GENERALIZED LELONG NUMBERS DETERMINE VALUATIVE TRANSFORMS

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Abstract. Boucksom, Favre, and Jonsson proved that equality of the valuative transforms of two plurisubharmonic germs implies equality of all generalized Lelong numbers with tame reference weights. We prove the converse. It is enough to test against the analytic weights defined by primary ideals. On each birational model, these tests determine an intersection polynomial on the relative ample cone. Its first derivative, together with a negative-definite exceptional intersection matrix, recovers the trace of the valuative divisor.

1. Introduction

Let o denote the origin of $\mathbb{C}^n$, and let $u, v : (\mathbb{C}^n, o) \to \mathbb{R} \cup \{-\infty\}$ be plurisubharmonic (psh) germs, neither identically equal to $-\infty$. If φ is a psh weight with an isolated pole at o, Demailly's generalized Lelong number is

$$\nu_\varphi(u) = \big(dd^c u \wedge (dd^c\varphi)^{n-1}\big)(\{o\}). \tag{1}$$

We normalize $d^c = (i/2\pi)(\bar\partial - \partial)$, so that $dd^c \log|f|$ is the divisor current of a nonzero holomorphic function f. The product in (1) is the generalized Lelong product of [Dem87]; it is not the non-pluripolar product, which discards mass on pluripolar sets. The definition, comparison theorem, and invariance under bounded changes of the weight are established in [Dem87].

We use the terminology of [BFJ08, Section 5.3]. A *psh weight* is a psh germ φ with an isolated singularity at o such that e^φ is continuous. It is *tame* if there is a constant $C > 0$ such that, for every $t > 0$ and every $f \in \mathcal{J}(t\varphi)$,

$$\log|f| \le (t - C)\varphi + O(1). \tag{2}$$

Because φ is negative near its pole, inequality (2) says that the singularities detected by the multiplier ideals approximate φ with an error uniformly bounded in t. We shall not use the definition directly; it is included to make clear which class of test weights occurs in the theorem.

Date: September 5, 2026.

2020 *Mathematics Subject Classification.* Primary 32U25; Secondary 14B05, 14C20, 32U35.

Key words and phrases. Generalized Lelong number, valuative transform, b-divisor, mixed multiplicity, relative Hodge index theorem.

Boucksom–Favre–Jonsson associate to u a valuative transform $\widehat{u}$. The same information can be written as a nef Weil b-divisor $Z(u)$ over o: the coefficient of $Z(u)$ along an exceptional prime divisor is the negative of the corresponding generic Lelong number. We recall the construction in section 2. The language of b-divisors goes back to Shokurov; the local form used here is developed in [BFJ08, Sections 1–4]; see also [Sho03].

The following four conditions appear in [BFJ08, Theorem A]:

(1) u and v have the same Lelong number at every infinitely near point;
(2) $\mathcal{J}_+(tu) = \mathcal{J}_+(tv)$ for every $t > 0$;
(3) the relative types of u and v with respect to every tame maximal psh weight agree;
(4) $\nu_\varphi(u) = \nu_\varphi(v)$ for every tame psh weight φ.

Here $\mathcal{J}_+(w)$ is the stationary value of $\mathcal{J}((1+\varepsilon)w)$ as $\varepsilon \downarrow 0$. Boucksom–Favre–Jonsson proved that (1), (2), and (3) are equivalent and imply (4). They left (4)⇒(1) open and proposed the construction of fundamental solutions for the formal Monge–Ampère operator as one possible approach; see [BFJ08, p. 455 and p. 457].

In dimension two, the valuative transform and its Laplacian are governed by the valuative tree; see [FJ05]. In higher dimension, the formal Monge–Ampère operator is instead built from intersections on all modifications. The strong openness theorem [GZ15] settles the openness problem mentioned alongside Theorem A in [BFJ08], but it does not invert the averaging operation in Theorem B of that paper. More recent intersection theories for nef b-divisor classes establish general approximation and Hodge index principles in a projective setting; see [DF22]. The local argument below uses the local intersection formalism of [BFJ08]. We give a matrix proof of the strictness needed for separation and distinguish the fixed-model intersection pairing from the general nef intersection product.

We prove this remaining implication. The statement below is slightly stronger: only weights with analytic singularities coming from primary ideals are needed.

Theorem 1.1. *Let $u, v : (\mathbb{C}^n, o) \to \mathbb{R} \cup \{-\infty\}$ be psh germs. The following conditions are equivalent, provided neither germ is identically $-\infty$.*

(i) *u and v have the same Lelong number at every infinitely near point.*
(ii) *$\nu_\varphi(u) = \nu_\varphi(v)$ for every tame psh weight φ.*
(iii) *For every $\mathfrak{m}_o$-primary ideal $\mathfrak{a} = (f_1, \ldots, f_N) \subset \mathcal{O}_{\mathbb{C}^n,o}$,*

$$\nu_{\varphi_\mathfrak{a}}(u) = \nu_{\varphi_\mathfrak{a}}(v), \qquad \varphi_\mathfrak{a} := \frac{1}{2}\log\Bigg(\sum_{j=1}^N |f_j|^2\Bigg).$$

Consequently, all four conditions of [BFJ08, Theorem A] *are equivalent.*

For $n \geq 2$, put $W = Z(u) - Z(v)$. On a model determining the divisor of a primary ideal $\mathfrak{a}$, the analytic weight $\varphi_\mathfrak{a}$ satisfies

$$\nu_{\varphi_\mathfrak{a}}(u) = -\left\langle Z(u), Z(\mathfrak{a})^{n-1} \right\rangle. \tag{3}$$

Thus condition (iii) gives diagonal equations

$$\left\langle W, Z(\mathfrak{a})^{n-1} \right\rangle = 0. \tag{4}$$

Fix a model X_π. Every rational relatively ample exceptional divisor A has a positive multiple equal to the divisor of a primary ideal determined on this same model. Consequently the polynomial $F_\pi(A) = W_\pi \cdot A^{n-1}$ vanishes on the relative ample cone. Differentiating at an ample divisor H in the direction W_π gives

$$W_\pi^2 \cdot H^{n-2} = 0.$$

A negative-definite intersection matrix forces $W_\pi = 0$. Since the model was arbitrary, $W = 0$. This argument uses only a finite polynomial on each model, not a differentiability assertion for a volume on the space of all Weil b-divisors.

Each input is separated below. Sections 2 and 3 review the b-divisor and intersection formalism. Section 4 proves strict negativity directly. Section 5 records the complementary polarization argument, including the mixed-multiplicity interpretation. Section 6 proves the theorem by ample perturbations; the proof does not depend on the polarization section.

