(tu)_0$ for $t < 1$ sufficiently close to one. Subcritical nonintegrability admits a strict divisorial test by Lemma 2.3; critical germs require the exact equality (1.5).

**Proposition 1.3** (Removing divisorial factors)**.** *Suppose $\mathcal{I}(u)_0 = (h)J_0$, where $J_0$ has no divisorial component. Then*

$$v = u - \log|h| \in \mathrm{PSH}, \qquad \mathcal{I}(v)_0 = J_0,$$

*and* (E) *holds for $u$ if and only if it holds for $v$.*

The proof is given in Section 2. Together with Corollary 1.1, the proposition treats ideals $(h)\mathfrak{m}$ in arbitrary dimension. On a smooth surface, $J_0$ is either the unit ideal or $\mathfrak{m}$-primary. In the latter case the following theorem applies; in the former, a constant upper bound gives (U).

**Theorem B** (Smooth surfaces)**.** *Let $u$ be psh on $(\mathbb{C}^2, 0)$, with proper $\mathfrak{m}$-primary $J = \mathcal{I}(u)_0$. Then* (E) *holds if and only if every critical germ admits a critical divisorial detector centered at the origin. The divisor may depend on the germ. Under this condition, at most $\dim_{\mathbb{C}} R/J$ divisorial tests, counting both exact tests for critical germs and strict tests for germs outside $J_-$, suffice to construct a complete ideal $\mathfrak{a}$ and a positive rational number $q$ such that $\psi = q\log|\mathfrak{a}|$ satisfies* (U)*.*

For a toric weight, set $g(t) = u(e^{t_1}, \dots, e^{t_n})$ and $P(u) = \operatorname{dom} g^*$, where $g^*$ is the convex conjugate. The convex upper set $P(u)$ need not be closed.

**Theorem C** (Toric sandwich)**.** *Let $u$ be toric on $(\mathbb{C}^n, 0)$, and suppose $u \le \psi + O(1)$, where $\psi = c\log|\mathfrak{a}| + O(1)$ has analytic singularities. There are a monomial ideal $\mathfrak{b}$ and an integer $N > 0$, depending only on $\mathfrak{a}$, such that*

$$\chi = \frac{c}{N}\log|\mathfrak{b}|, \qquad u \le \chi + O(1), \qquad \chi \le \psi + O(1). \tag{1.6}$$

Theorem C follows from convex duality and the Rees-valuation criterion for integral closure, independently of Theorems A and B. The sandwich preserves the multiplier ideal when $\mathcal{I}(\psi) = \mathcal{I}(u)$, giving the following corollary.

**Corollary 1.4** (Unrestricted An–Seo criterion)**.** *For toric $u$ on $(\mathbb{C}^n, 0)$, the following are equivalent:*

1. (E) *holds;*
2. (E) *holds with toric approximants;*
3. *there is a polyhedron $Q \subset \mathbb{R}^n_{\ge 0}$, rational after one positive dilation, such that*

$$P(u) \subset Q, \quad Q + \mathbb{R}^n_{\ge 0} \subset Q, \quad \operatorname{int} Q \cap \mathbb{Z}^n_{>0} = \operatorname{int} P(u) \cap \mathbb{Z}^n_{>0}. \tag{1.7}$$

An–Seo [1, Theorem 1.2] prove the equivalence of (2) and (3); Theorem C gives (1)⇒(2).

**Corollary 1.5** (Boundary lattice-point criterion)**.** *Let $u$ be toric on $(\mathbb{C}^n, 0)$, with proper $\mathfrak{m}$-primary $\mathcal{I}(u)_0$. The following are equivalent:*

(1) *Property (E) holds.*
(2) *Every critical germ admits a critical divisorial detector centered at the origin.*
(3) *For every $\beta \in \partial P(u) \cap \mathbb{Z}^n_{>0}$, there is a vector $w \in \mathbb{Z}^n_{>0}$ such that*

$$\inf_{p \in P(u)} \langle p, w \rangle = \langle \beta, w \rangle. \tag{1.8}$$

*In (2), it is enough to test critical monomials. Only finitely many boundary lattice points occur in (3).*

Thus, in the finite-colength toric case, the obstruction is the absence of an integral supporting normal at a boundary lattice point. The proof in Section 6 constructs the polyhedron in Corollary 1.4 directly from these supporting half-spaces.

**Corollary 1.6** (Guan's diagonal criterion)**.** *For $1 \le m \le n$ and $a_i > 0$, the weight $u = \log \sum_{i=1}^m |z_i|^{a_i}$ satisfies (E) if and only if either $(a_1, \dots, a_m) \in c\mathbb{Q}^m_{>0}$ for some $c > 0$, or*

$$\sum_{i=1}^m \frac{p_i}{a_i} = 1 \quad \textit{has no solution } p \in \mathbb{Z}^m_{>0}. \tag{1.9}$$

*The approximants need not be toric.*

This recovers [8, Theorem 1.1], including $m < n$.

The proofs construct one analytic upper bound with the prescribed multiplier ideal. For Theorem A, exact lc places admit a common klt boundary; Fujino's finite-generation theorem then produces a single ideal from their valuation filtrations. Bergman estimates give the upper bound, while the reducedness of the lc multiplier ideal, supplied by Kim–Kollár, gives equality of ideals. For Theorem B, finite colength reduces nonintegrability to finitely many divisorial tests. A quadratic optimization on their finite valuative subtree gives rational coefficients, which are realized by a product of simple complete ideals. Theorem C uses a separate construction: the Rees valuations of an arbitrary analytic upper bound define a rational polyhedron, whose vertices produce the monomial ideal in the sandwich.

## 2. Analytic lemmas

For $B_k w = k^{-1} \log |\mathcal{I}(kw)_0|$, the Bergman estimates [4, 3] give

$$\begin{aligned} B_k w &\ge w - C_k, \qquad e^{2k(B_k w - w)} \in L^1_{\mathrm{loc}}, \\ \nu_F(w) - A_F/k &\le k^{-1} v_F(\mathcal{I}(kw)_0) \le \nu_F(w). \end{aligned} \tag{2.1}$$

Finite generators are chosen on a smaller neighborhood.

