FLAT TAME WEIGHTS AND VALUATIVE SEPARATION

SHIJIE BAO, QI'AN GUAN, ZHITONG MI, XUN SUN, AND ZHENG YUAN

Abstract. We prove that a tame maximal weight is flat if and only if its holomorphic relative types define a centered real valuation. This yields a bijection between flat tame maximal weights modulo bounded functions and centered real valuations of finite log discrepancy. Zhou weights are flat as a consequence. Applying the criterion to the quasi-monomial weights constructed by Boucksom–Favre–Jonsson proves the remaining implication (4) ⇒ (1) in their Theorem A.

1. Introduction

1.1. **Flatness and valuations.** We work at the origin $o \in \mathbb{C}^n$, with $n \geq 2$, and put $R = \mathcal{O}_{\mathbb{C}^n,o}$ and $\mathfrak{m} = \mathfrak{m}_o$. Write $\mathrm{Val}_{R,o}$ for the space of centered real valuations $w : R \to [0, +\infty]$, with the topology of pointwise convergence on R. For a maximal plurisubharmonic weight Φ with isolated pole at o, write

$$\sigma(u, \Phi) = \sup\{c \geq 0 : u \leq c\Phi + O(1)\},$$
$$\nu_\Phi(u) = (dd^c u \wedge (dd^c \Phi)^{n-1})\{o\}, \qquad \tau_\Phi = (dd^c \Phi)^n\{o\}.$$

Demailly's comparison theorem [Dem93, Theorem 5.9] gives $\tau_\Phi \sigma(u, \Phi) \leq \nu_\Phi(u)$. Equality for every psh germ is the defining property of a *flat* weight [Ras10, Section 6]. We characterize flatness among tame maximal weights by their holomorphic relative types.

Theorem 1.1 (Flatness criterion)**.** *Let Φ be a tame maximal weight with isolated pole at o, and define*

$$w_\Phi(f) = \sigma(\log |f|, \Phi) \quad (0 \neq f \in R), \qquad w_\Phi(0) = +\infty.$$

Then Φ is flat if and only if $w_\Phi \in \mathrm{Val}_{R,o}$. In this case, its residual Monge–Ampère mass equals the volume of this valuation:

$$\tau_\Phi = \mathrm{vol}(w_\Phi) > 0. \tag{1.1}$$

For $w \in \mathrm{Val}_{R,o}$, we use the notation

$$\mathfrak{a}_t(w) = \{f \in R : w(f) \geq t\}, \qquad \mathrm{vol}(w) = \lim_{m\to\infty} \frac{n!\ell_R(R/\mathfrak{a}_m(w))}{m^n}.$$

Date: November 2, 2026.

2020 *Mathematics Subject Classification.* Primary 32U25; Secondary 13A18, 13H15, 14B05.

Key words and phrases. Flat weight, tame weight, generalized Lelong number, relative type, valuation.

Let $A(w)$ denote the log discrepancy of w.

Corollary 1.2 (Correspondence)**.** *The map $[\Phi] \mapsto w_\Phi$ is a bijection between flat tame maximal weights modulo bounded functions and $\{w \in \mathrm{Val}_{R,o} : A(w) < \infty\}$.*

The proof is given in theorem 6.2.

Zhou weights have tame maximal representatives whose holomorphic relative types are valuations [BGMY25b]. The criterion therefore gives the following consequence; their definition is recalled in section 6.3.

Corollary 1.3 (Zhou weights are flat)**.** *Every local Zhou weight Φ, represented by its tame maximal representative, is flat.*

1.2. **Valuative separation.** Boucksom–Favre–Jonsson compare the following four measurements of psh singularities.

Theorem 1.4 ([BFJ08, Theorem A])**.** *Let u_1, u_2 be psh germs at o. Consider the following conditions:*

(1) *On every proper modification $\pi : X_\pi \to \mathbb{C}^n$ above o, the ordinary Lelong numbers satisfy*
$$\nu(u_1 \circ \pi, p) = \nu(u_2 \circ \pi, p) \quad \text{for all } p \in \pi^{-1}(o).$$
(2) $\mathcal{J}_+(tu_1)_o = \mathcal{J}_+(tu_2)_o$ *for every $t > 0$.*
(3) $\sigma(u_1, \Phi) = \sigma(u_2, \Phi)$ *for every tame maximal weight Φ.*
(4) $\nu_\Phi(u_1) = \nu_\Phi(u_2)$ *for every tame weight Φ.*

Conditions (1)–(3) *are equivalent; each entails* (4).

Here
$$\mathcal{J}_+(u)_o = \bigcup_{\varepsilon > 0} \mathcal{J}((1+\varepsilon)u)_o.$$
The strong openness theorem of Guan–Zhou [GZ15, Theorem 1.1] gives $\mathcal{J}_+(u)_o = \mathcal{J}(u)_o$. Thus condition (2) can be replaced by $\mathcal{J}(tu_1)_o = \mathcal{J}(tu_2)_o$ for every $t > 0$. Guan–Zhou state this consequence in [GZ15, Corollary 3.3].

Bao–Guan–Mi–Yuan [BGMY25b, Theorem 1.1] give an alternative analytic proof of the equivalence of (2) and (3), and show that condition (3) can be restricted to local Zhou weights.

Theorem 1.5 (The BFJ converse)**.** *In theorem 1.4, condition* (4) *implies condition* (1)*. Consequently, all four conditions are equivalent.*

1.3. **Proof strategy.** The proof uses three ingredients. First, Cutkosky's limit theorem [Cut13] and the colon identity for valuation ideals yield the hypersurface multiplicity formula of proposition 3.2. Tame approximation then gives the mass identities on holomorphic logarithms. Second, the holomorphic testing lemma lemma 4.3 forces the associated BFJ probability measure to be a Dirac mass, extending the identities to all psh germs. Finally, a compactness argument and a finite product of separating functions identify the finite interpolation obstruction with log discrepancy (theorem 5.1). Together with valuative interpolation, this gives the correspondence in theorem 6.2.

1.4. **Organization of the paper.** Section 2 recalls weights and valuative transforms. Section 3 proves the hypersurface multiplicity formula used in the proof of theorem 1.1 in Section 4. Section 5 proves the discrepancy identity from jumping numbers. Section 6 establishes the correspondence with finite-discrepancy valuations and deduces flatness of Zhou weights. Section 7 proves the converse implication in Boucksom–Favre–Jonsson's Theorem A. Appendix A gives a continuity estimate for the interpolation envelope.