The one-dimensional case and the standing dimension convention. If $n = 1$, the exponent in (1) is zero. Hence for every test weight

$$\nu_\varphi(u) = (dd^c u)(\{o\}) = \nu^L(u, o).$$

The last equality is the one-variable description of the Lelong number as the atom of the Riesz measure, with the normalization chosen above. A proper bimeromorphic map between smooth curve germs is an isomorphism: its fibers are finite, so it is finite, and a finite birational map onto a normal curve is an isomorphism. Thus infinitely near points introduce no additional Lelong numbers. Conditions (i)–(iii) are all equivalent to $\nu^L(u, o) = \nu^L(v, o)$; for (iii) one may use the primary ideal (z). This proves the one-dimensional case before any exceptional divisor is used. *For the remainder of the paper, $n \geq 2$.*

2. Birational models and valuative divisors

2.1. **Models above the origin.** We use the local models of [BFJ08, Sections 1.1–1.2], written analytically as smooth projective modifications

$$\pi : X_\pi \longrightarrow (\mathbb{C}^n, o)$$

that are isomorphisms away from o. The valuations under consideration are divisorial valuations centered at o, not valuations centered on a positive-dimensional subvariety through o. We may use the algebraic local models over $\mathbb{C}[z_1, \ldots, z_n]_{(z_1,\ldots,z_n)}$ of that paper, and their analytifications when evaluating a psh germ. Primary analytic, algebraic, and formal ideals are identified as explained in Section 3.1. The valuative interpretation of infinitely near Lelong numbers is the one in [BFJ08, Section 5.6].

Any finitely many models admit a common smooth projective dominating model: take the component of their fiber product containing the common open set away from o, and resolve it. We may further principalize the pullback of $\mathfrak{m}_o$ and resolve the exceptional locus. Thus models on which this pullback is invertible and the exceptional support has simple normal crossings form a cofinal system. A statement about all traces of a Weil b-divisor can be checked on this system: every other trace is its pushforward from a dominating model. We use this observation when applying the finite-atomic formula of [BFJ08, Proposition 4.9].

Write $\mathrm{Exc}(\pi)$ for the exceptional locus and set

$$\mathrm{Div}(\pi) := \bigoplus_{E \subset \mathrm{Exc}(\pi)} \mathbb{R}E. \tag{5}$$

Every exceptional prime is projective over the point o. In this local setting over a smooth base, an exceptional divisor is determined by its relative numerical class, and

$$\mathrm{Div}(\pi) \simeq N^1(X_\pi/\mathbb{C}^n)_{\mathbb{R}}; \tag{6}$$

see [BFJ08, Section 1.2]. If $\mu : X_{\pi'} \to X_\pi$ is a higher model, exceptional divisors admit pushforward and pullback, with $\mu_*\mu^*D = D$.

We use $\mathbb{R}$-Cartier b-divisors throughout, and usually omit the prefix "$\mathbb{R}$-". The following elementary consequence of projectivity will be used twice.

Lemma 2.1 (An ample exceptional divisor)**.** *For every smooth projective modification $\pi : X_\pi \to (\mathbb{C}^n, o)$ there is an integral exceptional divisor A such that $\mathcal{O}_{X_\pi}(A)$ is π-ample. After replacing A by a positive multiple and shrinking the representative of the base germ, it may be taken π-very ample.*

Proof. If the model has no exceptional divisors, there is nothing to separate; the identity model may be omitted from subsequent arguments. On a nontrivial model, projectivity provides a relatively ample numerical class. The isomorphism (6) identifies the relative Néron–Severi space with the space of exceptional real divisors. Since the ample cone is open, it contains a rational exceptional class. After multiplication this class is represented by an integral exceptional divisor A. Relative ampleness depends only on the numerical class, and a sufficiently large positive multiple of a relatively ample line bundle is relatively very ample. These are the relative forms of the standard ampleness criteria; see [Laz04, Chapter I, Section 1.7]. More explicitly, choose the exceptional primes as a basis of $\mathrm{Div}(\pi)$. Rational coefficient vectors are dense in this finite-dimensional real space, so a small ball contained in the ample cone contains such a vector. Clearing its denominators produces A. Multiplication by a positive integer preserves ampleness. Relative Serre generation then gives a relatively very ample multiple. No claim is made that every nef exceptional divisor is ample or generated. $\square$

Definition 2.2. A *Weil b-divisor* Z over o is a compatible collection $Z = (Z_\pi)_\pi$ with $Z_\pi \in \mathrm{Div}(\pi)$ and $\mu_* Z_{\pi'} = Z_\pi$ whenever π' dominates π. The divisor Z_π is the *incarnation* or *trace* of Z on X_π.

A *Cartier b-divisor* C is a Weil b-divisor for which some model X_π satisfies $C_{\pi'} = \mu^* C_\pi$ on every higher model. Such a model is a *determination* of C.

Thus $D \in \mathrm{Div}(\pi)$ defines a Cartier b-divisor by pullback to higher models; denote it by $\overline{D}$ when needed. A Weil b-divisor need not be determined on any finite model. These are the projective- and inductive-limit definitions in [BFJ08, Section 1.2]. Two Weil b-divisors are equal precisely when their incarnations agree on every model.

For later use, we also recall positivity. A Cartier b-divisor C determined on X_π is *nef over o* if

$$C_\pi \cdot \Gamma \geq 0 \tag{7}$$

for every complete curve $\Gamma \subset \pi^{-1}(o)$. This criterion is [BFJ08, Proposition 2.2]. A nef Weil b-divisor is, by definition, a coefficientwise limit of nef Cartier b-divisors. Thus "nef" here is a relative condition over the germ, not nefness on a compact ambient variety.

2.2. **The divisor attached to a psh germ.** Let E be an exceptional prime on X_π. At a very general point of E, $u \circ \pi$ has a generic Lelong number $\nu_E(u)$. Define

$$\mathrm{ord}_E Z(u) := -\nu_E(u). \tag{8}$$

Equivalently, if

$$dd^c(u \circ \pi) = \sum_E \nu_E(u)[E] + R_\pi$$

is the Siu decomposition and R_π does not charge an exceptional prime, then $[Z(u)_\pi] = -\sum_E \nu_E(u)[E]$. This is the construction in [BFJ08, Section 5.2].

Proposition 2.3 (Boucksom–Favre–Jonsson)**.** *For every psh germ u, the Weil b-divisor $Z(u)$ is nef.*

Proof. The assertion is [BFJ08, Theorem 5.1]. We spell out how its approximation is compatible with the definition of the nef cone. For a divisorial valuation v centered at o, write $v(u)$ for the generic Lelong number with the same scaling as v. Demailly approximation, in the form of [BFJ08, equation (5.3) and the proof of Theorem 5.1], gives

$$\lim_{k\to\infty} \frac{v(\mathcal{J}(ku))}{k} = v(u).$$

The multiplier ideals here need not be primary. For each integer $k \geq 1$ replace them by

$$\mathfrak{b}_k = \mathcal{J}(ku) + \mathfrak{m}_o^{k^2}.$$

This ideal is either primary or the unit ideal. Its divisor is therefore nef Cartier; the unit ideal corresponds to the zero divisor. Indeed, primary ideals can be principalized by modifications supported over o, and their pullbacks are relatively generated invertible ideals, as detailed in Section 3.2. Valuations of sums of ideals are minima, so

$$\frac{v(\mathfrak{b}_k)}{k} = \min\left\{\frac{v(\mathcal{J}(ku))}{k},\, k\,v(\mathfrak{m}_o)\right\} \longrightarrow v(u).$$

Here $v(\mathfrak{m}_o) > 0$, whereas $v(u)$ is finite. Thus $k^{-1}Z(\mathfrak{b}_k)$ converges coefficientwise to $Z(u)$. The nef Weil cone is closed in this topology by definition, proving the assertion from the quoted approximation. Monotonicity of this particular sequence is not asserted or needed. □

Remark 2.4 (Why the primary truncation matters). For $u = \log|z_1|$ on $\mathbb{C}^2$, local integrability gives $\mathcal{J}(ku) = (z_1^k)$ for every positive integer k. Although this is a principal ideal on the base, its exceptional b-divisor in the present local system is not Cartier. To see this, take the normalized monomial divisorial valuations $v_m(z_1) = m$, $v_m(z_2) = 1$. Their values on this divisor are $-km$. They are unbounded, whereas the function of a Cartier b-divisor extends continuously to the compact normalized valuation space by [BFJ08, Proposition 1.14], and is therefore bounded. There is no contradiction with the fact that (z_1^k) defines a Cartier divisor on the base: our b-divisors retain only exceptional components over the point, not the hypersurface $\{z_1 = 0\}$.

