Counterexamples from Green sublevel families for higher order p-Bergman kernels

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Abstract

We construct explicit Green sublevel families that give three counterexamples for higher order and p-Bergman kernels. First, the logarithm of a higher order Bergman kernel need not be convex along the scaled Green sublevel sets, answering a question of Błocki–Zwonek in the negative. Second, the monotonicity of higher order p-Bergman kernels along those families fails for every $p > 2$, although it is known for $1 \le p \le 2$. Third, the natural L^p analogue of Guan's concavity theorem for minimal L^2 integrals fails for every $p > 2$. The constructions are based on the domains

$$D_q^{(r)} := \{x \in \mathbb{C} : \ |x|^r < |1 - qx^r|\},$$

whose Green functions, sublevel sets, and relevant extremal problems admit explicit reductions to weighted spaces on the unit disk.

Contents

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1 Introduction

Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $w \in D$. The Bergman kernel of D at w is defined by

$$K_D(w) := \sup\Big\{ |f(w)|^2 : \ f \in \mathcal{O}(D), \ \|f\|_D := \Big(\int_D |f|^2 \, d\lambda \Big)^{1/2} \leq 1 \Big\},$$

where $d\lambda$ denotes the Lebesgue measure. Following [4], for a homogeneous polynomial H of degree k, $H(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$, we write $P_H(f) := \sum_{|\alpha|=k} a_\alpha D^\alpha f$, and we define the higher order Bergman kernel

$$K_D^H(z) := \sup\Big\{ |P_H(f)(z)|^2 : \ f \in \mathcal{O}(D), \ f^{(j)}(z) = 0 \ (0 \leq j < k), \ \|f\|_D \leq 1 \Big\}.$$

In dimension one, $H(z) = z^k$ and $K_D^{z^k}(z)$ will also be denoted by $K_D^{(k)}(z)$.

Let $G_D(\cdot, w)$ be the (pluricomplex) Green function of D with pole at w, and assume $w = 0$. For $a \leq 0$ we denote

$$D_a := e^{-a}\big\{ G_D(\cdot, 0) < a \big\}.$$

The main theorem of [4] asserts that $a \mapsto K_{D_a}^H(0)$ is non-decreasing, and it is noted that

$$K_{D_a}^H(0) = e^{2(n+k)a} K_{\{G_D(\cdot,0)<a\}}^H(0).$$

1.1 The convexity question of Błocki–Zwonek

The main theorem of [4] also motivates the following question raised in [4, Remark 3].

Question 1.1 (Błocki–Zwonek, [4, Remark 3]). *Is the function*

$$(-\infty, 0] \ni a \longmapsto \log K_{D_a}^H(w)$$

convex, as it is in the case $H \equiv 1$?

The convexity in the case $H \equiv 1$ follows from Berndtsson's theorem on the plurisubharmonic variation of Bergman kernels (see the final remark in [2]). Our first result shows that Question 1.1 has a negative answer in general.

THEOREM 1.2. *For $Q > 1$ sufficiently large set $c := 2Q$ and*

$$D_c := \big\{ z \in \mathbb{C} : \ |z|^2 < |1 - cz^2| \big\}.$$

Then D_c is a domain in $\mathbb{C}$ (indeed $\mathbb{C}$ minus two disjoint Jordan regions, hence unbounded and doubly connected), $0 \in D_c$, and with $H(z) = z^6$ the function

$$(-\infty, 0] \ni a \longmapsto \log K_{(D_c)_a}^{z^6}(0)$$

is <u>*not*</u> *convex. More precisely, for*

$$a_- := -\log 2, \qquad a_0 := -\tfrac{1}{2}\log 2, \qquad a_+ := 0,$$

so that $a_0 = (a_- + a_+)/2$*, we have*

$$\log K^{z^6}_{(D_c)_{a_0}}(0) > \frac{1}{2}\Big(\log K^{z^6}_{(D_c)_{a_-}}(0) + \log K^{z^6}_{(D_c)_{a_+}}(0)\Big),$$

and the difference is

$$\frac{3}{64Q^2} + O(Q^{-4}) \qquad (Q \to \infty).$$

The proof of Theorem 1.2 is divided into three steps, each of independent interest:

1. an explicit formula for the Green function of D_c and the observation that the scaled domains $(D_c)_a$ are again of the same form (Subsection 2.1);
2. an exact reduction of the extremal problem defining $K^{z^6}_{D_q}(0)$ to the third Taylor coefficient functional on a weighted Bergman space of the unit disk (Subsection 2.2);
3. a rigorous asymptotic expansion of the associated infinite Gram matrix, valid up to order q^{-4}, which yields the precise size of the convexity defect (Subsection 2.3).