1.5. **Notations and conventions.**

- We write psh for plurisubharmonic and denote by PSH_o the set of psh germs at o that are not identically $-\infty$.
- The notation $O(1)$ denotes a function bounded near o; thus $u = v + O(1)$ means that $u - v$ is bounded near o.
- We use $d^c = (\partial - \bar\partial)/(2\pi i)$, so that $dd^c \log|f| = [\mathrm{div}(f)]$ for $0 \neq f \in R$. The masses are the classical generalized Lelong products, not non-pluripolar products.
- For a nonzero ideal $I = (f_1, \dots, f_r) \subset R$, set
$$\log|I| = \frac{1}{2}\log\big(|f_1|^2 + \cdots + |f_r|^2\big).$$
This germ is well defined modulo bounded functions.
- The multiplier ideal sheaf $\mathcal{J}(u)$ has stalk
$$\mathcal{J}(u)_o = \{f \in R : |f|^2 e^{-2u} \text{ is locally integrable at } o\}.$$
- For a Noetherian local ring $(S, \mathfrak{n})$ of dimension d, module length and Hilbert–Samuel multiplicity are given by
$$\ell_S(M) = \sup\{r \in \mathbb{Z}_{\geq 0} : 0 = M_0 \subsetneq \cdots \subsetneq M_r = M\},$$
$$e(I; S) = \lim_{m\to\infty} \frac{d!\,\ell_S(S/I^m)}{m^d},$$
where the M_i are S-submodules of M and I is $\mathfrak{n}$-primary.

2. Weights and valuative transforms

2.1. **Weights and logarithmic bounds.** A plurisubharmonic function Φ on a neighbourhood U of o has an *isolated singularity at* o if $\Phi(o) = -\infty$ and Φ is locally bounded on $U \setminus \{o\}$. Following [BFJ08, Section 5.3], such a function is called a *weight* if e^Φ is continuous on U. A weight Φ is called *maximal* if
$$(dd^c\Phi)^n = 0 \quad \text{on } U \setminus \{o\};$$
see [BFJ08, Section 5.4].

A psh germ Φ is said to be *tame* [BFJ08, Definition 5.8] if there exists a constant $C > 0$ such that, for every $t > 0$ and every germ $f \in \mathcal{J}(t\Phi)_o$,
$$\log|f| \leq (t - C)\Phi + O(1) \quad \text{near } o. \tag{2.1}$$

Lemma 2.1. *A tame weight with isolated pole has constants $0 < c < L$ such that*

$$L \log|z| + O(1) \le \Phi \le c \log|z| + O(1). \tag{2.2}$$

In particular, $0 < c^n \le \tau_\Phi \le L^n < \infty$.

Proof. Fix $t > C$. By (2.1),

$$1 \in \mathcal{J}(t\Phi)_o \quad \Longrightarrow \quad 0 \le (t-C)\Phi + O(1),$$

which contradicts $\Phi(o) = -\infty$. Thus $\mathcal{J}(t\Phi)_o \neq R$. Demailly approximation gives $\mathcal{J}(t\Phi)_o \neq 0$, and the sheaf $\mathcal{J}(t\Phi)$ is the unit ideal away from o. By coherence,

$$\mathfrak{m}^N \subset \mathcal{J}(t\Phi)_o \quad \Longrightarrow \quad N \log|z_j| \le (t-C)\Phi + O(1) \quad (1 \le j \le n).$$

Taking the maximum over j gives

$$\Phi \ge \frac{N}{t-C} \log|z| + O(1).$$

Writing $\nu(\Phi, o)$ for the ordinary Lelong number, Skoda's integrability theorem gives

$$\nu(\Phi, o) = 0 \quad \Longrightarrow \quad \mathcal{J}(s\Phi)_o = R \quad (s > 0),$$

contrary to the properness of $\mathcal{J}(t\Phi)_o$. Hence $c = \nu(\Phi, o) > 0$, and the standard logarithmic estimate gives $\Phi \le c \log|z| + O(1)$; see [Dem93]. Mass comparison now yields $c^n \le \tau_\Phi \le L^n$. □

2.2. **Comparison, approximation, and multiplicity.** We use Demailly's comparison and approximation theorems [Dem87, Dem93]; the relative-type attainment statement is discussed in [Ras06] and [BFJ08, Section 5.4].

Proposition 2.2. *The following statements hold.*

(i) *For maximal Φ, one has $u \le \sigma(u, \Phi)\Phi + O(1)$. The relative type is finite for $u \in \mathrm{PSH}_o$ when (2.2) holds.*
(ii) *For $t > 0$, Demailly approximation gives $u \le t^{-1} \log|\mathcal{J}(tu)_o| + O(1)$.*
(iii) *If weights $\alpha \le \beta + O(1)$ have isolated poles, then $\nu_\alpha(u) \ge \nu_\beta(u)$. Residual Monge–Ampère masses obey the same monotonicity. Scaling a weight by $b > 0$ multiplies these two quantities by b^{n-1} and b^n, respectively.*
(iv) *If I is $\mathfrak{m}$-primary, then*

$$(dd^c \log|I|)^n\{o\} = e(I; R), \qquad \nu_{\log|I|}(\log|f|) = e(I; R/(f)) \quad (0 \neq f \in \mathfrak{m}).$$

For irreducible f, Poincaré–Lelong identifies the second mass in (iv) with $(dd^c \log|I|)^{n-1} \wedge [f = 0]\{o\}$. The analytic multiplicity formula on the possibly singular reduced hypersurface is also a case of [DLLZ24, Theorem 1.1], with the same ideal repeated $n-1$ times. For general f, both sides are additive over its irreducible factors, with their multiplicities.

2.3. **The BFJ valuative formulas.** Let $\mathcal{V}$ be the compact space of semivaluations on the completed local ring, centered at o and normalized by $v(\mathfrak{m}) = 1$. Evaluations of nonzero elements may equal $+\infty$ at a semivaluation. For $w \in \mathrm{Val}_{R,o}$, the normalized valuation $w/w(\mathfrak{m})$ extends uniquely to a point of $\mathcal{V}$ by $\mathfrak{m}$-adic truncation. Write $\mathcal{V}_{\mathrm{qm}}$ for the quasi-monomial points. For an ideal $I = (f_i)$, $v(I) = \min_i v(f_i)$; in particular $v(IJ) = v(I) + v(J)$.

