Failure of fiberwise log-plurisubharmonicity for classical p-Bergman kernels

Shijie Bao
Department of Mathematics, Graduate School of Science
Osaka Metropolitan University
3-3-138 Sugimoto, Sumiyoshi-ku Osaka, Japan 558-8585
shijie.bao@omu.ac.jp

Date: August 24, 2026

Abstract

We show that fiberwise log-plurisubharmonicity of the classical p-Bergman kernel fails for every $p > 4$. A finite-dimensional weighted entire-function model is lifted to a pseudoconvex Hartogs family. In the simplest case $p = 6$, the Levi form of the logarithm of the fiberwise kernel along a fixed holomorphic section is exactly $-1/8$ at the origin. The construction also yields bounded pseudoconvex counterexamples by exhaustion.

Contents

1 Introduction

For a domain $D \subset \mathbb{C}^n$ and $p > 0$, the classical p-Bergman kernel is defined by

$$K_{D,p}(z) := \sup_{f \in \mathcal{O}(D) \cap L^p(D)} \frac{|f(z)|^p}{\int_D |f|^p \, d\lambda}.$$

The variation theory of Bergman kernels has its origins in the Hilbert-space case $p = 2$. For holomorphic families of open Riemann surfaces, Maitani–Yamaguchi [8] proved the plurisubharmonic variation of the logarithm of the Bergman kernel. Berndtsson [1] established the corresponding result for general pseudoconvex families, and subsequently developed its vector-bundle interpretation through positivity of direct images [2]; see also Berndtsson–Păun [4] for relative Bergman kernels and pseudoeffectivity.

The L^2 extension theory provides a closely related route to Bergman-kernel estimates. Starting from the Ohsawa–Takegoshi extension theorem [9], sharp forms and applications to Bergman kernels were obtained by Błocki [6] and Guan–Zhou [7]. A proof of the Ohsawa–Takegoshi theorem with

Mathematics Subject Classification (2020): 32A36, 32U35.
Keywords: p-Bergman kernel, fiberwise Bergman kernel, plurisubharmonic variation, pseudoconvex Hartogs domain

sharp estimates based on positivity and variation of domains was later given by Berndtsson–Lempert [3]. These results underscore the special role of the L^2 structure in the positivity theory.

For a pseudoconvex domain $\mathcal{X} \subset \mathbb{C}^n \times \mathbb{C}^m$ with fibers $\mathcal{X}_w$, the logarithm of the fiberwise kernel $K_{\mathcal{X}_w,p}(z)$ is plurisubharmonic for $0 < p \leq 2$; see [5]. The persistence of this property for $p > 2$, both in the classical case and for kernels with respect to a functional ξ, was left open there. We give a negative answer in the classical case for every $p > 4$.

THEOREM 1.1. *Let $p > 4$, choose*

$$\frac{5}{2} < \alpha \leq \frac{p+1}{2},$$

and put

$$S(z,w) := 1 + |z|^4 + |1 + wz^2|^2.$$

The Hartogs domain

$$\mathcal{X}_\alpha := \{(z,\eta,w) \in \mathbb{C}^2 \times \mathbb{D} : |\eta|^2 S(z,w)^\alpha < 1\}$$

is pseudoconvex. If

$$X_{\alpha,w} := \{(z,\eta) \in \mathbb{C}^2 : |\eta|^2 S(z,w)^\alpha < 1\},$$

then

$$\frac{\partial^2}{\partial w\,\partial \overline{w}} \log K_{X_{\alpha,w},p}(0,0)\Big|_{w=0} = \frac{5-2\alpha}{8} < 0.$$

Consequently the logarithm of the classical fiberwise p-Bergman kernel is not plurisubharmonic on $\mathcal{X}_\alpha$. In particular, for $p = 6$ one may take $\alpha = 3$, and the displayed Levi form is $-1/8$. Moreover, bounded pseudoconvex counterexamples exist for every $p > 4$.

The full proof is given in Section 2. It reduces the extremal problem to a weighted A^p space of entire functions. The polynomial decay of the weight forces that space to consist only of affine functions; an evenness argument then shows that the constant function is extremal. The negative Levi form follows from two beta integrals. Bounded examples are obtained by truncating the z-variable and using exhaustion of the p-Bergman kernels.

2 Failure of fiberwise log-plurisubharmonicity for classical p-Bergman kernels

In this section we prove Theorem 1.1. The construction is a Hartogs lift of a finite-dimensional weighted extremal problem on the complex plane. Throughout the section, let $p > 4$ and fix

$$\frac{5}{2} < \alpha \leq \frac{p+1}{2}. \tag{2.1}$$

Set

$$S(z,w) := 1 + |z|^4 + |1 + wz^2|^2 = |1|^2 + |z^2|^2 + |1 + wz^2|^2. \tag{2.2}$$

2.1 The Hartogs lift and the extremal function

Since the three entries in the last expression in (2.2) are holomorphic in (z,w),

$$\Phi_\alpha(z,w) := \alpha \log S(z,w)$$

is plurisubharmonic on $\mathbb{C} \times \mathbb{D}$. The complete Hartogs domain

$$\mathcal{X}_\alpha = \{(z, \eta, w) \in \mathbb{C}^2 \times \mathbb{D} : |\eta|^2 < e^{-\Phi_\alpha(z,w)}\} \tag{2.3}$$

is therefore pseudoconvex. Its fiber over w is

$$X_{\alpha,w} = \{(z, \eta) \in \mathbb{C}^2 : |\eta|^2 < S(z, w)^{-\alpha}\}.$$

Define

$$I_\alpha(w) := \int_{\mathbb{C}} S(z, w)^{-\alpha}\, d\lambda(z). \tag{2.4}$$

The integral is finite since $\alpha > 1/2$.