1.2 Monotonicity fails for $p > 2$

Recall that for $p > 0$, a homogeneous polynomial H of degree k, and a domain $D \subset \mathbb{C}^n$ containing the origin, the higher order p-Bergman kernel we define the higher order p-Bergman kernel by

$$K^{H,p}_D(0) := \sup\Big\{|P_H(f)(0)|^p : \ f \in \mathcal{O}(D), \ f^{(j)}(0) = 0 \ (0 \le j < k), \ \|f\|_{L^p(D)} \le 1\Big\}.$$

We also recall that $D_a := e^{-a}\{G_D(\cdot,0) < a\}$ for $a \le 0$. The main theorem of [4] asserts that, for $p = 2$,

$$(-\infty, 0] \ni a \longmapsto K^{H,p}_{D_a}(0) \quad \text{is non-decreasing.}$$

In [7] this was extended to all $p \in [1,2]$: by [7, Theorem 1.7] one has $K^{H,p}_D(z) = \inf_{\xi \in S_H} K_{\xi,D,p}(z)$ for $p \ge 1$, while by [7, Theorem 1.8] the functions

$$(-\infty, 0] \ni a \longmapsto \log K_{\xi,\{G_D(\cdot,0)<a\},p}(0)$$

are convex and the functions

$$(-\infty, 0] \ni a \longmapsto e^{(2n+pk)a} K_{\xi,\{G_D(\cdot,0)<a\},p}(0)$$

are non-decreasing for $p \in (0,2]$; combining the two results, [7, Corollary 1.10] yields the monotonicity of $a \mapsto K^{H,p}_{D_a}(0)$ for every $p \in [1,2]$. Since the two ingredients have complementary ranges ($p \ge 1$ for the inf–min identity and $p \le 2$ for the convexity of the p-Bergman kernels with respect to ξ), it is natural to ask whether the monotonicity persists in the two remaining regimes $p \in (0,1)$ and $p \in (2, +\infty)$.

This construction also answers this question in the negative for every $p > 2$. The simplest instance, treated in detail below, uses again the domain D_c and the polynomial $H(z) = z^6$ of Theorem 1.2 and covers $p > 3$; the full range $p > 2$ is then reached through the r-fold symmetric families of Theorem 1.4.

THEOREM 1.3. *Let $p > 3$ and $H(z) = z^6$. Then there is a constant $c > 1$ such that, with $D_{1/c} = (D_c)_{-\log c}$,*

$$K^{z^6,p}_{D_{1/c}}(0) > K^{z^6,p}_{D_c}(0).$$

Consequently the function

$$(-\infty, 0] \ni a \longmapsto K^{z^6,p}_{(D_c)_a}(0)$$

is <u>not</u> non-decreasing. More precisely, there are constants $0 < c_p < C_p < +\infty$, depending only on p, such that for every $q \geq 8$,

$$c_p\, q^{-(p-3)} \;\leq\; K^{z^6,p}_{D_q}(0) \;\leq\; C_p\, q^{-(p-3)},$$

while

$$\lim_{q\to 0^+} K^{z^6,p}_{D_q}(0) = \frac{(6!)^p(6p+2)}{2\pi} = K^{z^6,p}_{\mathbb{D}}(0) = K^{z^6,p}_{I_{D_c}(0)}(0),$$

where $\mathbb{D}$ is the unit disk and $I_{D_c}(0)$ is the Azukawa indicatrix of D_c at 0.