The valuative transform $\widehat{u}$ of a psh germ u is a nonpositive formal psh function on $\mathcal{V}$. It satisfies

$$\widehat{\log|I|}(v) = -v(I), \qquad \widehat{bu} = b\widehat{u} \quad (b > 0), \qquad \widehat{\max(u_1, u_2)} = \max(\widehat{u}_1, \widehat{u}_2).$$

At a divisorial valuation ord_E, $-\widehat{u}(\mathrm{ord}_E)$, interpreted homogeneously, is the generic Lelong number of the pullback along E. For $w \in \mathrm{Val}_{R,o}$, put

$$w[u] = w(\mathfrak{m})\big(-\widehat{u}(w/w(\mathfrak{m}))\big). \tag{2.3}$$

In particular, $w[\log|I|] = w(I)$.

Theorem 2.3 ([BFJ08])**.** *Let Φ be a tame maximal weight and $g = \widehat{\Phi}$. Then g is bounded, continuous, and strictly negative on $\mathcal{V}$, and*

$$\nu_\Phi(u) = \int_{\mathcal{V}} -\widehat{u}\, d\,\mathrm{MA}(g), \tag{2.4}$$

$$\sigma(u, \Phi) = \inf_{v \in \mathcal{V}} \frac{-\widehat{u}(v)}{-g(v)}. \tag{2.5}$$

Every formal psh function h admits a sequence $h_j = c_j \log|I_j|$ decreasing to h at every point of $\mathcal{V}$, where $c_j > 0$ and I_j are primary ideals. Finally, its values on all of $\mathcal{V}$ are determined by its values on $\mathcal{V}_{\mathrm{qm}}$:

$$h(v) = \inf_\pi h(r_\pi(v)). \tag{2.6}$$

Here $r_\pi(v)$ is the quasi-monomial retraction associated to a sufficiently high simple modification π.

Formula (2.4) is [BFJ08, Theorem B], and (2.5) follows from [BFJ08, Theorem 5.12]. The decreasing approximation is [BFJ08, Proposition 3.13]; (2.6) is [BFJ08, Definition 2.8]. Boundedness and strict negativity follow from (2.2). Continuity follows from the tame multiplier-ideal approximation [BFJ08, Lemma 5.9]: the continuous transforms of primary ideal weights converge uniformly, since their errors are bounded by a constant times $(-g)/m$. Primary ideals in the formal approximation may be regarded as analytic ideals: if $\mathfrak{m}^N \subset I$, truncating generators to polynomials modulo $\mathfrak{m}^N$ does not change the completed ideal or its valuation function.

3. Valuation ideals on hypersurfaces

Let $w \in \mathrm{Val}_{R,o}$. Set $\lambda = w(\mathfrak{m}) > 0$ and

$$\mathfrak{a}_t = \{f : w(f) \geq t\} \quad (t > 0), \qquad \mathfrak{a}_t = R \quad (t \leq 0), \qquad H(t) = \ell_R(R/\mathfrak{a}_t).$$

The family is decreasing and graded: $\mathfrak{a}_s\mathfrak{a}_t \subset \mathfrak{a}_{s+t}$. Moreover $\mathfrak{m}^{\lceil t/\lambda \rceil} \subset \mathfrak{a}_t \subset \mathfrak{m}$ for $t > 0$, so its positive-index ideals are primary.

Theorem 3.1 (Cutkosky)**.** *Let $(S, \mathfrak{n})$ be an analytically unramified equicharacteristic local ring of dimension $d > 0$, with perfect residue field. For a graded family $I_\bullet$ of $\mathfrak{n}$-primary ideals, both limits below exist and*

$$\lim_{m\to\infty} \frac{d!\ell_S(S/I_m)}{m^d} = \lim_{m\to\infty} \frac{e(I_m; S)}{m^d}.$$

This is [Cut13, Theorems 5.8 and 6.5]. Reduced complex analytic local rings are excellent with reduced completion, so the theorem applies to R and to $R/(f)$ for irreducible f. In particular,

$$\mathrm{vol}(w) = \lim_{m\to\infty} \frac{n!H(m)}{m^n} = \lim_{m\to\infty} \frac{e(\mathfrak{a}_m; R)}{m^n}. \tag{3.1}$$

The limit is finite. Monotonicity of H extends its asymptotic formula from integer to real indices by comparison with the floor and ceiling.

Proposition 3.2 (Hypersurface-volume formula)**.** *For $0 \neq f \in \mathfrak{m}$,*

$$\lim_{m\to\infty} \frac{e(\mathfrak{a}_m(w); R/(f))}{m^{n-1}} = w(f)\,\mathrm{vol}(w).$$

Proof. First suppose f is irreducible. Put $s = w(f) > 0$ and $S = R/(f)$. For every real t and $h \in R$,

$$h \in (\mathfrak{a}_t : f) \iff w(fh) = s + w(h) \geq t \iff h \in \mathfrak{a}_{t-s}.$$

Thus multiplication by f gives the exact sequence

$$0 \longrightarrow R/\mathfrak{a}_{t-s} \xrightarrow{\cdot f} R/\mathfrak{a}_t \longrightarrow R/(\mathfrak{a}_t + (f)) \longrightarrow 0. \tag{3.2}$$

Consequently

$$L_f(t) := \ell_S(S/\mathfrak{a}_t S) = H(t) - H(t-s). \tag{3.3}$$

The ideals $\mathfrak{a}_m S$ are a graded primary family in the reduced analytic local ring S of dimension $n-1$. By theorem 3.1, there is a finite number B_f such that

$$\lim_{m\to\infty} \frac{(n-1)!L_f(m)}{m^{n-1}} = B_f = \lim_{m\to\infty} \frac{e(\mathfrak{a}_m; S)}{m^{n-1}}. \tag{3.4}$$

Since L_f is increasing, its asymptotic formula again holds for real indices. Evaluating (3.3) at $t = s, 2s, \ldots, qs$ and using $H(0) = 0$, we obtain

$$\begin{aligned} H(qs) &= \sum_{j=1}^{q} L_f(js) \\ &= \frac{B_f s^{n-1}}{(n-1)!} \sum_{j=1}^{q} j^{n-1} + o(q^n) \\ &= \frac{B_f s^{n-1}}{n!} q^n + o(q^n). \end{aligned}$$

Here $\sum_{j\le q} o(j^{n-1}) = o(q^n)$. On the other hand, (3.1) yields

$$H(qs) = \frac{\mathrm{vol}(w)s^n}{n!}q^n + o(q^n).$$

Comparison gives $B_f = s\,\mathrm{vol}(w)$, proving the irreducible case.