Remark 2.5 (Signs). The coefficients in (8) are nonpositive. This is the convention of [BFJ08]. For example, on the blowup of o, $-E$ is relatively ample and has positive degree on curves contracted by the blowup.

2.3. **Infinitely near points.** Let $p \in \pi^{-1}(o)$ and let $\mu : \mathrm{Bl}_p X_\pi \to X_\pi$ be the blowup of the smooth point p, with exceptional divisor F_p. In local coordinates centered at p, the order along F_p is the ordinary multiplicity at p. Its psh counterpart is

$$\nu_{F_p}(u) = \nu^L(u \circ \pi, p), \tag{9}$$

where ν^L is the usual Lelong number. This equality can be checked using the same approximation as above. For a holomorphic germ f at p, the order of its pullback along F_p is its multiplicity at p: in a blowup chart, the lowest homogeneous term factors out precisely that power of the exceptional coordinate. For an ideal weight, both sides of (9) are consequently the minimum of these multiplicities. For general $w = u \circ \pi$ near p, apply the ideal case to Demailly approximants with singularities $k^{-1}\log|\mathcal{J}(kw)|$. Their ordinary Lelong numbers converge to that of w, and their generic Lelong numbers along F_p converge to that of its pullback by the divisorial approximation in proposition 2.3. Passing to the limit proves (9). This is the identification used in [BFJ08, Section 5.6]. Consequently,

$$Z(u) = Z(v) \Longleftrightarrow \nu^L(u \circ \pi, p) = \nu^L(v \circ \pi, p) \text{ for all } (\pi, p). \tag{10}$$

For the forward direction, choose any (π, p) and blow up p. Equality of the two Weil b-divisors gives equality of their coefficients along F_p on a smooth dominating model. Equation (9) then gives equality of the ordinary

Lelong numbers at this particular p; since p was arbitrary this proves the forward implication. For the reverse direction, fix an exceptional prime E and choose a very general smooth point $p \in E$ outside the other exceptional components. The ordinary Lelong number of $u \circ \pi$ at such a point is the generic Lelong number $\nu_E(u)$, and the point may be chosen to have this property for both u and v simultaneously. In fact the exceptional sets for the two generic values are countable unions of proper analytic subsets of E, so their union can be avoided. This is the generic-value property in the Siu decomposition used in [BFJ08, Section 5.2]. Equality at all infinitely near points therefore gives equality of the coefficient of every exceptional prime in (8).

3. Primary ideals, analytic weights, and intersections

3.1. The analytic and formal local rings. We use the three local rings

$$R = \mathbb{C}[z_1, \dots, z_n]_{(z_1,\dots,z_n)}, \qquad \mathcal{O} = \mathcal{O}_{\mathbb{C}^n,o}, \qquad \widehat{R} = \mathbb{C}[[z_1, \dots, z_n]].$$

The natural inclusions induce isomorphisms

$$R/\mathfrak{m}_o^N \simeq \mathcal{O}/\mathfrak{m}_o^N \simeq \widehat{R}/\mathfrak{m}_o^N \quad (N \geq 1).$$

An ideal containing $\mathfrak{m}_o^N$ is the inverse image of an ideal in this quotient. Primary ideals in the three rings therefore correspond under extension and contraction. Choose generators f_j of an analytic primary ideal containing $\mathfrak{m}_o^N$ and truncate them to polynomials p_j of degree less than N. Since $f_j - p_j \in \mathfrak{m}_o^N$, the p_j together with the monomials of degree N generate the same ideal. They also generate the same pulled-back ideal on every model, so all divisorial values agree. This explains the identification at the beginning of [BFJ08, Section 5] and its applicability to primary analytic weights. It does not assert that an arbitrary non-primary analytic ideal is algebraic.

3.2. The Cartier b-divisor of an ideal. Let $\mathfrak{a} \subset \mathcal{O}_{\mathbb{C}^n,o}$ be $\mathfrak{m}_o$-primary. For each exceptional prime E, set

$$\operatorname{ord}_E Z(\mathfrak{a}) := -\operatorname{ord}_E(\mathfrak{a}), \qquad \operatorname{ord}_E(\mathfrak{a}) := \min_{f\in\mathfrak{a}} \operatorname{ord}_E(f). \tag{11}$$

On a log resolution π of $\mathfrak{a}$,

$$\mathfrak{a}\mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}\big(Z(\mathfrak{a})_\pi\big). \tag{12}$$

Locally write the pulled-back generators as $f_j = sh_j$, where s generates the invertible ideal and the h_j generate the unit ideal. The order of s is the minimum of the orders of the f_j. The negative sign in (11) comes from $\mathfrak{a}\mathcal{O} = \mathcal{O}(-\operatorname{div}(s))$. Pullback preserves invertibility, so on higher models this divisor pulls back. Thus $Z(\mathfrak{a})$ is Cartier; see [BFJ08, Section 1.2]. For blowups and their tautological line bundles, see [Har77, Chapter II, Section 7]. The sheaf in (12) is generated relative to the base, so $Z(\mathfrak{a})$ is nef; compare [BFJ08, Proposition 2.2].

On a complete curve in a fiber, this relatively generated line bundle is globally generated and has nonnegative degree. Thus it satisfies exactly the relative nefness criterion used above.

For $\mathfrak{a} = (f_i)$ and $\mathfrak{b} = (g_j)$, the product is generated by $f_i g_j$, and any valuation satisfies

$$v(\mathfrak{a}\mathfrak{b}) = \min_{i,j}(v(f_i) + v(g_j)) = v(\mathfrak{a}) + v(\mathfrak{b}).$$

Consequently, for primary ideals $\mathfrak{a}, \mathfrak{b}$,

$$Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a}) + Z(\mathfrak{b}), \tag{13}$$

and more generally

$$Z(\mathfrak{a}_1^{k_1} \cdots \mathfrak{a}_r^{k_r}) = \sum_{i=1}^{r} k_i Z(\mathfrak{a}_i). \tag{14}$$

These are equations (2.1) of [BFJ08]. A product of $\mathfrak{m}_o$-primary ideals remains primary because its radical is $\mathfrak{m}_o$.