Lemma 2.1. *For every $w \in \mathbb{D}$, if f is entire and*

$$\int_{\mathbb{C}} |f(z)|^p S(z, w)^{-\alpha}\, d\lambda(z) < +\infty,$$

then f is affine.

Proof. Uniformly for $w \in \mathbb{D}$ one has

$$1 + |z|^4 \le S(z, w) \le 3(1 + |z|^4).$$

Write $f(z) = \sum_{k\ge 0} a_k z^k$. Since $p > 1$, the coefficient estimate on the circle of radius r gives

$$|a_k|^p r^{kp} \le \frac{1}{2\pi} \int_0^{2\pi} |f(re^{\mathrm{i}\theta})|^p\, d\theta.$$

If $a_k \ne 0$, integration in r and the lower bound $S^{-\alpha} \ge 3^{-\alpha}(1 + |z|^4)^{-\alpha}$ show that necessarily

$$kp - 4\alpha < -2, \qquad \text{or equivalently} \qquad k < \frac{4\alpha - 2}{p}.$$

By (2.1), $(4\alpha - 2)/p \le 2$. Thus $a_k = 0$ for every $k \ge 2$. ■

Proposition 2.2. *For every $w \in \mathbb{D}$,*

$$K_{X_{\alpha,w},p}(0, 0) = \frac{1}{\pi I_\alpha(w)}. \tag{2.5}$$

In particular, the constant function is the unique normalized extremal function at $(0, 0)$.

Proof. Let $F \in A^p(X_{\alpha,w})$ satisfy $F(0, 0) = 1$, and put $f(z) := F(z, 0)$. Applying the submean inequality to $|F(z, \cdot)|^p$ on the η-disk of radius $S(z, w)^{-\alpha/2}$ gives

$$\int_{X_{\alpha,w}} |F|^p\, d\lambda \ge \pi \int_{\mathbb{C}} |f(z)|^p S(z, w)^{-\alpha}\, d\lambda(z).$$

The right-hand side is finite, so Lemma 2.1 gives $f(z) = 1 + bz$ for some $b \in \mathbb{C}$.

The density $S(z,w)^{-\alpha}$ is even in z. Since $p>1$, convexity gives

$$\frac{|1+bz|^p+|1-bz|^p}{2}\geq 1.$$

Pairing z and $-z$ in the weighted integral, we obtain

$$\int_{\mathbb{C}}|f(z)|^p S(z,w)^{-\alpha}\,d\lambda(z)\geq I_\alpha(w).$$

Consequently every admissible F has p-th power norm at least $\pi I_\alpha(w)$. Equality is attained by $F\equiv 1$, which proves (2.5); strict convexity also gives uniqueness. ∎

2.2 The negative Levi form

We now compute the variation of (2.5). At $w=0$,

$$S(z,0)=2+|z|^4,\qquad S_w(z,0)=z^2,\qquad S_{w\overline{w}}(z,0)=|z|^4.$$

Differentiation under the integral sign in (2.4) gives

$$(I_\alpha)_w(0)=-\alpha\int_{\mathbb{C}}z^2(2+|z|^4)^{-\alpha-1}\,d\lambda(z)=0 \tag{2.6}$$

by the angular integration, and

$$\begin{aligned}(I_\alpha)_{w\overline{w}}(0)=\alpha(\alpha+1)&\int_{\mathbb{C}}|z|^4(2+|z|^4)^{-\alpha-2}\,d\lambda(z)\\ -\alpha&\int_{\mathbb{C}}|z|^4(2+|z|^4)^{-\alpha-1}\,d\lambda(z).\end{aligned} \tag{2.7}$$

For $j\geq 0$ and $\beta>(j+1)/2$, the substitution $u=r^4/2$ yields

$$\int_{\mathbb{C}}|z|^{2j}(2+|z|^4)^{-\beta}\,d\lambda(z)=\frac{\pi}{2}\,2^{(j+1)/2-\beta}B\left(\frac{j+1}{2},\beta-\frac{j+1}{2}\right). \tag{2.8}$$

Applying (2.8) to the two terms in (2.7) gives

$$\frac{\int_{\mathbb{C}}|z|^4(2+|z|^4)^{-\alpha-2}\,d\lambda(z)}{I_\alpha(0)}=\frac{\alpha-\frac{1}{2}}{4\alpha(\alpha+1)}, \tag{2.9}$$

$$\frac{\int_{\mathbb{C}}|z|^4(2+|z|^4)^{-\alpha-1}\,d\lambda(z)}{I_\alpha(0)}=\frac{1}{2\alpha}. \tag{2.10}$$