For general f, unique factorization in R gives $f = a\prod_{i=1}^r f_i^{b_i}$, with a a unit and f_i pairwise nonassociate irreducibles. The associativity formula for multiplicity [HS06, Chapter 11] gives

$$e(\mathfrak{a}_m; R/(f)) = \sum_{i=1}^r b_i e(\mathfrak{a}_m; R/(f_i)).$$

Indeed, $R_{(f_i)}$ is a discrete valuation ring and $\ell_{R_{(f_i)}}((R/(f))_{(f_i)}) = b_i$. The irreducible case therefore gives

$$\lim_{m\to\infty}\frac{e(\mathfrak{a}_m; R/(f))}{m^{n-1}} = \sum_{i=1}^r b_i w(f_i)\,\mathrm{vol}(w) = w(f)\,\mathrm{vol}(w). \qquad \square$$

4. The flatness criterion

Throughout the next two subsections, Φ is a tame maximal weight whose holomorphic relative type satisfies $w = w_\Phi \in \mathrm{Val}_{R,o}$.

4.1. **The valuation-ideal approximation.**

Lemma 4.1. *There is an integer $C > 0$ independent of m such that, with $\Psi_m = m^{-1}\log|\mathfrak{a}_m(w)|$,*

$$(1 + C/m)\Phi \le \Psi_m \le \Phi \quad \textit{modulo bounded functions.} \tag{4.1}$$

Proof. Choose an integer C at least the tameness constant. By (2.1) and Demailly approximation,

$$\mathcal{J}((m+C)\Phi)_o \subset \mathfrak{a}_m(w),$$
$$(m+C)\Phi \le \log|\mathcal{J}((m+C)\Phi)_o| + O(1) \le \log|\mathfrak{a}_m(w)| + O(1).$$

Write $\mathfrak{a}_m(w) = (f_{m,1}, \dots, f_{m,r_m})$. Attainment of relative type gives

$$w(f_{m,j}) = \sigma(\log|f_{m,j}|, \Phi) \ge m \quad \Longrightarrow \quad \log|f_{m,j}| \le m\Phi + O(1).$$

Consequently,

$$\log|\mathfrak{a}_m(w)| = \max_j \log|f_{m,j}| + O(1) \le m\Phi + O(1).$$

Dividing by m proves (4.1). $\square$

Proposition 4.2. *One has $\tau_\Phi = \mathrm{vol}(w) > 0$ and*

$$\nu_\Phi(\log|f|) = \mathrm{vol}(w)w(f) \quad \textit{for every } f \in R\setminus\{0\}. \tag{4.2}$$

Proof. By proposition 2.2 and lemma 4.1,

$$\tau_\Phi \le \frac{e(\mathfrak{a}_m(w);R)}{m^n} \le (1+C/m)^n \tau_\Phi. \tag{4.3}$$

Here $e(\mathfrak{a}_m(w);R)/m^n = (dd^c\Psi_m)^n\{o\}$. Equations (3.1) and (2.2) give

$$0 < c^n \le \tau_\Phi = \lim_{m\to\infty} \frac{e(\mathfrak{a}_m(w);R)}{m^n} = \mathrm{vol}(w).$$

For irreducible f, the hypersurface multiplicity formula gives

$$\nu_{\Psi_m}(\log|f|) = \frac{e(\mathfrak{a}_m(w);R/(f))}{m^{n-1}}.$$

Comparison in the weight variable yields

$$\nu_\Phi(\log|f|) \le \frac{e(\mathfrak{a}_m(w);R/(f))}{m^{n-1}} \le (1+C/m)^{n-1}\nu_\Phi(\log|f|). \tag{4.4}$$

Hence proposition 3.2 gives

$$\nu_\Phi(\log|f|) = \lim_{m\to\infty} \frac{e(\mathfrak{a}_m(w);R/(f))}{m^{n-1}} = w(f)\,\mathrm{vol}(w).$$

For $f = a\prod_{i=1}^r f_i^{b_i}$ with $a \in R^\times$, additivity gives

$$\nu_\Phi(\log|f|) = \sum_{i=1}^r b_i\nu_\Phi(\log|f_i|) = \mathrm{vol}(w)\sum_{i=1}^r b_i w(f_i) = \mathrm{vol}(w)w(f).$$

The proof is complete. □

4.2. **From holomorphic tests to all psh germs.**

Lemma 4.3 (Holomorphic testing)**.** *Let Φ be tame and maximal, and suppose*

$$\nu_\Phi(\log|f|) = \tau_\Phi\sigma(\log|f|,\Phi) \quad \textit{for every nonzero } f \in R.$$

Then there is a unique $v_0 \in \mathcal{V}$ such that

$$\mathrm{MA}(g) = \frac{\tau_\Phi}{-g(v_0)}\delta_{v_0}, \qquad \frac{\nu_\Phi(u)}{\tau_\Phi} = \sigma(u,\Phi) = \frac{-\widehat{u}(v_0)}{-g(v_0)}, \qquad g = \widehat{\Phi}, \tag{4.5}$$

for every psh germ u.

Proof. By (2.4),

$$dP = \frac{-g}{\tau_\Phi}\,d\,\mathrm{MA}(g), \qquad P(\mathcal{V}) = \frac{1}{\tau_\Phi}\int_{\mathcal{V}}(-g)\,d\,\mathrm{MA}(g) = 1.$$

Thus P is a Radon probability measure. For $0 \ne f \in R$, set $R_f(v) = v(f)/(-g(v))$ and $a_f = \sigma(\log|f|,\Phi)$. The hypotheses and theorem 2.3 give

$$R_f - a_f \ge 0, \qquad \int_{\mathcal{V}}(R_f - a_f)\,dP = 0.$$

The function R_f is continuous with values in $[0,+\infty]$: evaluation of f is continuous, and $-g$ is continuous and bounded away from zero. For $\varepsilon>0$, the open set $U_{f,\varepsilon}=\{v : R_f(v) > a_f+\varepsilon\}$ satisfies

$$0=\int_{\mathcal{V}}(R_f-a_f)\,dP \geq \varepsilon P(U_{f,\varepsilon})\geq 0.$$

Hence $U_{f,\varepsilon}\cap \operatorname{supp} P=\varnothing$, and

$$\frac{v(f)}{-g(v)}=a_f \quad (v\in\operatorname{supp} P,\ 0\neq f\in R). \tag{4.6}$$

For $v_1,v_2\in\operatorname{supp} P$, taking minima in (4.6) over generators of $\mathfrak{m}$ gives

$$\frac{1}{-g(v_1)}=\frac{1}{-g(v_2)} \quad\Longrightarrow\quad v_1|_R=v_2|_R.$$

For a formal series F and its degree-N truncation F_N,

$$v_i(F-F_N)\geq N+1, \qquad v_i(F)=\lim_{N\to\infty}\min\{v_i(F_N),N+1\} \quad (i=1,2).$$

Thus $v_1=v_2$ on the completion. Since P is a Radon probability measure, $\operatorname{supp} P=\{v_0\}$ and $P=\delta_{v_0}$.