THEOREM 1.4. *For every $p > 2$ there is an integer $r > 2/(p-2)$ with the following property. For every integer $m \geq 1$, with $H(z) = z^{rm}$ and*

$$D^{(r)}_q := \big\{x \in \mathbb{C} : \ |x|^r < |1 - qx^r|\big\},$$

there are constants $0 < c_{p,r,m} < C_{p,r,m} < +\infty$, depending only on p, r, m, such that for every $q \geq 8$,

$$c_{p,r,m}\, q^{-(p-2-2/r)} \;\leq\; K^{z^{rm},p}_{D^{(r)}_q}(0) \;\leq\; C_{p,r,m}\, q^{-(p-2-2/r)},$$

while

$$\lim_{q\to 0^+} K^{z^{rm},p}_{D^{(r)}_q}(0) = \frac{(rm)!^p(rmp+2)}{2\pi}.$$

Consequently, for every sufficiently large $c > 1$, with $D^{(r)}_{1/c} = (D^{(r)}_c)_{-\frac{2}{r}\log c}$, one has

$$K^{z^{rm},p}_{D^{(r)}_{1/c}}(0) > K^{z^{rm},p}_{D^{(r)}_c}(0),$$

and the function $a \mapsto K^{z^{rm},p}_{(D^{(r)}_c)_a}(0)$ is not non-decreasing.

Thus the monotonicity part of the Błocki–Zwonek and Bao–Guan–Sun results has no analogue for any $p > 2$. For the family D_q of Theorem 1.3 the borderline exponent q^{3-p} flips sign exactly at $p = 3$; the r-fold families of Theorem 1.4 push this threshold arbitrarily close to 2 by taking $r > 2/(p-2)$ large. The limiting comparison with the Azukawa indicatrix, however, still holds with equality for all these families. The regime $p \in (0, 1)$ remains open.

1.3 The L^p concavity of minimal integrals fails for $p > 2$

Following [5], let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing the origin o, let I be an ideal of $\mathcal{O}_o$, and let $(F, o) \in \mathcal{O}_o$. The <u>minimal L^2 integral</u> of D is

$$C_{F,I}(D) := \inf\Big\{\int_D |\tilde{F}|^2 : \ (\tilde{F} - F, o) \in I, \ \tilde{F} \in \mathcal{O}(D)\Big\}.$$

Guan [5] established a concavity property of the minimal L^2 integrals on the sublevel sets of a plurisubharmonic function: if φ is a negative plurisubharmonic function on D with $\varphi(o) = -\infty$, and

$$G(t) := C_{F,\mathcal{I}(\varphi)_o}\big(\{\varphi < -t\} \cap D\big), \qquad t \geq 0,$$

where $\mathcal{I}(\varphi)_o$ denotes the multiplier ideal of φ at o (the ideal of germs $f \in \mathcal{O}_o$ with $|f|^2 e^{-\varphi}$ locally integrable near o), then $G(0) < +\infty$ implies that $G(-\log r)$ is concave in $r \in (0, 1]$. This concavity has been a basic tool in the characterization of the equality cases of the optimal L^2 extension theorem and related sharp extension problems.

The monotonicity results of [4] and [7] for the higher order p-Bergman kernels along the Green sublevel sets, recalled in the previous subsection, are the kernel-side counterparts of this structural property. In their spirit, it is natural to ask whether the L^p analogue of Guan's concavity holds: for $p > 0$ define the minimal L^p integral

$$C_{F,I,p}(D) := \inf\Big\{\int_D |\tilde{F}|^p : \ (\tilde{F} - F, o) \in I, \ \tilde{F} \in \mathcal{O}(D)\Big\},$$

set

$$G_p(t) := C_{F,\mathcal{I}(\varphi)_o,p}\big(\{\varphi < -t\} \cap D\big), \qquad t \geq 0,$$

and ask whether $G_p(0) < +\infty$ implies that $G_p(-\log r)$ is concave in $r \in (0, 1]$. For $p = 2$ this is exactly Guan's theorem.

This construction also answers this question in the negative for every $p > 2$: the r-fold families $D_q^{(r)}$ and the polynomial $H(z) = z^{rm}$ of Theorem 1.4 are transplanted into the minimal L^p integral setting through the multiplier ideal of the Green function.