3.3. Realizing ample divisors by primary ideals.

Lemma 3.1 (Strict signs on an ample exceptional divisor)**.** *Let A be a relatively ample exceptional real divisor on a nontrivial model. Writing the exceptional primes as $E_1, \dots, E_r$, we have $A = -\sum_i a_i E_i$ with $a_i > 0$ for every i.*

Proof. The Cartier b-divisor $\overline{A}$ is nef by [BFJ08, Proposition 2.2]. It is nonzero. Indeed a nontrivial model has a fiber of positive dimension: otherwise the map would be finite birational onto a normal base and therefore an isomorphism. This projective fiber contains a curve, on which an ample class has positive degree. Proposition 2.4 of [BFJ08] supplies $c > 0$ with $\overline{A} \leq cZ(\mathfrak{m}_o)$. For each exceptional prime,

$$\operatorname{coeff}_{E_i}(A) \leq -c \operatorname{ord}_{E_i}(\mathfrak{m}_o) < 0.$$

The last inequality holds because the center is o, so all generators of $\mathfrak{m}_o$ have positive order along E_i. Thus every coefficient is strictly negative, not merely nonpositive. ☐

Lemma 3.2 (Generated exceptional divisors)**.** *Let B be an integral exceptional divisor with strictly negative coefficients on a nontrivial model. If $\mathcal{O}_{X_\pi}(B)$ is relatively generated, then $\mathfrak{a}_B = \pi_* \mathcal{O}_{X_\pi}(B)$ is primary and*

$$\mathfrak{a}_B \mathcal{O}_{X_\pi} = \mathcal{O}_{X_\pi}(B), \qquad Z(\mathfrak{a}_B) = \overline{B}.$$

Proof. The inequality $B \leq 0$ embeds $\mathcal{O}(B)$ into $\mathcal{O}_{X_\pi}$. Normality of the base and proper birationality give $\pi_* \mathcal{O}_{X_\pi} = \mathcal{O}$, so the pushforward is a coherent ideal in the base ring. Off o, the map is an isomorphism and B has no support; the ideal is the unit ideal there. It is proper at o, since the function 1 cannot be a section of $\mathcal{O}(B)$ along a prime with negative coefficient. Its zero set is therefore exactly $\{o\}$. An ideal with maximal radical in a Noetherian local ring contains a power of that maximal ideal and is primary, proving the first assertion.

Relative generation is the surjectivity of

$$\pi^*\pi_*\mathcal{O}(B) \longrightarrow \mathcal{O}(B).$$

The image inside $\mathcal{O}_{X_\pi}$ is precisely $\mathfrak{a}_B\mathcal{O}_{X_\pi}$, proving the first equality. Pulling back to any higher model again gives the extended ideal and the pullback of B. This proves equality as Cartier b-divisors, not only on the given model. □

Proposition 3.3 (Ample rational classes are ideal tests)**.** *For a rational relatively ample exceptional divisor A on a nontrivial model X_π, there are an integer $m > 0$ and a primary ideal $\mathfrak{a}$ such that $Z(\mathfrak{a}) = \overline{mA}$, determined on X_π.*

Proof. Clear the denominators of A. Relative Serre generation [Laz04, Chapter I, Section 1.7] gives a sufficiently high power of the resulting ample line bundle which is relatively generated. Thus for sufficiently divisible m, the divisor mA is integral and $\mathcal{O}(mA)$ is relatively generated. Its coefficients are strictly negative by lemma 3.1. Apply lemma 3.2 to $B = mA$. The ideal and m may depend on A; no uniform choice over the ample cone is needed. □

Proposition 3.4 (Ideal divisors span)**.** *The real vector space of Cartier b-divisors is spanned by the divisors $Z(\mathfrak{a})$ of $\mathfrak{m}_o$-primary ideals. Thus every Cartier b-divisor has an expression*

$$C = \sum_{j=1}^{N} \lambda_j Z(\mathfrak{a}_j), \qquad \lambda_j \in \mathbb{R}. \tag{15}$$

Proof. This is the real form of [BFJ08, Proposition 1.14]. We give the construction on a fixed model, keeping track of determinations. First let D be integral and choose an integral ample exceptional divisor A on its determination. Write $D = \sum_i d_i E_i$ and $A = -\sum_i a_i E_i$ with $a_i > 0$. For $m > d_i/a_i$ for every i, the coefficients of $D + mA$ and mA are strictly negative. For sufficiently large m, relative Serre generation applied to the coherent sheaves $\mathcal{O}(D)$ and $\mathcal{O}$ also makes $\mathcal{O}(D+mA)$ and $\mathcal{O}(mA)$ relatively generated. Choose m satisfying both conditions. Applying lemma 3.2 to these two divisors gives primary ideals with

$$\overline{D} = Z(\mathfrak{a}_1) - Z(\mathfrak{a}_2).$$

Both terms are determined on the original model.

For real $D = \sum_i d_i E_i$, apply the integral construction to each E_i, multiply the resulting differences by d_i, and sum. This is a finite sum because a model has finitely many exceptional primes. Every Cartier b-divisor has a determination, so the construction proves the assertion. The zero divisor requires no terms. □

Remark 3.5. The coefficients λ_j may be negative. Thus proposition 3.4 concerns the full vector space, not only the nef cone. This permits the later test against the signed Cartier divisor $\overline{W_\pi}$.

3.4. **The analytic weight of an ideal.** For generators $\mathfrak{a} = (f_1, \dots, f_N)$, set

$$\varphi_{\mathfrak{a}} = \frac{1}{2} \log \left(\sum_{j=1}^{N} |f_j|^2 \right). \tag{16}$$

If f and g are two vectors of generators, there are holomorphic matrices M, N with $f = Mg$ and $g = Nf$. Their operator norms are bounded on a smaller neighborhood. Hence $\|f\| \leq C\|g\|$ and $\|g\| \leq C'\|f\|$, so the logarithmic weights differ by a bounded function. Demailly's comparison theorem implies that their generalized Lelong numbers coincide [Dem87].

Since the ideal is primary, its generators have no common zero off o. Thus the weight has an isolated pole. The vector of generators is locally Lipschitz, and the norm satisfies $|\|f(z)\| - \|f(w)\|| \leq \|f(z) - f(w)\|$. Consequently $e^{\varphi_{\mathfrak{a}}} = \|f\|$ is locally Lipschitz. Lemma 5.10 of [BFJ08] states that a psh weight with Hölder-continuous exponential is tame, so this weight is an admissible test.

On a principalizing model, use the local factorization $f_j = sh_j$ to get

$$\varphi_{\mathfrak{a}} \circ \pi = \log|s| + \tfrac{1}{2} \log \sum_j |h_j|^2.$$

The second term is smooth because the h_j have no common zero. The generic Lelong number along E is therefore the vanishing order of s, namely $\operatorname{ord}_E(\mathfrak{a})$. This description remains valid after pullback to any higher model. Hence

$$Z(\varphi_{\mathfrak{a}}) = Z(\mathfrak{a}). \tag{17}$$

3.5. **Intersection products.** For $D_1, \dots, D_n \in \operatorname{Div}(\pi)$, their local intersection number $D_1 \cdots D_n$ is computed on X_π. It is defined because the divisors are supported over o, so the resulting zero-cycle is proper. The product is symmetric, multilinear, and preserved on higher models by the projection formula; see [BFJ08, Section 4.1, Proposition 4.2] and [Ful98, Section 2.3]. Hence for Cartier b-divisors determined on X_π,

$$\langle C_1, \dots, C_n \rangle := C_{1,\pi} \cdots C_{n,\pi}. \tag{18}$$

Boucksom–Favre–Jonsson extend this product to nef Weil b-divisors by monotone approximation; see [BFJ08, Definition 4.3 and Proposition 4.4]. We need only the following finite case.

More explicitly, if $Z_1, \dots, Z_n$ are nef Weil b-divisors, they set

$$\langle Z_1, \dots, Z_n \rangle := \inf_{C_i \geq Z_i} \langle C_1, \dots, C_n \rangle, \tag{19}$$

where the infimum is taken over nef Cartier b-divisors C_i dominating Z_i coefficientwise. Monotonicity of intersection for nef Cartier divisors makes this compatible with (18). The value can in general be $-\infty$, but it is finite in the one-Weil-factor situation used below.