For a primary ideal $I=(f_1,\dots,f_r)$, (4.6) gives

$$\frac{v_0(I)}{-g(v_0)}=\min_i\frac{v_0(f_i)}{-g(v_0)}\leq\min_i\frac{v(f_i)}{-g(v)}=\frac{v(I)}{-g(v)} \quad (v\in\mathcal{V}).$$

Choose $h_j=c_j\log|I_j|\downarrow\widehat{u}$ as in theorem 2.3. Positive scaling and passage to the limit give

$$\frac{-\widehat{u}(v_0)}{-g(v_0)}=\lim_j\frac{-h_j(v_0)}{-g(v_0)}\leq\lim_j\frac{-h_j(v)}{-g(v)}=\frac{-\widehat{u}(v)}{-g(v)}.$$

Therefore (2.5) and $P=\delta_{v_0}$ yield

$$\sigma(u,\Phi)=\inf_{v\in\mathcal{V}}\frac{-\widehat{u}(v)}{-g(v)}=\frac{-\widehat{u}(v_0)}{-g(v_0)},$$
$$\frac{\nu_\Phi(u)}{\tau_\Phi}=\int_{\mathcal{V}}\frac{-\widehat{u}}{-g}\,dP=\frac{-\widehat{u}(v_0)}{-g(v_0)}.$$

The definition of P gives the measure identity in (4.5). □

Proof of theorem 1.1. First suppose Φ is flat. Linearity in the first argument gives

$$w_\Phi(fg)=\frac{\nu_\Phi(\log|fg|)}{\tau_\Phi}=\frac{\nu_\Phi(\log|f|)+\nu_\Phi(\log|g|)}{\tau_\Phi}=w_\Phi(f)+w_\Phi(g).$$

Since relative type satisfies $\sigma(\max\{u_1,u_2\},\Phi)=\min_i\sigma(u_i,\Phi)$,

$$\log|f+g|\leq\max\{\log|f|,\log|g|\}+\log 2,$$
$$w_\Phi(f+g)\geq\sigma(\max\{\log|f|,\log|g|\},\Phi)=\min\{w_\Phi(f),w_\Phi(g)\}.$$

For $0 \neq f \in R$, put $k = \operatorname{ord}_o(f)$. The bounds (2.2) give

$$\log|f| \le k \log|z| + O(1) \le \frac{k}{L}\Phi + O(1),$$

whereas

$$\log|f| \le a\Phi + O(1) \quad \Longrightarrow \quad \log|f| \le ac\log|z| + O(1) \quad \Longrightarrow \quad ac \le k.$$

Thus

$$\frac{k}{L} \le w_\Phi(f) \le \frac{k}{c}, \qquad w_\Phi|_{R^\times} = 0, \qquad w_\Phi(f) > 0 \quad (0 \neq f \in \mathfrak{m}).$$

These identities show that $w_\Phi \in \mathrm{Val}_{R,o}$.

Conversely, suppose $w_\Phi \in \mathrm{Val}_{R,o}$. By proposition 4.2 and lemma 4.3,

$$\nu_\Phi(u) = \tau_\Phi \sigma(u, \Phi) \quad (u \in \mathrm{PSH}_o), \qquad \tau_\Phi = \mathrm{vol}(w_\Phi) > 0.$$

This proves flatness and (1.1). □

5. Log discrepancy from jumping numbers

For $u \in \mathrm{PSH}_o$ and $0 \neq f \in R$, the *jumping number of u with respect to (f)* is

$$c_o^f(u) = \sup\{t \ge 0 : |f|^2 e^{-2tu} \text{ is locally integrable at } o\};$$

see [JM14, Section 3.1]. The case $f = 1$ is the *complex singularity exponent* [Tia87, DK01]. For a nonzero ideal $I \subset R$, its *jumping number with respect to (f)* and its *log canonical threshold* are denoted by

$$\mathrm{lct}^{(f)}(I) = c_o^f(\log|I|), \qquad \mathrm{lct}(I) = c_o^1(\log|I|);$$

see [ELSV04] and [JM12, Section 1.4]. These analytic definitions agree with the algebraic ones by log resolution; see [JM14, Section 3.1].

Let $I_\bullet = (I_m)_{m\ge1}$ be a graded sequence of nonzero ideals, so $I_pI_q \subset I_{p+q}$ for $p, q \ge 1$. Its asymptotic vanishing order along $v \in \mathrm{Val}_{R,o}$ and its *asymptotic jumping number with respect to (f)* are

$$v(I_\bullet) = \inf_{m\ge1} \frac{v(I_m)}{m}, \qquad \mathrm{lct}^{(f)}(I_\bullet) = \sup_{m\ge1} m\,\mathrm{lct}^{(f)}(I_m);$$

see [JM12, Section 2.1].

Let $A(v)$ denote log discrepancy [JM12, Section 5]. For a prime divisor E on a smooth modification $Y \to X = (\mathbb{C}^n, o)$, $A(\mathrm{ord}_E) = 1 + \mathrm{ord}_E(K_{Y/X})$. The function A is homogeneous and linear on quasi-monomial cones, and is defined on general valuations by the supremum over quasi-monomial retractions.

For $v \in \mathrm{Val}_{R,o}$, consider the interpolation obstruction

$$\mathscr{A}(v) = \sup_{0\neq f\in R} \big(\mathrm{lct}^{(f)}(\mathfrak{a}_\bullet(v)) - v(f)\big). \tag{5.1}$$

This is the finiteness condition used in [BGMY25a, Theorem 1.11], where it is identified with the jumping-number invariant introduced by Bao–Guan–Yuan [BGY23]. Bao–Guan–Zhou [BGZ25] obtain related algebraic

jumping-number formulas using log discrepancy. Since $v(\mathfrak{a}_\bullet(v)) = 1$ [JM12, Lemma 2.4], the valuative threshold inequality [JM12, Corollary 6.9] gives

$$\mathscr{A}(v) \le A(v). \tag{5.2}$$

We prove equality under the finiteness hypothesis.