**Lemma 2.1.** *Properties (E) and (U) are equivalent. Given $\psi$ in (U), one may take $u_j = \max\{u, \psi - j\}$.*

*Proof.* If (E) holds, the first approximant gives (U) after absorbing its bounded remainder. Conversely, suppose $u \le \psi + C$ and define $u_j = \max\{u, \psi - j\}$. These weights are psh, and

$$\psi - j \le u_j \le \psi + \max\{C, 0\}, \qquad \mathcal{I}(u_j) = \mathcal{I}(\psi) = \mathcal{I}(u), \qquad u_j \downarrow u.$$

The two bounds show that each $u_j$ has the same analytic singularities as $\psi$. The sequence decreases to $u$ wherever $\psi$ is finite; convergence also holds on $\{\psi = -\infty\}$, where $u = -\infty$. □

**Lemma 2.2.** *If $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-\sum b_i E_i)$ on a log resolution, then*

$$u \le q \log|\mathfrak{a}| + O(1) \quad \Longleftrightarrow \quad \nu_{E_i}(u) \ge q b_i \quad (b_i > 0).$$

*Proof.* Necessity follows by restriction transverse to $E_i$. For sufficiency, Siu decomposition gives

$$dd^c(u \circ \pi) - q \sum_i b_i [E_i] \ge 0.$$

Here $s_i$ is a local defining equation of $E_i$. Thus $u \circ \pi - q \sum_i b_i \log|s_i|$ extends locally as a psh function. A finite cover of the compact fiber gives a uniform upper bound. Since $\log|\mathfrak{a}| \circ \pi = \sum_i b_i \log|s_i| + O(1)$, the resulting inequality descends to $X$. □

**Lemma 2.3** (Subcritical approximation)**.** *For every $0 < t < 1$, there is an analytic weight $\psi_t$ such that*

$$u \le \psi_t + O(1), \qquad \mathcal{I}(\psi_t)_0 = \mathcal{I}(tu)_0.$$

*If $f \notin \mathcal{I}(tu)_0$, there is a prime divisor $E$ over $X$ satisfying the strict inequality*

$$\nu_E(u) > v_E(f) + A_E. \tag{2.2}$$

*If $\mathcal{I}(u)_0$ is $\mathfrak{m}$-primary, then $c_X(E) = \{0\}$.*

*Proof.* Normalize $u \le 0$, put $w = tu$, and choose rational $\delta > 0$ with

$$(1+\delta)t < 1, \qquad \mathcal{I}((1+\delta)w)_0 = \mathcal{I}(w)_0$$

by strong openness [9]. For $k \ge (1+\delta)/\delta$, Hölder's inequality applied to

$$|f|^2 e^{-2w} = \left(|f|^2 e^{-2(1+\delta)B_k w}\right)^{1/(1+\delta)} |f|^{2\delta/(1+\delta)} e^{2(B_k w - w)}$$

and (2.1) gives

$$\mathcal{I}((1+\delta)w)_0 \subset \mathcal{I}((1+\delta)B_k w)_0 \subset \mathcal{I}(w)_0.$$

Hence $\psi_t = (1+\delta)B_k w$ has the required ideal and $\psi_t \ge (1+\delta)tu - C \ge u - C$. If $f \notin \mathcal{I}(\psi_t)_0$, a divisor on its log resolution satisfies

$$v_E(f) + A_E \le \nu_E(\psi_t) \le (1+\delta)t\nu_E(u) < \nu_E(u).$$

If the center were positive-dimensional and $\mathcal{I}(u)_0$ were $\mathfrak{m}$-primary, generic integrability along the center would give $\nu_E(u) < A_E$. This contradicts (2.2), since $v_E(f) \ge 0$. □

*Proof of Proposition 1.3.* The generic codimension-one multiplier-ideal formula gives $\lfloor \nu_H(u) \rfloor = v_H(h)$ for every prime hypersurface $H$. Siu subtraction and the principal-ideal shift formula therefore give

$$v = u - \log|h| \in \mathrm{PSH}, \qquad \mathcal{I}(u) = (h)\mathcal{I}(v), \qquad \mathcal{I}(v)_0 = J_0.$$

We first observe that an analytic upper bound can be chosen with rational coefficient without changing its multiplier ideal. Indeed, if $\psi = c\log|\mathfrak{a}|$ and $\mathfrak{a}\mathcal{O}_Y = \mathcal{O}_Y(-D)$ on a log resolution, choose $q \in \mathbb{Q}_{>0}$ with

$$q \le c, \qquad \lfloor qD \rfloor = \lfloor cD \rfloor.$$

For rational $c$, take $q = c$; otherwise every nonzero coefficient of $cD$ is nonintegral, so a sufficiently close rational $q < c$ works. The log-resolution formula gives

$$\mathcal{I}(q\log|\mathfrak{a}|) = \mathcal{I}(c\log|\mathfrak{a}|), \qquad c\log|\mathfrak{a}| \le q\log|\mathfrak{a}| + O(1).$$

Now let $\psi$ satisfy (U) for $u$, and write $q = p/r$ as above. Equality of multiplier ideals along each hypersurface implies

$$qv_H(\mathfrak{a}) \ge v_H(h), \qquad \mathfrak{a}^p \subset (h^r).$$

Thus $\mathfrak{c} = h^{-r}\mathfrak{a}^p \subset R$ is an ideal and

$$v \le \frac{1}{r}\log|\mathfrak{c}| + O(1), \qquad (h)\mathcal{I}\left(\frac{1}{r}\log|\mathfrak{c}|\right) = \mathcal{I}(q\log|\mathfrak{a}|) = (h)J_0.$$

Canceling $(h)$ gives the residual upper bound with multiplier ideal $J_0$. Conversely, rationalize a certificate for $v$ and write it as $r^{-1}\log|\mathfrak{c}|$ by taking a power of its ideal. Then

$$u \le \frac{1}{r}\log|h^r\mathfrak{c}| + O(1), \qquad \mathcal{I}\left(\frac{1}{r}\log|h^r\mathfrak{c}|\right) = (h)J_0.$$

Lemma 2.1 proves the equivalence of the approximation properties. □

## 3. Log canonical weights

We use the following finite-generation theorem in the complex analytic category.

**Theorem 3.1** (Fujino [6, Theorem 1.18])**.** *Let $\pi : Y \to X$ be a projective morphism of complex analytic spaces, and let $(Y, B)$ be klt with $K_Y + B$ $\mathbb{Q}$-Cartier. Then*

$$R(Y/X, K_Y + B) = \bigoplus_{m\ge 0} \pi_*\mathcal{O}_Y(\lfloor m(K_Y + B)\rfloor)$$

*is locally finitely generated over $\mathcal{O}_X$.*

Here log resolutions are chosen as compositions of blowups, hence projective over a representative of the analytic germ. We apply the theorem after shrinking that representative around the origin.