Proposition 3.6 (One Weil factor on a fixed model)**.** *Let Z be any real Weil b-divisor and let $C_2, \dots, C_n$ be Cartier b-divisors, determined on X_π. The formula*

$$\langle Z, C_2, \dots, C_n \rangle := Z_\pi \cdot C_{2,\pi} \cdots C_{n,\pi} \tag{20}$$

defines a finite pairing independent of the common determining model. It is real-linear in Z, multilinear and symmetric in the Cartier variables, and agrees with the BFJ nef intersection product when all factors are nef. In particular it applies to $Z = Z(u) - Z(v)$.

Proof. Step 1: independence of the model. Let $\mu : X_{\pi'} \to X_\pi$ dominate the common determination. By the definition of a Cartier b-divisor, $C_{i,\pi'} = \mu^* C_{i,\pi}$, whereas the Weil compatibility condition is $\mu_* Z_{\pi'} = Z_\pi$. The ordinary projection formula [Ful98, Section 2.3] gives

$$\begin{aligned} Z_{\pi'} \cdot \mu^* C_{2,\pi} \cdots \mu^* C_{n,\pi} &= (\mu_* Z_{\pi'}) \cdot C_{2,\pi} \cdots C_{n,\pi} \\ &= Z_\pi \cdot C_{2,\pi} \cdots C_{n,\pi}. \end{aligned}$$

Only the Cartier factors are pulled back in this equation. For two unrelated determinations, apply it on a common dominating model. This proves independence without assuming that Z is Cartier.

Step 2: finiteness and linearity. On a fixed model, Z_π is a finite real linear combination of proper exceptional primes, and each $C_{i,\pi}$ is a real Cartier divisor. Their intersection is a finite real number. Ordinary intersection theory is multilinear, so the formula is linear in Z and multilinear in the Cartier factors. These statements pass to all models by Step 1. In particular the value for a difference $Z^+ - Z^-$ is the difference of the two values and is independent of the presentation.

Step 3: comparison with the BFJ product for nef factors. Suppose Z and the C_i are nef. Proposition 3.13 of [BFJ08] provides nef Cartier divisors Z_k decreasing coefficientwise to Z (the case $Z = 0$ is immediate). On a common determination of Z_k and the C_i, ordinary intersection theory computes the BFJ product. Step 1 therefore gives

$$\langle Z_k, C_2, \dots, C_n \rangle_{\mathrm{BFJ}} = (Z_k)_\pi \cdot C_{2,\pi} \cdots C_{n,\pi}.$$

The right side converges to the finite expression in (20): there are only finitely many coefficients on X_π, and they converge. By [BFJ08, Proposition 4.4], the left side decreases to the BFJ product with Z. This proves agreement and also proves its finiteness in this case.

Step 4: comparison with the signed extension. For clarity, signed Cartier factors can be decomposed on the same model. Fix an ample exceptional H. For each $C_{i,\pi}$, openness of the ample cone implies that $C_{i,\pi} + tH$ is ample when t is large: $H + t^{-1}C_{i,\pi}$ lies in a sufficiently small neighborhood of H. Thus

$$C_{i,\pi} = (C_{i,\pi} + tH) - tH$$

is a difference of nef divisors determined on X_π. Expanding the finite fixed-model expression gives the alternating sum of the nef products from Step 3. It follows that this expansion is independent of the decompositions and agrees with the usual signed extension. This supplies the extension in all Cartier variables, rather than assuming that a citation allowing one test variable already covers every signed factor. □

Remark 3.7 (Scope of this pairing). The proposition allows exactly one arbitrary Weil factor. The other factors must be Cartier so that they pull back from one common model. It does not define a finite multilinear product for several arbitrary Weil factors. Nor does it identify the trace of a nef Weil divisor with a nef Cartier divisor: such a trace need not be relatively nef.

Remark 3.8 (Mixed multiplicity). For primary ideals,

$$-\langle Z(\mathfrak{a}_1), \dots, Z(\mathfrak{a}_n)\rangle = e(\mathfrak{a}_1, \dots, \mathfrak{a}_n). \tag{21}$$

This is formula (4.3) of [BFJ08], based on [Ram74, Ree84]. It is not logically needed below, but explains why ideal multiplication supplies the polarization data.

3.6. **Generalized Lelong numbers as intersections.**

Proposition 3.9 (The fixed-model Lelong formula). *Let $\mathfrak{a}$ be primary, and suppose its Cartier divisor is determined on X_π, with trace $A = Z(\mathfrak{a})_\pi$. Then for every psh germ u,*

$$\nu_{\varphi_{\mathfrak{a}}}(u) = -Z(u)_\pi \cdot A^{n-1} = -\langle Z(u), Z(\mathfrak{a})^{n-1}\rangle. \tag{22}$$

No isolated-singularity assumption is imposed on u.

Proof. We keep the normalization of divisorial valuations explicit. On a model where $\mathfrak{m}_o$ is principalized, put

$$b_E = \mathrm{ord}_E(\mathfrak{m}_o) > 0, \qquad v_E = b_E^{-1}\,\mathrm{ord}_E\,.$$

Thus $v_E(\mathfrak{m}_o) = 1$ and $\widehat{u}(v_E) = \mathrm{coeff}_E(Z(u)_\pi)/b_E$. We may first take a higher simple determining model so that the formula of [BFJ08, Proposition 4.9] applies. That formula is

$$\mathrm{MA}(\widehat{\varphi}_{\mathfrak{a}}) = \sum_{E\subset \mathrm{Exc}(\pi)} b_E\,(E\cdot A^{n-1})\,\delta_{v_E}. \tag{23}$$

Here δ_{v_E} denotes the unit point mass at v_E. There are $n-1$ copies of the weight in the Monge–Ampère measure. The sum is finite, and its coefficients are nonnegative because A is relatively nef and restricts to a nef class on each projective E. The relevant valuative function is that of A by (17).

Theorem B of [BFJ08] gives, for a tame weight φ,

$$\nu_\varphi(u) = \int_{\mathcal{V}} -\widehat{u}\ \mathrm{MA}(\widehat{\varphi}). \tag{24}$$

Apply it to the tame weight $\varphi_{\mathfrak{a}}$ and substitute (23). If $z_E = \operatorname{coeff}_E Z(u)_\pi$, we obtain

$$\begin{aligned}\nu_{\varphi_{\mathfrak{a}}}(u) &= -\sum_E \frac{z_E}{b_E}\, b_E(E \cdot A^{n-1}) \\ &= -\sum_E z_E(E \cdot A^{n-1}) = -Z(u)_\pi \cdot A^{n-1}.\end{aligned}$$

This displays both the cancellation of the normalization factors and the source of the minus sign. Every z_E is finite; hence there is no subtraction of infinite quantities.