Theorem 5.1 (Discrepancy identity)**.** *Let $v \in \mathrm{Val}_{R,o}$ with $\mathscr{A}(v) < \infty$. Then*

$$A(v) = \mathscr{A}(v) < \infty. \tag{5.3}$$

For $v, w \in \mathrm{Val}_{R,o}$, set

$$w \ge v \quad \Longleftrightarrow \quad w(f) \ge v(f) \quad \forall f \in R \setminus \{0\},$$

and

$$D_v = \{w \in \mathrm{Val}_{R,o} : w \ge v\}.$$

Lemma 5.2. *For every $L > 0$, the set $K_L = \{w \in D_v : A(w) \le L\}$ is compact.*

Proof. Set

$$M_f = \begin{cases} 0, & f \in R^\times, \\ L/\operatorname{lct}((f)), & 0 \ne f \in \mathfrak{m}. \end{cases}$$

The single-ideal valuative inequality gives, for $w \in K_L$,

$$0 \le w(f) \le M_f < \infty, \qquad K_L \subset \mathcal{P}_L := \prod_{0 \ne f \in R} [0, M_f].$$

Let w be a pointwise limit of a net (w_α) in K_L, and put $w(0) = +\infty$. Passing to the limit gives

$$w(fg) = w(f) + w(g), \qquad w(f+g) \ge \min\{w(f), w(g)\},$$
$$w(f) \ge v(f) \quad (0 \ne f \in R).$$

Thus $w \in D_v$, with $w(f/g) = w(f) - w(g)$ on $\mathrm{Frac}(R)$. Lower semicontinuity [JM12, Lemma 5.7] gives

$$A(w) \le \liminf_\alpha A(w_\alpha) \le L.$$

Hence K_L is closed in the compact product $\mathcal{P}_L$. This is the compactness argument of [JM12, Proposition 5.9]. □

Proof of theorem 5.1. We first rewrite the valuative formula as

$$\operatorname{lct}^{(f)}(\mathfrak{a}_\bullet(v)) = \inf_{w \in D_v} \big(A(w) + w(f)\big), \qquad 0 \ne f \in R. \tag{5.4}$$

For $u \in \mathrm{Val}_{R,o}$, [JM12, Lemma 2.4] gives

$$u(\mathfrak{a}_\bullet(v)) = \inf_{\substack{0 \ne h \in R \\ v(h) > 0}} \frac{u(h)}{v(h)}, \qquad u(\mathfrak{a}_\bullet(v)) \ge 1 \Longleftrightarrow u \ge v.$$

If $b = u(\mathfrak{a}_\bullet(v)) > 0$, put $w = u/b$. Then

$$w \in D_v, \qquad A(w) + w(f) = \frac{A(u) + u(f)}{u(\mathfrak{a}_\bullet(v))}.$$

Conversely,

$$\frac{A(w)+w(f)}{w(\mathfrak{a}_\bullet(v))} \le A(w)+w(f) \quad (w \in D_v).$$

Taking infima proves (5.4).

Suppose $\mathscr{A}(v) < L < A(v)$. If $K_L = \varnothing$, (5.4) with $f = 1$ gives

$$\mathscr{A}(v) \ge \inf_{w \in D_v} A(w) \ge L > \mathscr{A}(v).$$

Thus $K_L \neq \varnothing$. Since $v \notin K_L$, the open sets

$$U_f = \{w \in K_L : w(f) > v(f)\} \quad (0 \neq f \in R)$$

cover K_L. By lemma 5.2, there are $f_1, \dots, f_r$ such that

$$K_L = \bigcup_{i=1}^{r} U_{f_i}.$$

For $g = \prod_{i=1}^r f_i \neq 0$,

$$w(g) - v(g) = \sum_{i=1}^{r} \big(w(f_i) - v(f_i)\big) > 0 \quad (w \in K_L).$$

Compactness and continuity give

$$\delta = \min_{w \in K_L} \big(w(g) - v(g)\big) > 0.$$

Choose $k \in \mathbb{N}$ with $k\delta > L$. Then

$$A(w) + k\big(w(g) - v(g)\big) \ge \begin{cases} k\delta > L, & w \in K_L, \\ A(w) > L, & w \in D_v \setminus K_L. \end{cases}$$

Equations (5.1) and (5.4) now give

$$\begin{aligned} \mathscr{A}(v) &\ge \mathrm{lct}^{(g^k)}(\mathfrak{a}_\bullet(v)) - kv(g) \\ &= \inf_{w \in D_v} \big(A(w) + k(w(g) - v(g))\big) \ge L > \mathscr{A}(v). \end{aligned}$$

Hence $A(v) \le \mathscr{A}(v)$, and (5.2) proves the theorem. □

6. Finite-discrepancy valuations and flat tame weights

6.1. **Valuative interpolation.** With the notation of section 5, we use the following representation theorem.

Theorem 6.1 (Valuative interpolation)**.** *Let $w \in \mathrm{Val}_{R,o}$. The condition $\mathscr{A}(w) < \infty$ is equivalent to the existence of a tame maximal weight Φ_w satisfying*

$$w(f) = \sigma(\log|f|, \Phi_w) \quad (f \neq 0).$$

When this holds, one also has

$$c_o^f(\Phi_w) = \mathrm{lct}^{(f)}(\mathfrak{a}_\bullet(w)) \quad (f \neq 0).$$

This is [BGMY25a, Theorem 1.11]. By theorem 5.1 and (5.2), its finiteness condition is equivalent to $A(w) < \infty$. An estimate for the continuity of the representing envelope is proved in appendix A.

6.2. **The correspondence.**

Theorem 6.2 (Correspondence)**.** *The map $[\Phi] \mapsto w_\Phi$ is a bijection between flat tame maximal weights modulo bounded functions and $\{w \in \mathrm{Val}_{R,o} : A(w) < \infty\}$. The corresponding weight Φ_w satisfies, for $u \in \mathrm{PSH}_o$,*

$$\tau_{\Phi_w} = \mathrm{vol}(w) > 0, \qquad \sigma(u, \Phi_w) = w[u], \qquad \nu_{\Phi_w}(u) = \mathrm{vol}(w) w[u].$$

Proof. If Φ is tame, flat, and maximal, theorem 1.1 gives $w_\Phi \in \mathrm{Val}_{R,o}$. By theorems 5.1 and 6.1,

$$\mathscr{A}(w_\Phi) < \infty, \qquad A(w_\Phi) = \mathscr{A}(w_\Phi) < \infty.$$