**Lemma 3.2** (Simultaneous finite generation)**.** *Let $E_1, \dots, E_s$ be divisors over a smooth analytic germ, with centers of codimension at least two. Suppose there is an effective rational divisor $\Delta$ such that the pair $(X, \Delta)$ is klt and its log discrepancies satisfy*

$$0 < a_i := A(E_i, X, \Delta) < 1 \qquad (1 \le i \le s).$$

*For any positive integers $d_i$, set $\mathfrak{b}_k = \bigcap_i \{f : v_{E_i}(f) \ge k d_i\}$. Then $\bigoplus_k \mathfrak{b}_k$ is finitely generated and, for some $r > 0$,*

$$\mathfrak{b}_{rj} = \mathfrak{b}_r^j \qquad (j \ge 1). \tag{3.1}$$

*The hypothesis on $\Delta$ holds whenever an lc weight $u$ satisfies $\nu_{E_i}(u) = A_{E_i}$ for all $i$.*

*Proof.* Take a projective log resolution $\pi : Y \to X$ extracting all $E_i$. In this proof, $F$ ranges over the prime $\pi$-exceptional divisors, and $a_F = A(F, X, \Delta) > 0$. Thus

$$K_Y + \Delta_Y^{\mathrm{str}} = \pi^*(K_X + \Delta) + \sum_F (a_F - 1)F.$$

Choose rational $\epsilon > 0$ and $\delta_F$ with $\epsilon d_i < 1 - a_i$ and $0 < \delta_F < \min\{1, a_F\}$. Define the snc boundary

$$B = \Delta_Y^{\mathrm{str}} + \sum_i (1 - a_i - \epsilon d_i)E_i + \sum_{F \notin \{E_i\}} (1 - \delta_F)F.$$

All coefficients lie in $[0, 1)$. Moreover,

$$K_Y + B - \pi^*(K_X + \Delta) = -\epsilon \sum_i d_i E_i + \underbrace{\sum_{F \notin \{E_i\}} (a_F - \delta_F)F}_{F_+ \ge 0}. \tag{3.2}$$

By Theorem 3.1, $R(Y/X, K_Y + B)$ is locally finitely generated. Choose $M$ clearing denominators, with $L = M\epsilon \in \mathbb{Z}_{>0}$. Locally trivializing $M(K_X + \Delta)$, its $M$-th Veronese has degree $j$ part

$$\pi_* \mathcal{O}_Y\big(-Lj \sum_i d_i E_i + MjF_+\big) = \mathfrak{b}_{Lj}.$$

A meromorphic section extends holomorphically across the exceptional image, which has codimension at least two. For a holomorphic function the displayed divisor imposes exactly $v_{E_i}(f) \ge Ljd_i$; the effective divisor $MjF_+$ imposes no further condition. Passing to a further Veronese subalgebra gives (3.1).

Put $\mathcal{B} = \bigoplus_j \mathfrak{b}_r^j T^{rj}$ and $\mathcal{A} = \bigoplus_k \mathfrak{b}_k T^k$. Each $fT^k \in \mathcal{A}$ is integral over $\mathcal{B}$, since $(fT^k)^r \in \mathfrak{b}_{rk} T^{rk} = \mathfrak{b}_r^k T^{rk}$. The excellent ring $\mathcal{B}$ has finite normalization in $\mathrm{Frac}(R)(T)$, a finite extension of $\mathrm{Frac}(\mathcal{B}) = \mathrm{Frac}(R)(T^r)$. Hence $\mathcal{A}$, as a submodule of this normalization, is finite over $\mathcal{B}$.

It remains to construct a common divisor $\Delta$ when the $E_i$ are lc places of $u$. The Bergman estimates give

$$\eta_m = (1 - 1/m)B_m u, \qquad \mathcal{I}(\eta_m) \supset \mathcal{I}((1 - 1/m)u) = \mathcal{O},$$
$$0 < A_{E_i} - \nu_{E_i}(\eta_m) \le A_{E_i}(2/m - 1/m^2) < 1 \quad (m \gg 1).$$

Fix $m$ large enough for these inequalities to hold for every $i$, and write $\eta_m = c\log|\mathfrak{a}|$, with $c \in \mathbb{Q}\cap(0,1)$. This weight is klt. On a common log resolution, a general $g \in \mathfrak{a}$ has smooth residual divisor transverse to the boundary. Thus $\Delta = c\,\mathrm{div}(g)$ defines a klt pair and satisfies $A(E_i, X, \Delta) = A_{E_i} - \nu_{E_i}(\eta_m)$. □

*Proof of Theorem A.* Suppose (U) holds. Since $u \le \psi + O(1)$ and $u$ is lc, the weight $\psi$ is also lc. Its non-klt locus has the same irreducible components $Z_i$. On a log resolution, each such component is the image of a prime divisor $E_i$ for which

$$A_{E_i} = \nu_{E_i}(\psi) \le \nu_{E_i}(u) \le A_{E_i}.$$

This proves necessity.

Conversely, first suppose $\operatorname{codim} Z_i \ge 2$. By [10, Corollary 1.2],

$$J = \mathcal{I}(u)_0 = \bigcap_i I_{Z_i,0}. \tag{3.3}$$

Apply Lemma 3.2 with $d_i = A_{E_i}$. By (2.1),

$$\mathcal{I}((k+1)u)_0 \subset \mathfrak{b}_k, \qquad v_F(\mathfrak{b}_k) \le (k+1)\nu_F(u).$$

Taking $k = rj$ gives, for every divisor $F$,

$$\frac{1}{r}v_F(\mathfrak{b}_r) = \frac{1}{rj}v_F(\mathfrak{b}_{rj}) \le \left(1 + \frac{1}{rj}\right)\nu_F(u) \longrightarrow \nu_F(u).$$

For $\psi = r^{-1}\log|\mathfrak{b}_r|$, Lemma 2.2 and the defining inequalities of $\mathfrak{b}_r$ yield

$$u \le \psi + O(1), \qquad A_{E_i} \le \nu_{E_i}(\psi) \le \nu_{E_i}(u) = A_{E_i}.$$

Consequently,

$$J \subset \mathcal{I}(\psi)_0 \subset \bigcap_i \{f : v_{E_i}(f) + A_{E_i} > A_{E_i}\} = \bigcap_i I_{Z_i,0} = J.$$

To treat codimension-one components, note that each such component $H$ satisfies $1 \le \nu_H(u) \le A_H = 1$. Let $h$ be their reduced equation. Removing these divisorial factors, Siu subtraction, the principal-ideal shift formula, and (3.3) give

$$v = u - \log|h| \in \mathrm{PSH}, \qquad \mathcal{I}(v) = \mathcal{I}(u) : (h) = \bigcap_{Z_i \not\subset \{h=0\}} I_{Z_i}.$$