If the given determination was not the simple model used above, push down using proposition 3.6. Because A is Cartier and pulls back, the intersection is unchanged. This proves the formula on the original determination as well. The last equality in (22) is precisely the definition of that pairing. Equivalently, compatibility with the BFJ product follows from [BFJ08, Proposition 4.6]:

$$\int_{\mathcal{V}} \widehat{u}\ \mathrm{MA}(\widehat{\varphi}_{\mathfrak{a}}) = \langle Z(u), Z(\mathfrak{a})^{n-1}\rangle. \tag{25}$$

□

Remark 3.10 (Analytic input versus finite-dimensional algebra). Theorem B is the analytic input; its proof in [BFJ08] uses Demailly approximation. It is not reproved here. Once (22) is established, testing with primary ideals uses only finite intersections on their determining models. In particular, non-isolated singularities of u do not prevent the use of the formula, even though their valuative divisors need not be Cartier.

4. Strict relative Hodge separation

We need negative definiteness on the vector space of exceptional divisors, not only negative semidefiniteness. The latter is the conclusion of [BdFF12, Lemma A.2] for nef remaining factors. For the repeated ample factor needed here, the stronger statement follows from an elementary matrix identity and positivity on each exceptional prime. This avoids any use of a simultaneous surface slice.

Lemma 4.1 (A matrix criterion). *Let $Q = (q_{ij})$ be a real symmetric $r \times r$ matrix. Suppose $q_{ij} \geq 0$ for $i \neq j$, and suppose there is a vector $a = (a_1, \dots, a_r)$ with $a_i > 0$ and $(Qa)_i < 0$ for every i. Then Q is negative definite.*

Proof. Take any $x \in \mathbb{R}^r$ and write $x_i = a_i y_i$, which is possible since $a_i > 0$. We claim the exact identity

$$x^\mathsf{T} Q x = \sum_i a_i (Qa)_i y_i^2 - \sum_{i<j} q_{ij} a_i a_j (y_i - y_j)^2. \tag{26}$$

Indeed the first sum on the right is

$$\sum_i q_{ii} a_i^2 y_i^2 + \sum_{i<j} q_{ij} a_i a_j (y_i^2 + y_j^2).$$

Expanding the square in the second sum cancels its $y_i^2 + y_j^2$ terms and leaves $2\sum_{i<j} q_{ij} a_i a_j y_i y_j$. Together with the diagonal terms, this is $x^\mathsf{T} Q x$.

If $x \neq 0$, at least one $y_i \neq 0$. All coefficients $a_i (Qa)_i$ in the first sum of (26) are strictly negative, so that sum is strictly negative. The second sum is nonnegative before its minus sign, because $q_{ij} a_i a_j \geq 0$. Consequently $x^\mathsf{T} Q x < 0$ for every nonzero x. No connectedness assumption on the graph of the nonzero q_{ij} is needed. □

Lemma 4.2 (Strict relative Hodge index)**.** *Let $\pi : X_\pi \to (\mathbb{C}^n, o)$ be a smooth projective modification and let H be a relatively ample exceptional rational divisor. Then*

$$Q_H(D_1, D_2) = D_1 \cdot D_2 \cdot H^{n-2} \tag{27}$$

is negative definite on $\mathrm{Div}(\pi)$*. In particular,*

$$D \neq 0 \quad \Longrightarrow \quad D^2 \cdot H^{n-2} < 0. \tag{28}$$

For $n = 2$, the factor H^{n-2} is omitted.

Proof. If $\mathrm{Div}(\pi) = 0$, the assertion is vacuous. Otherwise enumerate its prime basis as $E_1, \ldots, E_r$. By lemma 3.1 write

$$H = -\sum_i a_i E_i, \qquad a_i > 0,$$

and set $q_{ij} = E_i E_j H^{n-2}$. Symmetry of intersection gives $q_{ij} = q_{ji}$. We verify the two other hypotheses of lemma 4.1 separately.

Step 1: off-diagonal entries are nonnegative. For $i \neq j$, the smooth ambient variety makes E_i and E_j Cartier divisors with no common irreducible component. Restricting a local equation of E_j to E_i gives an effective divisor cycle, representing $E_i E_j$. It has dimension $n-2$ if it is nonzero and is supported on the proper fiber over o. The restriction of H to every projective component of this cycle is ample. Its top degree on a nonzero such component is positive. For a rational H, multiply by an integer to make it integral and divide the intersection number by the positive power of that integer. It follows that

$$q_{ij} = E_i E_j H^{n-2} \geq 0.$$

For $n = 2$, this just says that intersections of distinct exceptional curves are sums of nonnegative local intersection multiplicities.

Step 2: a strictly negative weighted row sum. Multilinearity and the expression for H give, for each i,

$$(Qa)_i = \sum_j a_j E_i E_j H^{n-2} = E_i\Big(\sum_j a_j E_j\Big) H^{n-2} = -E_i H^{n-1}.$$

The variety E_i is projective of dimension $n-1$, and $H|_{E_i}$ is ample because $\pi(E_i) = \{o\}$. Therefore $E_i H^{n-1} = (H|_{E_i})^{n-1} > 0$. Positivity of the top degree of an ample bundle follows, for example, by taking a very ample power and computing the degree of its projective embedding; see [Laz04, Chapter I]. Thus $(Qa)_i < 0$ for every i.

Step 3: apply the matrix criterion to the divisor coefficients. For $D = \sum_i x_i E_i$, the value $D^2 H^{n-2}$ equals $x^\mathsf{T} Q x$. The primes are a basis of the space of actual exceptional divisors, so $D \neq 0$ means $x \neq 0$. Lemma 4.1 gives strict negativity, proving the assertion. □

Remark 4.3 (The role of ampleness). The strict inequality $E_i H^{n-1} > 0$ is required for every exceptional prime. Replacing ample by nef may change some of these numbers to zero, and the matrix proof then gives no strict inequality for the corresponding directions. This is why a semidefinite statement alone would not prove separation. In dimension two the resulting assertion is the classical exceptional-curve negativity of [Mum61]. In all dimensions the argument above uses BFJ negativity for the signs of H, followed by the explicit matrix calculation; it does not assume a higher-dimensional strict Hodge theorem as an extra input.

Example 4.4 (Blowing up the origin)**.** On $\mathrm{Bl}_o \mathbb{C}^n$, let $E \simeq \mathbb{P}^{n-1}$ be the exceptional divisor and take $H = -E$. Since $\mathcal{O}(E)|_E = \mathcal{O}_{\mathbb{P}^{n-1}}(-1)$, if h is the hyperplane class then

$$E^2 H^{n-2} = \int_E c_1(\mathcal{O}(E)|_E) c_1(\mathcal{O}(H)|_E)^{n-2} = \int_{\mathbb{P}^{n-1}} (-h) h^{n-2} = -1.$$

Thus the matrix is (-1), in agreement with the signs of the general proof. The ampleness of H does not make the square of an exceptional divisor positive.

5. Polarization of diagonal identities

Lemma 5.1 (Polynomial vanishing on a positive lattice)**.** *If a polynomial P in r real variables vanishes at every point of $\mathbb{N}^r_{>0}$, then $P = 0$.*

Proof. We induct on the number of variables. For $r = 1$, a polynomial of degree d which is not zero has at most d distinct roots, whereas the positive integers give infinitely many. Now suppose the assertion is known for $r-1$ variables and write

$$P(x_1, \dots, x_r) = \sum_{j=0}^{d} P_j(x_1, \dots, x_{r-1}) x_r^j.$$

Fix arbitrary positive integers $k_1, \dots, k_{r-1}$. The resulting polynomial in x_r vanishes at every positive integer. The one-variable case says it is identically zero, so each coefficient $P_j(k_1, \dots, k_{r-1})$ is zero. Since the first $r-1$ integers were arbitrary, each polynomial P_j vanishes on the whole positive lattice in $r-1$ variables. The induction hypothesis gives $P_j = 0$ for every j, and hence $P = 0$. $\square$

Lemma 5.2 (Polarization by products of ideals)**.** *Assume $n \geq 2$. Let W be a real Weil b-divisor, paired with Cartier factors as in proposition 3.6. If*

$$\langle W, Z(\mathfrak{a})^{n-1} \rangle = 0 \tag{29}$$

for every $\mathfrak{m}_o$-primary ideal $\mathfrak{a}$, then

$$\langle W, Z(\mathfrak{a}_1), \dots, Z(\mathfrak{a}_{n-1}) \rangle = 0 \tag{30}$$

for all primary ideals $\mathfrak{a}_i$.