Conversely, $A(w) < \infty \Longrightarrow \mathscr{A}(w) \le A(w) < \infty$. Theorem 6.1 provides a tame maximal Φ_w with holomorphic relative type w. By theorem 1.1, Φ_w is flat and $\tau_{\Phi_w} = \mathrm{vol}(w) > 0$. Put $g = \widehat{\Phi_w}$. The point v_0 in lemma 4.3 satisfies

$$w(f) = \frac{v_0(f)}{-g(v_0)} \quad (0 \neq f \in R),$$
$$w(\mathfrak{m}) = \frac{1}{-g(v_0)}, \qquad v_0 = \frac{w}{w(\mathfrak{m})}.$$

Here the second line follows by taking minima over generators of $\mathfrak{m}$. Equations (4.5) and (2.3) therefore give

$$\sigma(u, \Phi_w) = w[u], \qquad \nu_{\Phi_w}(u) = \mathrm{vol}(w) w[u] \quad (u \in \mathrm{PSH}_o).$$

Suppose Φ_1, Φ_2 are tame maximal weights defining the same w. Their valuation-ideal approximants Ψ_m are identical, and lemma 4.1 gives

$$(1 + C_i/m)\Phi_i \le \Psi_m \le \Phi_i \quad (i = 1, 2)$$

modulo bounded functions. Thus

$$\Phi_1 \le \frac{1}{1 + C_1/m}\Phi_2 + O(1), \qquad \sigma(\Phi_1, \Phi_2) \ge \sup_{m \ge 1} (1 + C_1/m)^{-1} = 1.$$

Attainment for Φ_2 gives $\Phi_1 \le \Phi_2 + O(1)$. Interchanging the indices yields $\Phi_1 = \Phi_2 + O(1)$. □

6.3. **Zhou weights.** Let F_0 be a nonzero holomorphic tuple and ψ_0 a psh germ such that $|F_0|^2 e^{-2\psi_0}$ is locally integrable. A local Zhou weight Φ for this datum satisfies:

(i) $|F_0|^2 e^{-2\psi_0} |z|^{2N} e^{-2\Phi}$ is integrable for some N;
(ii) $|F_0|^2 e^{-2\psi_0} e^{-2\Phi}$ is not integrable;
(iii) if $\Psi \ge \Phi + O(1)$ is psh and the expression in (ii) with Ψ in place of Φ is not integrable, then $\Psi = \Phi + O(1)$.

Bao–Guan–Mi–Yuan [BGMY25b, Section 1.1, Proposition 1.7, and Section 1.2] prove that these weights admit tame maximal representatives with $w_\Phi \in \mathrm{Val}_{R,o}$.

Proof of corollary 1.3. Apply theorem 1.1 to the two properties from [BGMY25b] just stated. □

7. Quasi-monomial tests and the BFJ converse

The construction in [BFJ08, Theorem 5.13] gives, for each normalized quasi-monomial v, a tame maximal weight Φ_v satisfying

$$\sigma(u, \Phi_v) = -\widehat{u}(v) \quad (u \in \mathrm{PSH}_o). \tag{7.1}$$

Taking $u = \log|f|$ gives $\sigma(\log|f|, \Phi_v) = v(f)$. Consequently theorem 1.1 gives

$$\tau_{\Phi_v} = \mathrm{vol}(v) > 0, \qquad \nu_{\Phi_v}(u) = \mathrm{vol}(v)(-\widehat{u}(v)). \tag{7.2}$$

Thus these generalized Lelong tests are positive multiples of point evaluations.

Proof of theorem 1.5. Condition (4) of theorem 1.4, (7.2), and $\mathrm{vol}(v) > 0$ give

$$\widehat{u}_1(v) = -\frac{\nu_{\Phi_v}(u_1)}{\mathrm{vol}(v)} = -\frac{\nu_{\Phi_v}(u_2)}{\mathrm{vol}(v)} = \widehat{u}_2(v) \quad (v \in \mathcal{V}_{\mathrm{qm}}).$$

For $\mu \in \mathcal{V}$, (2.6) gives

$$\widehat{u}_1(\mu) = \inf_\pi \widehat{u}_1(r_\pi(\mu)) = \inf_\pi \widehat{u}_2(r_\pi(\mu)) = \widehat{u}_2(\mu).$$

By [BFJ08, Sections 5.1 and 5.6], this is equivalent to condition (1). The other implications follow from theorem 1.4. □

The divisorial subfamily of (7.1) suffices. Indeed, on each quasi-monomial simplex, rational points are normalized divisorial valuations and are dense, while the restrictions of $\widehat{u}_i$ are finite and continuous [BFJ08, Proposition 5.6]. Equality at these rational points therefore gives equality throughout every simplex. The retraction formula (2.6) gives equality on $\mathcal{V}$.

Appendix A. A continuity estimate for the interpolation envelope

We prove a local uniform estimate for the Bergman approximations of the envelope in [BGMY25a, Lemma 2.32].

Let Φ denote the negative envelope called φ_2 in [BGMY25a, Section 2.5], on a bounded hyperconvex domain D containing o. The function Φ is locally bounded on $D \setminus \{o\}$, satisfies logarithmic bounds, and tends to zero at ∂D. The boundary limit follows from the exhaustion-barrier argument in [BGMY25a, Lemma 2.32]. Lemmas 2.29–2.31 there also give a constant $C > 0$ with

$$c_o^f(\Phi) - \sigma(\log|f|, \Phi) \le C \quad (f \not\equiv 0), \tag{A.1}$$

and the following envelope property. For $t > 0$, let $D_t = \{\Phi < -t\}$. The corresponding envelope on D_t is $\Phi + t$, and every negative psh function h on D_t satisfies

$$h \le \sigma(h, \Phi)(\Phi + t). \tag{A.2}$$

Here the relative type is unchanged upon adding the constant t.

For the Hilbert space

$$\mathcal{H}_m = \left\{ f \in \mathcal{O}(D) : \int_D |f|^2 e^{-2m\Phi}\, dV < \infty \right\},$$

let ψ_m be its Bergman approximation:

$$\psi_m(z) = \frac{1}{m} \log \sup_{\|f\|_{\mathcal{H}_m} \le 1} |f(z)|.$$

Demailly's estimate gives $\psi_m \ge \Phi - c_1/m$, with c_1 independent of m. We claim that for every fixed $t > 0$ there is $B_t > 1$ such that

$$\Phi - \frac{c_1}{m} \le \psi_m \le \left(1 - \frac{C}{m}\right)(\Phi + t) + \frac{\log B_t}{m} \quad \text{on } D_t, \qquad m > C. \tag{A.3}$$

By the boundary limit, $\overline{D_t} \Subset D$. Since $\Phi < 0$,

$$\|f\|_{\mathcal{H}_m} \le 1 \quad \Longrightarrow \quad \int_D |f|^2\, dV \le \int_D |f|^2 e^{-2m\Phi}\, dV \le 1.$$

The interior mean-value estimate gives $B_t > 1$, independent of m, such that

$$\sup_{D_t} |f| < B_t \quad (\|f\|_{\mathcal{H}_m} \le 1).$$

For $f \neq 0$, put $h = \log(|f|/B_t) < 0$ on D_t. By (A.1),

$$c_o^f(\Phi) \ge m, \qquad \sigma(h, \Phi) = \sigma(\log|f|, \Phi) \ge c_o^f(\Phi) - C \ge m - C.$$

Since $\Phi + t < 0$, (A.2) yields

$$\log|f| - \log B_t \le \sigma(h, \Phi)(\Phi + t) \le (m - C)(\Phi + t).$$

Taking the supremum over the unit ball proves the upper estimate in (A.3).