The weight $v$ is lc. For the remaining components, $v_{E_i}(h) = 0$, so $\nu_{E_i}(v) = A_{E_i}$. A residual certificate $r^{-1}\log|\mathfrak{b}|$ therefore gives

$$u \le \frac{1}{r}\log|h^r\mathfrak{b}| + O(1), \qquad \mathcal{I}\left(\frac{1}{r}\log|h^r\mathfrak{b}|\right) = \mathcal{I}(u).$$

If the residual ideal is $\mathcal{O}$, a constant upper bound serves as the residual certificate. In either case, Lemma 2.1 gives the required decreasing sequence. □

### 3.1. **The maximal ideal.**

**Lemma 3.3.** *Let $E$ be a prime divisor with $P = \nu_E(u) > 0$, and put $\mathfrak{a}_k(E) = \{f : v_E(f) \geq k\}$. Then, for every prime divisor $F$,*

$$v_F(\mathfrak{a}_\bullet(E)) := \inf_k \frac{v_F(\mathfrak{a}_k(E))}{k} \leq \frac{\nu_F(u)}{P}, \qquad v_E(\mathfrak{a}_\bullet(E)) = 1. \tag{3.4}$$

*Proof.* Put $\ell_k = \lceil (k + A_E)/P \rceil$. Then (2.1) gives

$$\mathcal{I}(\ell_k u)_0 \subset \mathfrak{a}_k(E), \qquad \frac{v_F(\mathfrak{a}_k(E))}{k} \leq \frac{\ell_k}{k}\nu_F(u) \longrightarrow \frac{\nu_F(u)}{P}.$$

If $v_E(g) = e > 0$, then $g^j \in \mathfrak{a}_{je}(E)$ has value $je$; this proves the last assertion. □

*Proof of Corollary 1.1.* For necessity in part (1), let $\psi$ satisfy (U). Since $1 \notin \mathcal{I}(\psi)_0$, its log resolution contains a divisor $E$ with $\nu_E(u) \geq \nu_E(\psi) \geq A_E$. Its center is the origin because $\mathcal{I}(\psi)_0 = \mathfrak{m}$. For sufficiency, first suppose $c_0(u) < 1$ and choose $c_0(u) < t < 1$. Then

$$\mathfrak{m} \subset \mathcal{I}(tu)_0 \subsetneq R \quad \Longrightarrow \quad \mathcal{I}(tu)_0 = \mathfrak{m},$$

and Lemma 2.3 gives (U). If $c_0(u) = 1$, the weight is lc near the origin, so Theorem A gives sufficiency. This proves part (1).

For part (2), assume $c_0(u) = 1$ and fix any prime divisor $E$ centered at the origin satisfying (1.2). The weight is lc near zero because $\mathcal{I}(u)$ is the unit ideal away from zero. Hence $\nu_E(u) \leq A_E$, and (1.2) gives $\nu_E(u) = A_E$. Lemma 3.2, applied to this fixed divisor $E$ with $d_1 = 1$, gives finite generation and

$$\mathfrak{a}_{rj}(E) = \mathfrak{a}_r(E)^j. \tag{3.5}$$

Combining (3.4) and (3.5),

$$\nu_F\left(\frac{A_E}{r}\log|\mathfrak{a}_r(E)|\right) = A_E v_F(\mathfrak{a}_\bullet(E)) \leq \nu_F(u).$$

Lemma 2.2 gives the upper bound; its coefficient at $E$ equals $A_E$. Hence $\mathfrak{m} \subset \mathcal{I}(\psi)_0 \subsetneq R$, proving (1.3). □

For a single lc place, the finite-generation conclusion also follows from [2, Propositions 1.3, 1.5].

## 4. The surface criterion

We first prove sufficiency in Theorem B, so assume its detection condition. For a prime divisor $E$ centered at the origin, put $V_E(u) = \{f : v_E(f) + A_E > \nu_E(u)\}$. Every such ideal contains $J$. Conversely, each $f \notin J$ is excluded by at least one of these ideals: use (1.5) when $f \in J_-$, and Lemma 2.3 otherwise.

Starting with $K_0 = R$, choose $f_i \in K_i \setminus J$, a divisor $E_{i+1}$ excluding $f_i$, and $K_{i+1} = K_i \cap V_{E_{i+1}}(u)$. Since $\dim K_{i+1}/J < \dim K_i/J$, this gives distinct divisors with

$$J = \bigcap_{i=1}^{s} V_{E_i}(u), \qquad s \leq \dim_{\mathbb{C}} R/J. \tag{4.1}$$

We realize these tests using the valuative tree.

4.1. **Tree potentials.** Let $\mathcal{V}$ be the valuative tree of $R$, normalized by $\nu(\mathfrak{m}) = 1$, with root $\nu_{\mathfrak{m}}$ and skewness $\alpha$. Its kernel is

$$k(\nu, \mu) = \alpha(\nu \wedge \mu), \qquad \alpha(\nu_{\mathfrak{m}}) = 1.$$

The following facts specify the tree-potential input to the construction. A complete ideal means an integrally closed ideal.

**Proposition 4.1** (Favre–Jonsson [5, Sections 1.2.9–1.3, 1.4.3, 1.4.5])**.** *For $R = \mathcal{O}_{\mathbb{C}^2,0}$, the following hold.*

(1) *A finite positive Borel measure $\rho$ on $\mathcal{V}$ determines a tree potential*

$$g_\rho(\nu) = \int_{\mathcal{V}} k(\nu, \mu)\, d\rho(\mu), \qquad \Delta g_\rho = \rho, \qquad \rho(\mathcal{V}) = g_\rho(\nu_{\mathfrak{m}}).$$

*This correspondence is one-to-one. Potentials are finite on quasi-monomial valuations and are extended to the whole tree by monotone limits.*

(2) *Every psh germ $u$ has a positive tree potential $g_u$. For a prime divisor $E$ centered at the origin, put $b_E = v_E(\mathfrak{m})$ and $\eta_E = b_E^{-1} v_E$. Then*

$$g_u(\eta_E) = \nu_E(u)/b_E.$$

(3) *For each such $E$, there is a simple complete $\mathfrak{m}$-primary ideal $\mathfrak{p}_E \subset R$ with*

$$g_{\mathfrak{p}_E}(\nu) = \nu(\mathfrak{p}_E) = b_E k(\nu, \eta_E), \qquad \Delta g_{\mathfrak{p}_E} = b_E \delta_{\eta_E}. \tag{4.2}$$

*Finite products of these ideals are complete, and their tree potentials add.*

These statements apply to convergent holomorphic germs: [5, Section 1.1] includes $\mathcal{O}_{\mathbb{C}^2,0}$ among its regular local rings, and its psh transforms are defined in this analytic setting. The ideals in (3) thus lie in $R$. For our selected divisors, write

$$b_i = v_{E_i}(\mathfrak{m}), \qquad \eta_i = b_i^{-1} v_{E_i}, \qquad P_i = \nu_{E_i}(u), \qquad g_u(\eta_i) = P_i/b_i,$$

and let $\mathfrak{p}_i = \mathfrak{p}_{E_i}$.