Proof. Set $r = n-1$ and $A_i = Z(\mathfrak{a}_i)$. For $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r_{>0}$, the ideal $\mathfrak{a}(\mathbf{k}) = \mathfrak{a}_1^{k_1} \cdots \mathfrak{a}_r^{k_r}$ is primary and (14) gives $Z(\mathfrak{a}(\mathbf{k})) = \sum_i k_i A_i$. Hence

$$P(\mathbf{k}) := \langle W, (k_1 A_1 + \cdots + k_r A_r)^r \rangle = 0. \tag{31}$$

All the A_i can be determined on one model, since there are finitely many. Formula (20) then shows that the expression is a finite polynomial with real coefficients. In particular, no undefined product of several arbitrary Weil factors occurs. By multilinearity,

$$P(\mathbf{k}) = \sum_{\alpha_1 + \cdots + \alpha_r = r} \frac{r!}{\alpha_1! \cdots \alpha_r!} k_1^{\alpha_1} \cdots k_r^{\alpha_r} \langle W, A_1^{\alpha_1}, \dots, A_r^{\alpha_r} \rangle . \tag{32}$$

By lemma 5.1, this polynomial is identically zero. The coefficient of $k_1 \cdots k_r$ is $r! \langle W, A_1, \dots, A_r \rangle$: each of the $r!$ orderings selects every A_i once and symmetry makes their values equal. Every coefficient of the zero polynomial is zero. Dividing this coefficient by $r! > 0$ proves the desired mixed identity. The case $r = 1$, or $n = 2$, is included: the coefficient is then simply $\langle W, A_1 \rangle$. $\square$

Example 5.3 (Dimension three)**.** For $n = 3$, put $A = Z(\mathfrak{a})$ and $B = Z(\mathfrak{b})$. The diagonal identities for A, B, and $A + B = Z(\mathfrak{a}\mathfrak{b})$ give

$$0 = \langle W, (A+B)^2 \rangle = \langle W, A^2 \rangle + 2\langle W, A, B \rangle + \langle W, B^2 \rangle.$$

The first and last terms vanish, so $\langle W, A, B \rangle = 0$. The general argument is the degree-$(n-1)$ multinomial version of this calculation.

Proposition 5.4 (Extension to all Cartier tests)**.** *Assume $n \geq 2$. Under the hypothesis of lemma 5.2,*

$$\langle W, C_1, \dots, C_{n-1} \rangle = 0 \tag{33}$$

for arbitrary Cartier b-divisors C_i.

Proof. For each i, proposition 3.4 gives a finite expansion

$$C_i = \sum_{j=1}^{N_i} \lambda_{ij} Z(\mathfrak{a}_{ij})$$

with real coefficients, possibly negative. The pairing in proposition 3.6 is multilinear without any sign restriction, so

$$\langle W, C_1, \dots, C_{n-1} \rangle = \sum_{j_1=1}^{N_1} \cdots \sum_{j_{n-1}=1}^{N_{n-1}} \left(\prod_{i=1}^{n-1} \lambda_{i j_i} \right) \cdot \langle W, Z(\mathfrak{a}_{1 j_1}), \dots, Z(\mathfrak{a}_{n-1, j_{n-1}}) \rangle.$$

Every displayed mixed intersection vanishes by lemma 5.2. The sum has finitely many finite terms; there is no convergence issue. This proves the assertion, including for signed Cartier tests. □

6. Ample perturbations and the main theorem

6.1. Recovering a trace from diagonal tests.

Proposition 6.1 (Separation on a fixed model)**.** *Let W be a real Weil b-divisor satisfying*

$$\langle W, Z(\mathfrak{a})^{n-1} \rangle = 0 \quad \textit{for every primary ideal } \mathfrak{a}. \tag{34}$$

Then $W_\pi = 0$ on every model, and consequently $W = 0$.

Proof. Fix a nontrivial model X_π. The identity model has no exceptional trace and requires no argument. Let $\mathcal{A}_\pi$ be the cone of relatively ample exceptional real divisors. By lemma 2.1, this is a nonempty open cone in the finite-dimensional space $\mathrm{Div}(\pi)$. Define the homogeneous polynomial of degree $n-1$

$$F_\pi(A) = W_\pi \cdot A^{n-1}, \qquad A \in \mathrm{Div}(\pi).$$

All coefficients of this polynomial are finite ordinary intersections on the chosen model.

Step 1: vanishing on rational ample divisors. For $A \in \mathcal{A}_\pi$ with rational coefficients, choose $m > 0$ and a primary ideal $\mathfrak{a}$ as in proposition 3.3. Both $\overline{mA}$ and $Z(\mathfrak{a})$ are determined on X_π and are equal. The hypothesis and proposition 3.6 give

$$0 = \langle W, Z(\mathfrak{a})^{n-1} \rangle = W_\pi \cdot (mA)^{n-1} = m^{n-1} F_\pi(A).$$

Since $m^{n-1} > 0$, this proves $F_\pi(A) = 0$.

Step 2: vanishing on all real ample divisors. For any $A \in \mathcal{A}_\pi$, openness gives a ball around its coefficient vector contained in the cone. Approximate that vector by rational vectors inside the ball to get ample rational $A_k \to A$. Since F_π is a polynomial, it is continuous; hence $F_\pi(A) = \lim_k F_\pi(A_k) = 0$. Only continuity on this fixed finite-dimensional space is used.

Step 3: differentiation in an arbitrary divisor direction. Choose a rational $H \in \mathcal{A}_\pi$ and any real $D \in \mathrm{Div}(\pi)$. There is $\varepsilon > 0$ with $H + tD \in \mathcal{A}_\pi$ for $|t| < \varepsilon$; this follows from openness, also when D has mixed signs. Step 2 implies that the polynomial $F_\pi(H+tD)$ is zero on this interval. Its binomial expansion is

$$F_\pi(H + tD) = \sum_{j=0}^{n-1} \binom{n-1}{j} t^j \, W_\pi D^j H^{n-1-j}. \tag{35}$$

Its derivative at $t = 0$ must therefore vanish:

$$0 = (n-1) W_\pi D H^{n-2}. \tag{36}$$

There is no interchange of a derivative and an infinite limit in this step: (35) is a finite sum. The extension from rational to real ample divisors in Step 2 is what allows the direction D itself to have irrational coefficients.

Step 4: test in the direction of the unknown trace. Take $D = W_\pi$ in (36). Since $n - 1 > 0$,

$$W_\pi^2 H^{n-2} = 0. \tag{37}$$

The strict negative definiteness in lemma 4.2 implies $W_\pi = 0$. For $n = 2$, the same calculation reads $W_\pi^2 = 0$; no H factor remains.