It follows that

$$\frac{\partial^2}{\partial w\,\partial\overline{w}}\log I_\alpha(w)\bigg|_{w=0}=\frac{2\alpha-5}{8}. \tag{2.11}$$

Combining (2.5), (2.6), and (2.11), we conclude that

$$\frac{\partial^2}{\partial w\,\partial\overline{w}}\log K_{X_{\alpha,w},p}(0,0)\bigg|_{w=0}=\frac{5-2\alpha}{8}<0. \tag{2.12}$$

This proves the explicit part of Theorem 1.1. Notice that for $p=6$ and $\alpha=3$, the right-hand side of (2.12) is exactly $-1/8$.

2.3 Bounded pseudoconvex counterexamples

Although the explicit family (2.3) is unbounded in the z-direction, the failure persists on a sufficiently large bounded truncation. For $R > 0$ put

$$\mathcal{X}_\alpha^{(R)} := \mathcal{X}_\alpha \cap \{|z| < R\}, \qquad X_{\alpha,w}^{(R)} := X_{\alpha,w} \cap \{|z| < R\}.$$

Then $\mathcal{X}_\alpha^{(R)}$ is bounded and pseudoconvex.

Corollary 2.3. *For every $p > 4$, there exists $R > 0$ such that*

$$(z, \eta, w) \longmapsto \log K_{X_{\alpha,w}^{(R)},p}(z, \eta)$$

is not plurisubharmonic on $\mathcal{X}_\alpha^{(R)}$.

Proof. Write

$$u_R(w) := \log K_{X_{\alpha,w}^{(R)},p}(0,0), \qquad u_\infty(w) := \log K_{X_{\alpha,w},p}(0,0).$$

The standard exhaustion property of the p-Bergman kernel gives, for every fixed w,

$$u_R(w) \searrow u_\infty(w) \qquad (R \to +\infty).$$

Indeed, monotonicity gives the existence of the limit; normalized extremal functions on the increasing domains form a normal family, and Fatou's lemma identifies the limit with the extremal problem on $X_{\alpha,w}$.

Both u_R and u_∞ are radial functions of w. In fact, if $w' = e^{i\theta}w$, the unitary map

$$(z, \eta) \longmapsto (e^{-i\theta/2}z, \eta)$$

identifies the corresponding fibers and fixes $(0, 0)$. By (2.12), there is a sufficiently small $r > 0$ such that $u_\infty(r) < u_\infty(0)$. Pointwise exhaustion at these two parameter values then gives $u_R(r) < u_R(0)$ for all sufficiently large R.

If the fiberwise logarithmic kernel on $\mathcal{X}_\alpha^{(R)}$ were plurisubharmonic, its restriction to the section $(0, 0, w)$ would be subharmonic. Radiality would then imply the submean inequality $u_R(0) \le u_R(r)$, a contradiction. ■

Thus the classical fiberwise log-plurisubharmonicity question has a negative answer for every $p > 4$, even when the total space is required to be bounded and pseudoconvex. The range $2 < p \le 4$, as well as the corresponding sublevel-set convexity question, is not settled by this construction.

AI declaration

The construction and proof presented in this paper were obtained with GPT-5.6-sol (high). The author independently verified all statements and computations and takes full responsibility for the content.

References

[1] B. Berndtsson. Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains. Ann. Inst. Fourier (Grenoble), 56(6):1633–1662, 2006. doi:10.5802/aif.2223.

[2] B. Berndtsson. Curvature of vector bundles associated to holomorphic fibrations. Ann. of Math. (2), 169(2):531–560, 2009. doi:10.4007/annals.2009.169.531.

[3] B. Berndtsson and L. Lempert. A proof of the Ohsawa–Takegoshi theorem with sharp estimates. J. Math. Soc. Japan, 68(4):1461–1472, 2016. doi:10.2969/jmsj/06841461.

[4] B. Berndtsson and M. Păun. Bergman kernels and the pseudoeffectivity of relative canonical bundles. Duke Math. J., 145(2):341–378, 2008. doi:10.1215/00127094-2008-054.

[5] S. J. Bao, Q. A. Guan and X. Sun, On the p-Bergman kernel with respect to a functional ξ, arXiv:2510.13144, 2025.

[6] Z. Błocki. Suita conjecture and the Ohsawa–Takegoshi extension theorem. Invent. Math., 193(1):149–158, 2013. doi:10.1007/s00222-012-0423-2.

[7] Q. Guan and X. Zhou. A solution of an L^2 extension problem with an optimal estimate and applications. Ann. of Math. (2), 181(3):1139–1208, 2015. doi:10.4007/annals.2015.181.3.6.

[8] F. Maitani and H. Yamaguchi. Variation of Bergman metrics on Riemann surfaces. Math. Ann., 330(3):477–489, 2004. doi:10.1007/s00208-004-0556-8.

[9] T. Ohsawa and K. Takegoshi. On the extension of L^2 holomorphic functions. Math. Z., 195(2):197–204, 1987. doi:10.1007/BF01166457.