**Lemma 4.2** (Domination)**.** *Let $h$ and $g$ be positive tree potentials, with $\Delta h$ supported on finitely many divisorial valuations. If $h \le g$ on* $\operatorname{supp} \Delta h$*, then $h \le g$ on $\mathcal{V}$.*

*Proof.* The case $h = 0$ is immediate. Otherwise let $T$ be the finite rooted subtree spanned by $\operatorname{supp} \Delta h$, and let $r_T : \mathcal{V} \to T$ be the retraction. Put

$$\rho_T = (r_T)_*(\Delta g), \qquad g_T(\nu) = \int_T k(\nu, \mu)\, d\rho_T(\mu) = g(r_T \nu).$$

Then $g_T = g$ on $T$, $g_T \le g$, and $h = h \circ r_T$. Attach an edge of length one below the root, with lower endpoint $o$, and extend $\alpha$ to zero at $o$. On the resulting finite metric tree $\widehat{T}$, extend both potentials linearly to zero at $o$.

For $G(\nu) = \int_T k(\nu,\mu)\,d\sigma(\mu)$ and any absolutely continuous test function $w$ with $w(o) = 0$,

$$\int_{\widehat{T}} G'w'\,d\alpha = \int_T \left(\int_{[o,\mu]} w'\,d\alpha\right) d\sigma(\mu) = \int_T w\,d\sigma.$$

Derivatives are taken along edges directed away from $o$. The potentials are Lipschitz on $\widehat{T}$, so this identity applies to $z = (h - g_T)_+$. Since $z = 0$ on $\operatorname{supp}\Delta h$,

$$\int_{\widehat{T}} |z'|^2\,d\alpha = \int_T z\,d(\Delta h - \rho_T) = -\int_T z\,d\rho_T \le 0.$$

Thus $z = 0$, proving $h \le g_T \le g$. □

4.2. **Rational interpolation.** To preserve the selected integrability tests, take the integer parts before normalizing:

$$d_i = \frac{\lfloor P_i \rfloor}{b_i}, \qquad K_{ij} = k(\eta_i,\eta_j). \tag{4.3}$$

Each selected test is nontrivial, so $d_i > 0$, and $d_i \le g_u(\eta_i)$.

**Lemma 4.3.** *The quadratic problem*

$$\max_{c\in\mathbb{R}^s_{\ge 0}} \left(d^T c - \frac{1}{2}c^T K c\right) \tag{4.4}$$

*has a unique rational solution $c$. The positive tree potential $h = \sum_i c_i k(\,\cdot\,,\eta_i)$ satisfies*

$$h \le g_u, \qquad h(\eta_i) \ge d_i \quad (1 \le i \le s). \tag{4.5}$$

*Proof.* The matrix $K$ is rational, since the meet of two divisorial valuations is divisorial and has rational skewness. It is positive definite: on the finite rooted hull of the $\eta_i$, with the extra edge $[o,\nu_{\mathfrak{m}}]$ as above,

$$x^T K x = \int_{\widehat{T}} \left(\sum_i x_i \mathbf{1}_{[o,\eta_i]}(\tau)\right)^2 d\alpha(\tau).$$

If the integral vanishes, the terminal edges give $x_i = 0$ at each maximal marked vertex. Removing these vertices and repeating gives $x = 0$. Thus the objective in (4.4) is strictly concave and tends to $-\infty$ as $|c| \to \infty$. Its unique maximizer satisfies

$$c \ge 0, \qquad Kc \ge d, \qquad c_i((Kc)_i - d_i) = 0.$$

For $I = \{i : c_i > 0\}$, these conditions give

$$c_I = K_{II}^{-1} d_I \in \mathbb{Q}^I, \qquad c_{I^c} = 0.$$

Moreover, $h(\eta_i) = (Kc)_i \ge d_i$, with equality whenever $c_i > 0$. Hence $h \le g_u$ on $\operatorname{supp}\Delta h$, and Lemma 4.2 gives (4.5). □

*Sufficiency in Theorem B.* Choose a positive integer $N$ such that $m_i = Nc_i/b_i$ is integral for every $i$, and set

$$\mathfrak{a} = \prod_{i=1}^{s} \mathfrak{p}_i^{m_i}, \qquad \psi = N^{-1} \log |\mathfrak{a}|. \tag{4.6}$$

This is a complete $\mathfrak{m}$-primary ideal, and (4.2) gives

$$g_\psi = \frac{1}{N} \sum_i m_i g_{\mathfrak{p}_i} = \sum_i c_i k(\,\cdot\,, \eta_i) = h \le g_u.$$

Applying Lemma 2.2 on a log resolution of $\mathfrak{a}$, we obtain $u \le \psi + O(1)$, hence $J \subset \mathcal{I}(\psi)_0$. Conversely, for every selected divisor,

$$f \in \mathcal{I}(\psi)_0 \Longrightarrow v_{E_i}(f) + A_{E_i} > \nu_{E_i}(\psi) = b_i h(\eta_i) \ge \lfloor P_i \rfloor.$$

Since $v_{E_i}(f) + A_{E_i}$ is an integer, it is strictly greater than $P_i$. Equation (4.1) now gives $\mathcal{I}(\psi)_0 \subset J$. Lemma 2.1 supplies the decreasing equisingular sequence. □

*Necessity in Theorem B.* Let $\psi$ satisfy (U), and fix $f \in \mathcal{C}(u)$. Since $f \in \mathcal{I}(tu)_0$ for every $t < 1$, letting $t$ increase to one in the divisorial integrability inequalities gives

$$v_E(f) + A_E \ge \nu_E(u)$$

for every prime divisor $E$. On the other hand, $f \notin \mathcal{I}(\psi)_0$, so a divisor $E$ on a log resolution of $\psi$ satisfies

$$v_E(f) + A_E \le \nu_E(\psi) \le \nu_E(u) \le v_E(f) + A_E.$$

All inequalities are therefore equalities, which gives (1.5). The center of $E$ is the origin because $\mathcal{I}(\psi)$ is the unit ideal away from zero. This necessity argument is valid in every dimension. □