The chosen model was arbitrary. Alternatively, it is enough to run the argument on a cofinal family and push forward to every other model. All traces are zero, which is exactly the assertion $W = 0$. □

Remark 6.2 (What is actually differentiated). The derivative above concerns the polynomial $A \mapsto W_\pi A^{n-1}$ on one model. It is not a derivative of the volume of a general filtration, nor of a Monge–Ampère mass along an arbitrary path of psh germs. The integer multiples realizing rational ample divisors as ideals may vary with A. They disappear by homogeneity in Step 1 before the derivative is taken. Thus a differentiable family of ideals is neither constructed nor required.

6.2. **The analytic conclusion.**

Proof of theorem 1.1. The case $n = 1$ was proved at the end of the introduction. We assume $n \geq 2$ and prove each implication.

Step 1: (i) implies (ii). Equality of the Lelong numbers at all infinitely near points implies, by (10), equality of every coefficient of $Z(u)$ and $Z(v)$. Hence the two Weil b-divisors are equal. The associated formal psh functions are then equal on the normalized valuation space by the BFJ correspondence [BFJ08, Proposition 3.11 and Definition 5.2]; that is, $\widehat{u} = \widehat{v}$. For every tame φ, Theorem B gives

$$\nu_\varphi(u) = \int_{\mathcal{V}} -\widehat{u} \; \mathrm{MA}(\widehat{\varphi}) = \int_{\mathcal{V}} -\widehat{v} \; \mathrm{MA}(\widehat{\varphi}) = \nu_\varphi(v).$$

Integrability is part of that theorem, so the equality of integrals is legitimate.

Step 2: (ii) implies (iii). For every primary ideal, the weight $\varphi_{\mathfrak{a}}$ has an isolated pole and a locally Lipschitz exponential. As checked in Section 3, it is tame by [BFJ08, Lemma 5.10]. It is therefore among the tests allowed in (ii), which gives (iii). The choice of generators does not matter by the bounded-change argument.

Step 3: (iii) implies equality of valuative divisors. Set

$$W = Z(u) - Z(v). \tag{38}$$

For each primary $\mathfrak{a}$, choose a common determining model for its divisor. The fixed-model formula (22) gives two finite real numbers, and their difference is

$$\begin{aligned} 0 &= \nu_{\varphi_{\mathfrak{a}}}(u) - \nu_{\varphi_{\mathfrak{a}}}(v) \\ &= -Z(u)_\pi Z(\mathfrak{a})_\pi^{n-1} + Z(v)_\pi Z(\mathfrak{a})_\pi^{n-1} \\ &= -\langle W, Z(\mathfrak{a})^{n-1}\rangle. \end{aligned}$$

These are exactly the hypotheses of proposition 6.1. That proposition gives $W = 0$, hence $Z(u) = Z(v)$.

Step 4: recover condition (i) and the stated consequence. For any infinitely near point p on a smooth model, blow up p and compare the coefficients along its exceptional divisor. Equality of $Z(u)$ and $Z(v)$ and (9) give equality of the ordinary Lelong numbers at p. Thus (i) holds. We have proved the cycle (i)⇒(ii)⇒(iii)⇒(i). Finally, BFJ Theorem A already identifies its conditions (1), (2), and (3), and shows that they imply (4). Our implication (ii)⇒(i) is precisely the missing implication (4)⇒(1) in that formulation, proving the final assertion. □

Remark 6.3 (The polarization proof gives the same separation). There is also a direct completion using Section 5. From the same diagonal identities, proposition 5.4 gives $\langle W, C_1, \ldots, C_{n-1}\rangle = 0$ for all Cartier tests. On a fixed model choose $C_1 = \overline{W_\pi}$ and take the remaining $n-2$ factors to be $\overline{H}$. Formula (20) then gives

$$\langle W, \overline{W_\pi}, \overline{H}^{\,n-2}\rangle = W_\pi^2 H^{n-2} = 0.$$

Lemma 4.2 again forces $W_\pi = 0$. For $n = 2$, the original diagonal identity is already linear and no polarization is necessary. The two proofs differ only in how they obtain the mixed test: polarization expands products of ideals, whereas the main proof differentiates a polynomial on the ample cone.

Remark 6.4 (Inputs and limitations). The local positivity input is [BFJ08, Propositions 2.2 and 2.4]; the analytic input is its Theorem B together with nefness of $Z(u)$. The finite-atomic formula in its Proposition 4.9 turns the ideal tests into fixed-model intersections. Relative generation of ample multiples then supplies an open cone of tests. The strictness needed for separation is proved in Section 4 by an explicit matrix identity. Neither the existence of a fundamental solution for the formal Monge–Ampère operator nor a surface-slicing assertion is assumed. The conclusion is equality of valuative transforms; it does not claim $u - v = O(1)$, or equality of generalized Lelong numbers for arbitrary non-tame weights.

References

[BdFF12] S. Boucksom, T. de Fernex, and C. Favre, *The volume of an isolated singularity*, Duke Math. J. **161** (2012), no. 8, 1455–1520.

[BFJ08] S. Boucksom, C. Favre, and M. Jonsson, *Valuations and plurisubharmonic singularities*, Publ. Res. Inst. Math. Sci. **44** (2008), no. 2, 449–494.

[Dem87] J.-P. Demailly, *Nombres de Lelong généralisés, théorèmes d'intégralité et d'analyticité*, Acta Math. **159** (1987), no. 3–4, 153–169.

[DF22] N.-B. Dang and C. Favre, *Intersection theory of nef b-divisor classes*, Compos. Math. **158** (2022), no. 7, 1563–1594.

[FJ05] C. Favre and M. Jonsson, *Valuative analysis of planar plurisubharmonic functions*, Invent. Math. **162** (2005), no. 2, 271–311.

[Ful98] W. Fulton, *Intersection theory*, second ed., Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 2, Springer-Verlag, Berlin, 1998.

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer-Verlag, New York–Heidelberg, 1977.

[GZ15] Q. Guan and X. Zhou, *A proof of Demailly's strong openness conjecture*, Ann. of Math. (2) **182** (2015), no. 2, 605–616.

[Laz04] R. Lazarsfeld, *Positivity in algebraic geometry I*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 48, Springer-Verlag, Berlin, 2004.

[Mum61] D. Mumford, *The topology of normal singularities of an algebraic surface and a criterion for simplicity*, Publ. Math. Inst. Hautes Ètudes Sci. **9** (1961), 5–22.

[Ram74] C. P. Ramanujam, *On a geometric interpretation of multiplicity*, Invent. Math. **22** (1973/74), 63–67.

[Ree84] D. Rees, *Generalizations of reductions and mixed multiplicities*, J. London Math. Soc. (2) **29** (1984), no. 3, 397–414.

[Sho03] V. V. Shokurov, *Prelimiting flips*, Proc. Steklov Inst. Math. **240** (2003), no. 1, 75–213.

Declaration on the use of artificial intelligence

The mathematical ideas, research directions, arguments, proof details, and computations in this article originated in AI-assisted work, not in ideas proposed by the author. The original manuscript was generated by GPT-5.6 Sol (high). A subsequent revision with GPT (Astra) re-examined the arguments, corrected errors, expanded the proofs, and supplied the fixed-model ample-perturbation and intersection-matrix presentation used here. The author takes responsibility for the mathematical content. AI-assisted checking is not a substitute for independent expert review.

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