To conclude, fix $K \Subset D \setminus \{o\}$. Since $\max_K \Phi < 0$, all sufficiently small $t > 0$ satisfy $K \subset D_t$. The function Φ is bounded on K, and (A.3) implies

$$\limsup_{m\to\infty} \|\psi_m - \Phi\|_{L^\infty(K)} \le t.$$

Let $t \downarrow 0$. Thus $\psi_m \to \Phi$ locally uniformly off o. The Bergman kernel is smooth and positive there: smoothness follows from local convergence of the holomorphic kernel series, and positivity from Demailly's lower estimate. Hence Φ is continuous off o. The logarithmic upper bound and the boundary limit give

$$e^\Phi \in C(D), \qquad e^\Phi(o) = 0, \qquad \lim_{z\to\partial D} e^{\Phi(z)} = 1.$$

Together with (A.1), this gives the required tame maximal representative.

References

[BGMY25a] S. Bao, Q. Guan, Z. Mi, and Z. Yuan, The existence of valuative interpolation, preprint (2025), arXiv:2510.22244v1.

[BGMY25b] S. Bao, Q. Guan, Z. Mi, and Z. Yuan, Tame maximal weights, relative types and valuations, *Adv. Math.* **477** (2025), 110364. doi:10.1016/j.aim.2025.110364.

[BGY23] S. Bao, Q. Guan, and Z. Yuan, Zhou valuations and jumping numbers, preprint (2023), arXiv:2311.06565.

[BGZ25] S. Bao, Q. Guan, and L. Zhou, Algebraic Zhou valuations, preprint (2025), arXiv:2505.19451.

[BFJ08] S. Boucksom, C. Favre, and M. Jonsson, Valuations and plurisubharmonic singularities, *Publ. Res. Inst. Math. Sci.* **44** (2008), no. 2, 449–494. doi:10.2977/PRIMS/1210167334.

[Cut13] S. D. Cutkosky, Multiplicities associated to graded families of ideals, *Algebra Number Theory* **7** (2013), no. 9, 2059–2083. doi:10.2140/ant.2013.7.2059.

[Dem87] J.-P. Demailly, Nombres de Lelong généralisés, théorèmes d'intégralité et d'analyticité, *Acta Math.* **159** (1987), no. 3–4, 153–169. doi:10.1007/BF02392558.

[Dem93] J.-P. Demailly, Monge–Ampère operators, Lelong numbers and intersection theory, in *Complex Analysis and Geometry* (V. Ancona and A. Silva, eds.), Plenum Press, New York, 1993, pp. 115–193.

[DK01] J.-P. Demailly and J. Kollár, Semi-continuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds, *Ann. Sci. École Norm. Sup.* (4) **34** (2001), no. 4, 525–556. doi:10.1016/S0012-9593(01)01069-2.

[DLLZ24] F. Deng, Y. Li, Q. Liu, and X. Zhou, Multiplicities and modifications, and singularities associated to blowing down negative vector bundles, preprint (2024), arXiv:2409.09407v1.

[ELSV04] L. Ein, R. Lazarsfeld, K. E. Smith, and D. Varolin, Jumping coefficients of multiplier ideals, *Duke Math. J.* **123** (2004), no. 3, 469–506. doi:10.1215/S0012-7094-04-12333-4.

[GZ15] Q. Guan and X. Zhou, A proof of Demailly's strong openness conjecture, *Ann. of Math.* (2) **182** (2015), no. 2, 605–616. doi:10.4007/annals.2015.182.2.5.

[HS06] C. Huneke and I. Swanson, *Integral closure of ideals, rings, and modules*, London Mathematical Society Lecture Note Series 336, Cambridge University Press, 2006. doi:10.1017/CBO9780511617089.

[JM12] M. Jonsson and M. Mustaţă, Valuations and asymptotic invariants for sequences of ideals, *Ann. Inst. Fourier (Grenoble)* **62** (2012), no. 6, 2145–2209. doi:10.5802/aif.2746.

[JM14] M. Jonsson and M. Mustaţă, An algebraic approach to the openness conjecture of Demailly and Kollár, *J. Inst. Math. Jussieu* **13** (2014), no. 1, 119–144. doi:10.1017/S1474748013000091.

[Ras06] A. Rashkovskii, Relative types and extremal problems for plurisubharmonic functions, *Int. Math. Res. Not.* (2006), Art. ID 76283, 26 pp. doi:10.1155/IMRN/2006/76283.

[Ras10] A. Rashkovskii, An extremal problem for generalized Lelong numbers, *Math. Z.* **266** (2010), no. 2, 345–362. doi:10.1007/s00209-009-0572-1.

[Tia87] G. Tian, On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$, *Invent. Math.* **89** (1987), no. 2, 225–246. doi:10.1007/BF01389077.

Declaration on the use of artificial intelligence

The authors proposed the research questions and reviewed the mathematical arguments. GPT-6 Astra and GPT-5.6-sol (high) assisted with the mathematical development and exposition. These models and DeepSeek 4.1 Flash 1017 also assisted with language editing and manuscript organization. The authors reviewed and edited the text and take full responsibility for the content of this article.

Department of Mathematics, Graduate School of Science, Osaka Metropolitan University, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka 558-8585, Japan
Email address: shijie.bao@omu.ac.jp

School of Mathematical Sciences, Peking University, Beijing 100871, China
Email address: guanqian@math.pku.edu.cn

School of Mathematics and Statistics, Beijing Jiaotong University, Beijing 100044, China
Email address: zhitongmi@amss.ac.cn

State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email address: sunxun@amss.ac.cn

State Key Laboratory of Mathematical Sciences, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
Email address: yuanzheng@amss.ac.cn