## 5. Simultaneous construction in higher dimension

**Theorem 5.1.** *Let $J = \mathcal{I}(u)_0$ be proper and $\mathfrak{m}$-primary on $(\mathbb{C}^n, 0)$. Suppose there are finitely many prime divisors $E_1, \dots, E_s$ centered at the origin such that*

$$J = \bigcap_i \{f : v_{E_i}(f) + A_{E_i} > P_i\}, \qquad P_i = \nu_{E_i}(u).$$

*Discard trivial tests, set $d_i = \lfloor P_i \rfloor$, and put*

$$\mathfrak{b}_k = \bigcap_i \{f : v_{E_i}(f) \geq k d_i\}. \tag{5.1}$$

*If $\bigoplus_k \mathfrak{b}_k$ is finitely generated, then $r^{-1} \log |\mathfrak{b}_r|$ satisfies (U) for sufficiently divisible $r$. In particular, the algebra is finitely generated if all the valuations are monomial in one common system of holomorphic coordinates. It is also finitely generated whenever*

$$\max_i \frac{A_{E_i} - 1}{P_i} < c_0(u). \tag{5.2}$$

*Proof.* Choose an integer $C \geq \max_i A_{E_i}/P_i$. By (2.1),

$$\mathcal{I}((k+C)u)_0 \subset \mathfrak{b}_k, \qquad v_F(\mathfrak{b}_k) \leq (k+C)\nu_F(u).$$

Finite generation allows us to choose $r$ such that $\mathfrak{b}_{rj} = \mathfrak{b}_r^j$ for every $j \geq 1$. Taking $k = rj$, dividing by $rj$, and letting $j \to \infty$, we obtain the divisorial inequalities required by Lemma 2.2. Hence $u \leq r^{-1} \log |\mathfrak{b}_r| + O(1)$. Moreover, for each selected divisor $E_i$,

$$f \in \mathcal{I}\left(r^{-1} \log |\mathfrak{b}_r|\right)_0 \Longrightarrow v_{E_i}(f) + A_{E_i} > d_i \Longrightarrow v_{E_i}(f) + A_{E_i} > P_i,$$

The upper bound gives $J \subset \mathcal{I}(r^{-1} \log |\mathfrak{b}_r|)_0$, and the divisorial tests give the reverse inclusion.

For common monomial weights $w_i$, the algebra is the analytic monomial algebra of

$$\{(\alpha, k) \in \mathbb{Z}^{n+1}_{\geq 0} : w_i \cdot \alpha \geq k d_i \text{ for every } i\},$$

which is finitely generated by Gordan's lemma. Under (5.2), choose rational $\max_i (A_{E_i} - 1)/P_i < \sigma < c_0(u)$. For $m \gg 1$,

$$\sigma B_m u \text{ is klt}, \qquad 0 < A_{E_i} - \sigma \nu_{E_i}(B_m u) < 1.$$

A general member of $\mathcal{I}(mu)_0$, with coefficient $\sigma/m < 1$, gives the common klt divisor required by Lemma 3.2. □

For $n \geq 3$, the sufficiency of (1.5) for general non-lc weights remains open. Theorem 5.1 gives a sufficient finite-generation condition.

## 6. Toric weights

An–Seo [1, Proposition 3.1 and Section 5] represent toric analytic weights by monomial ideals and use the Newton polyhedron of the first toric approximant. For an arbitrary analytic upper bound $c\log|\mathfrak{a}|$, we instead form a rational polyhedron from the Rees valuations of $\mathfrak{a}$. Clearing the denominators of its vertices gives a monomial ideal satisfying both inequalities in (1.6), and hence removes the toric-approximant hypothesis.

**Lemma 6.1** (Polyhedral upper bound)**.** *Suppose the Newton convex set $P(u)$ is contained in a polyhedron*

$$Q=\operatorname{conv}\{v_1,\dots,v_l\}+\mathbb{R}^n_{\geq 0}\subset\mathbb{R}^n_{\geq 0}.$$

*Then the vectors $v_j$ give the upper bound*

$$u(z)\leq\max_j\langle v_j,\log|z|\rangle+O(1). \tag{6.1}$$

*Proof.* Fix $s$ in the logarithmic domain. For $t_i<s_i$, convex duality gives

$$\begin{aligned} g(t)&=\sup_{p\in P(u)}\left(\langle p,t\rangle-g^*(p)\right)\\ &\leq g(s)+\sup_{p\in P(u)}\langle p,t-s\rangle\leq g(s)+\max_j\langle v_j,t-s\rangle\\ &\leq\max_j\langle v_j,t\rangle+g(s)+\max_j(-\langle v_j,s\rangle). \end{aligned}$$

The last two terms give a finite constant independent of $t$. This proves the inequality away from the coordinate hyperplanes; the psh extension principle gives it across them as well. □

*Proof of Theorem C.* If $\mathfrak{a}=R$, take $\mathfrak{b}=R$ and $N=1$. Otherwise let $E_i$ be the Rees divisors of $\mathfrak{a}$, and put

$$b_i=v_{E_i}(\mathfrak{a})>0,\qquad w_i=(v_{E_i}(z_1),\dots,v_{E_i}(z_n)).$$

The integral-closure criterion on the normalized blowup [11, Theorems 2.1, 4.6] states that

$$f\in\overline{\mathfrak{a}^k}\quad\Longleftrightarrow\quad v_{E_i}(f)\geq kb_i\quad\text{for every } i. \tag{6.2}$$

The corresponding monomial inequalities define the rational polyhedron

$$Q_{\mathfrak{a}}=\{p\in\mathbb{R}^n_{\geq 0}:\langle p,w_i\rangle\geq b_i\text{ for every }i\}. \tag{6.3}$$

Each $p\in P(u)$ gives a logarithmic lower bound for $u$. Comparing its divisorial values with those of the given analytic upper bound yields

$$\langle p,\log|z|\rangle-C_p\leq u\leq c\log|\mathfrak{a}|+O(1)\Longrightarrow\langle p,w_i\rangle\geq cb_i.$$

Thus $P(u)\subset cQ_{\mathfrak{a}}$. The polyhedron $Q_{\mathfrak{a}}$ is nonempty and has recession cone $\mathbb{R}^n_{\geq 0}$, so we can write $Q_{\mathfrak{a}}=\operatorname{conv}\{v_1,\dots,v_l\}+\mathbb{R}^n_{\geq 0}$. Choose $N$ with $Nv_j\in\overline{\mathbb{Z}}^n_{\geq 0}$ and set

$$\mathfrak{b}=(z^{Nv_1},\dots,z^{Nv_l}),\qquad\chi=\frac{c}{N}\log|\mathfrak{b}|. \tag{6.4}$$

By (6.2) and the integral-closure estimate,

$$z^{Nv_j} \in \overline{\mathfrak{a}^N} \Longrightarrow \log|\mathfrak{b}| \le N\log|\mathfrak{a}| + O(1) \Longrightarrow \chi \le \psi + O(1).$$

Conversely, Lemma 6.1 gives

$$u \le c\max_j \langle v_j, \log|z|\rangle + O(1) = \chi + O(1).$$

□

The constructed ideal satisfies

$$\overline{\mathfrak{b}} = \mathrm{Core}_T(\overline{\mathfrak{a}^N}), \qquad \mathrm{Core}_T(\mathfrak{c}) = (z^\alpha : z^\alpha \in \mathfrak{c}), \tag{6.5}$$

since both sides have exponent set $NQ_{\mathfrak{a}} \cap \mathbb{Z}^n_{\ge 0}$.

*Proof of Corollary 1.4.* Suppose (1) holds. Take the first approximant $\psi$ and apply Theorem C to obtain a toric analytic weight $\chi$ between $u$ and $\psi$, up to bounded terms. Then

$$\mathcal{I}(u)_0 \subset \mathcal{I}(\chi)_0 \subset \mathcal{I}(\psi)_0 = \mathcal{I}(u)_0,$$

Lemma 2.1, applied to $u$ and $\chi$, now gives the toric approximants required in (2). Conversely, (2) implies (1) directly. For toric $v$, [1, Theorem 2.10] gives

$$z^\alpha \in \mathcal{I}(v)_0 \quad \Longleftrightarrow \quad \alpha + \mathbf{1} \in \mathrm{int}\, P(v). \tag{6.6}$$

The constructed weight has $P(\chi) = cQ_{\mathfrak{a}}$. Its upper bound for $u$ gives $P(u) \subset P(\chi)$, and equality of multiplier ideals gives the interior lattice-point condition. Thus (1) implies (3). Conversely, suppose (3) holds. Choose $\lambda > 0$ such that $Q' = \lambda^{-1}Q$ is rational, and write

$$Q' = \mathrm{conv}\{v'_1, \dots, v'_l\} + \mathbb{R}^n_{\ge 0}, \qquad v'_j \in \mathbb{Q}^n_{\ge 0}.$$

Choose an integer $D > 0$ with $Dv'_j \in \mathbb{Z}^n_{\ge 0}$, and set

$$\chi = \frac{\lambda}{D}\log|(z^{Dv'_1}, \dots, z^{Dv'_l})|.$$

Then $P(\chi) = Q$; (6.1), (1.7), and (6.6) give (U). □

*Proof of Corollary 1.5.* The necessity argument for Theorem B gives (1)⇒(2) in every dimension. Put $J = \mathcal{I}(u)_0$ and

$$B = \{\beta \in \mathbb{Z}^n_{>0} : z^{\beta-\mathbf{1}} \notin J\}, \qquad |B| = \dim_{\mathbb{C}} R/J < \infty.$$

Every positive integral boundary point belongs to $B$. If $\beta \in \partial P(u)$, the upper-set property and $\beta > 0$ imply $\beta/t \in \mathrm{int}\, P(u)$ for every $t < 1$. By (6.6), $z^{\beta-\mathbf{1}}$ is critical. Assuming (2) only for these monomials, take a detector $E$, put $w_i = v_E(z_i) > 0$, and set $\delta = A_E - \sum_i w_i \ge 0$. The last inequality is the discrepancy inequality for the snc coordinate boundary. Then

$$\langle \beta, w\rangle + \delta = \nu_E(u) \le \inf_{p \in P(u)} \langle p, w\rangle \le \langle \beta, w\rangle.$$

Thus $\delta = 0$ and $w \in \mathbb{Z}^n_{>0}$ satisfies (1.8), proving (3).

Conversely, assume (3). A point $\beta \in B$ is either on $\partial P(u)$ or outside $\overline{P(u)}$. In the first case, use (1.8). In the second, strict separation gives $w \geq 0$ with

$$\inf_{p \in P(u)} \langle p, w \rangle > \langle \beta, w \rangle.$$

Choose a positive rational vector $w' \geq w$ sufficiently close to $w$. Since $P(u) \subset \mathbb{R}^n_{\geq 0}$, the infimum cannot decrease, whereas $\langle \beta, w' \rangle$ remains below it. We therefore have, for each $\beta \in B$, a positive rational vector $w_\beta$ such that

$$P(u) \subset \{p : \langle p, w_\beta \rangle \geq \langle \beta, w_\beta \rangle\}.$$

Set

$$Q = \mathbb{R}^n_{\geq 0} \cap \bigcap_{\beta \in B} \{p : \langle p, w_\beta \rangle \geq \langle \beta, w_\beta \rangle\}.$$

This rational upper polyhedron contains $P(u)$ and excludes every $\beta \in B$ from its interior. All other positive lattice points lie in $\operatorname{int} P(u) \subset \operatorname{int} Q$. Thus (1.7) holds, and Corollary 1.4 proves (1). □

The same necessary equality, with $w \geq 0$, shows that a positive integral boundary point admitting no nonzero rational supporting normal obstructs (E), without a colength assumption.

### 6.1. **Guan's criterion.**

*Proof of Corollary 1.6.* Monomial integration gives

$$z^\alpha \in \mathcal{I}(tu)_0 \quad \Longleftrightarrow \quad \sum_{i=1}^m \frac{\alpha_i + 1}{a_i} > t.$$

The multiplier ideal is the extension to $R$ of the ideal in the first $m$ variables. Suppose first that (1.9) has no positive integer solution. The finite set $\{p \in \mathbb{Z}^m_{>0} : \sum p_i/a_i \leq 1\}$ is then separated from level one. We can choose $t < 1$ sufficiently close to one so that $\mathcal{I}(tu)_0 = \mathcal{I}(u)_0$, and Lemma 2.3 gives (U). If instead the $a_i$ are commensurable, $u$ already has analytic singularities and hence satisfies (E).

It remains to rule out approximation when the exponents are not commensurable and (1.9) has a positive integer solution $p$. The monomial $f = z^{p-\mathbf{1}}$ then belongs to $\mathcal{I}(tu)_0$ for every $t < 1$, but not to $\mathcal{I}(u)_0$. For a prime divisor $E$ whose center contains the origin, put $w_i = v_E(z_i)$. The snc coordinate boundary gives

$$A_E \geq \sum_i w_i, \qquad \nu_E(u) = \min_i a_i w_i.$$

If some $w_i = 0$, then $\nu_E(u) = 0 < v_E(f) + A_E$. Otherwise the $a_i w_i$ are not all equal, and

$$\nu_E(u) < \sum_i \frac{p_i}{a_i}(a_i w_i) = \sum_i p_i w_i \leq v_E(f) + A_E.$$

If an analytic upper certificate existed, its multiplier ideal would exclude $f$. A divisor on its log resolution would then satisfy $v_E(f) + A_E \leq \nu_E(\psi) \leq \nu_E(u)$, a contradiction. □

## 7. Examples

*Example* 7.1. Let $a = 1+\sqrt{2}$, $b = 1+1/\sqrt{2}$, and $u = \log(|x|^a + |y|^b)$. Then

$$\frac{1}{a}+\frac{1}{b} = 1, \quad \frac{a}{b} = \sqrt{2}, \quad c_0(u) = 1, \quad \mathcal{I}(u)_0 = \mathcal{I}(2\log|(x,y)|)_0 = \mathfrak{m}.$$

Corollary 1.6 excludes (E). The monomial valuation $(1/a, 1/b)$ computes $c_0(u)$. For primitive convergents $s_j/r_j \to \sqrt{2}$, the monomial divisors satisfy

$$0 < A_{E_j} - \nu_{E_j}(u) = \begin{cases} s_j - \sqrt{2}r_j, & s_j > \sqrt{2}r_j, \\ (\sqrt{2}r_j - s_j)/\sqrt{2}, & s_j < \sqrt{2}r_j \end{cases} \longrightarrow 0,$$

whereas $A_E - \nu_E(u) > 0$ for every prime divisor $E$.

*Example* 7.2 (A three-dimensional certificate). Let $(s,t,w)$ be holomorphic coordinates on $(\mathbb{C}^3, 0)$, and let $E$ be the monomial divisor with weights $(1,2,3)$. If $\mathcal{I}(u)_0 = \mathfrak{m}$ and $\nu_E(u) \geq 6$, then

$$u \leq \log|(s^6, t^3, w^2)| + O(1).$$

Here $A_E = 6$ and $\mathfrak{a}_{6j}(E) = \overline{(s^6,t^3,w^2)^j}$, so (3.4) and Lemma 2.2 give the upper bound. The monomial integrability condition $(i+1)/6 + (j+1)/3 + (k+1)/2 > 1$ excludes exactly the constant monomial, so this upper bound has multiplier ideal $\mathfrak{m}$.

*Example* 7.3 (Clearing denominators). For $\mathfrak{a} = (x^5, (y-x^2)^2)$, the Rees divisor has $w_E = (2,4)$, $v_E(\mathfrak{a}) = 10$. Hence

$$Q_{\mathfrak{a}} = \{p \geq 0 : p_1 + 2p_2 \geq 5\}, \quad N = 2, \quad \mathfrak{b} = (x^{10}, y^5), \quad \chi = \frac{c}{2}\log|\mathfrak{b}|.$$

The first-power core is $(x^5, x^3y, xy^2, y^3)$. On $x = 0$, its logarithm has coefficient $3c > 5c/2$, so it cannot replace $\chi$ as an upper bound for $u = \chi$.

*Example* 7.4 (A single divisorial detector). Let $E$ be the monomial divisor of weights $(3,2)$ in coordinates $(s,t)$. Suppose $\mathcal{I}(u)_0 = \mathfrak{m}$ and $\nu_E(u) \geq 5$. The lower threshold 5 suffices to exclude constants, and the normalized kernel data are

$$b_E = 2, \qquad \alpha(b_E^{-1}v_E) = \frac{3}{2}, \qquad d = \frac{5}{2}, \qquad K = \left(\frac{3}{2}\right), \qquad c = \frac{5}{3}.$$

Here $\mathfrak{p}_E = \overline{(s^2,t^3)}$. Taking $N = 6$ gives $m = Nc/b_E = 5$, hence

$$\mathfrak{a} = \mathfrak{p}_E^5 = \overline{(s^2,t^3)^5}, \qquad \psi = \frac{5}{6}\log|(s^2,t^3)|, \qquad \mathcal{I}(\psi)_0 = (s,t). \tag{7.1}$$

The weight $N^{-1}\log|\mathfrak{a}|$ differs from $\psi$ by a bounded function. Domination and Lemma 2.2 give $u \leq \psi + O(1)$.

*Example* 7.5 (Two critical divisors). Suppose

$$\mathcal{I}(u)_0 = J = (s^4, s^2t, st^2, t^4), \qquad \nu_{(1,2)}(u) \geq 6, \quad \nu_{(2,1)}(u) \geq 6.$$

The two normalized divisorial valuations have skewness 2 and meet at $\nu_{\mathfrak{m}}$. Using the lower thresholds 6, we obtain

$$K = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad d = \begin{pmatrix} 6 \\ 6 \end{pmatrix}, \qquad c = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Their simple complete ideals are $\mathfrak{p}_1 = (s^2, t)$ and $\mathfrak{p}_2 = (s, t^2)$. Thus $N = 1$ and

$$\mathfrak{a} = \mathfrak{p}_1^2 \mathfrak{p}_2^2 = \overline{(s^6, s^2t^2, t^6)}.$$

Consequently $\psi = \log |(s^6, s^2t^2, t^6)|$ is an upper bound for $u$, and

$$s^{\alpha_1} t^{\alpha_2} \in \mathcal{I}(\psi)_0 \iff \begin{cases} \alpha_1 + 2\alpha_2 \geq 4, \\ 2\alpha_1 + \alpha_2 \geq 4 \end{cases} \iff s^{\alpha_1} t^{\alpha_2} \in J.$$

For $u = \psi$, the monomials $s^3, st, t^3$ are critical. Taking $s = x + y$, $t = y$ gives non-toric coordinates for both surface examples.

## Declaration on the use of artificial intelligence

The author proposed the research problem and guided the direction and revisions. OpenAI Codex contributed substantially to the mathematical constructions, arguments, and proof development, as well as literature comparisons and the preparation of the manuscript.

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