THEOREM 1.5. *Let $p > 2$, let $r \geq 1$ and $m \geq 1$ be integers, and set $k := rm$. Fix $c \in (0, 1)$, put*

$$D := D_c^{(r)} = \big\{x \in \mathbb{C} : \ |x|^r < |1 - cx^r|\big\},$$

which is a bounded pseudoconvex domain containing 0, and define

$$\varphi := 2(k + 1)\, G_D(\cdot, 0), \qquad F(x) := \frac{x^k}{k!},$$

where $G_D(\cdot, 0)$ is the Green function of D with pole at 0. Then $G_p(0) < +\infty$ and, as $\rho \to 0^+$,

$$G_p(-\log \rho) \sim \frac{2\pi}{(k!)^p(kp + 2)}\, \rho^{\mu}, \qquad \mu := \frac{pk + 2}{2(k + 1)} = 1 + \frac{k(p - 2)}{2(k + 1)} > 1.$$

Consequently $G_p(-\log \rho)$ is not concave in $\rho \in (0, 1]$, and the L^p analogue of Guan's concavity theorem fails for every $p > 2$.

The exponent μ makes the threshold $p = 2$ visible: for $p = 2$ one has $\mu = 1$, and the asymptotic is the linear growth compatible with Guan's concavity; for $0 < p < 2$ one has $\mu < 1$, so the same construction produces no contradiction, and the question remains open in that range. The proof is given in Section 4.

2 Non-convexity of the logarithm of a higher order Bergman kernel

The proof of Theorem 1.2 is divided into three steps, which are treated in the next three subsubsections.

2.1 The Green function of D_c and the scaling law

Throughout this subsubsection we write $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

PROPOSITION 2.1. *Let $c > 0$ and*

$$F_c(z) := \frac{z^2}{1 - cz^2}.$$

Then the domain $D_c = \{|F_c| < 1\}$ satisfies $0 \in D_c$, and the mapping

$$F_c : D_c \longrightarrow \mathbb{D} \setminus \{-1/c\}$$

is proper of degree 2, vanishing to order 2 at 0. Consequently the (negative) pluricomplex Green function of D_c with pole at 0 is

$$G_{D_c}(z, 0) = \frac{1}{2} \log |F_c(z)| = \log |z| - \frac{1}{2} \log |1 - cz^2|.$$

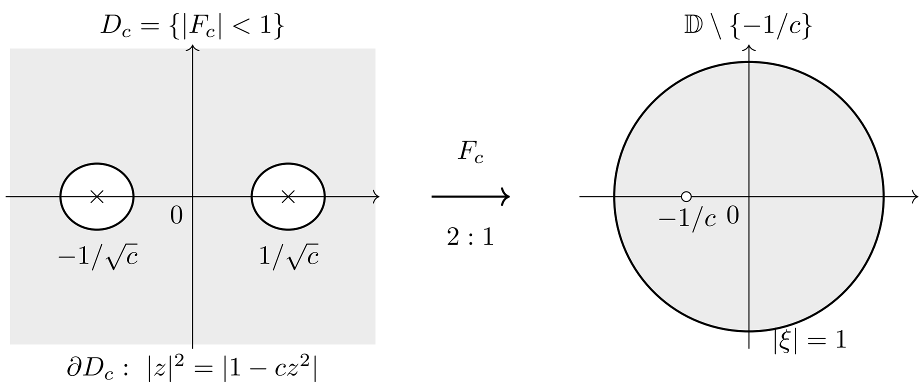

Figure 1: The domain $D_c = \{z \in \mathbb{C} : |z|^2 < |1-cz^2|\}$ in the z-plane (shaded; the two complementary components surround the poles $\pm 1/\sqrt{c}$ and the boundary is $|z|^2 = |1-cz^2|$) and its image $F_c(D_c) = \mathbb{D} \setminus \{-1/c\}$ in the ξ-plane. The map F_c is proper of degree 2 with $F_c(0) = 0$ (a double zero) and $F_c(\infty) = -1/c$.

Proof. The identity $D_c = \{|F_c| < 1\}$ is immediate from the definitions. The rational function F_c is holomorphic on D_c: the poles $\pm c^{-1/2}$ are not contained in D_c, since $|F_c| \to +\infty$ near them. The equation $F_c(z) = \xi$ reads $z^2 = \xi(1 - cz^2)$, i.e. $z^2 = \xi/(1 + c\xi)$, which has two solutions for every $\xi \notin \{-1/c, \infty\}$, so F_c is proper of degree 2 and $F_c^{-1}(0) = \{0\}$, with multiplicity 2 at 0.

The last multiplicity statement is essential here: the pullback formula for a general proper map would contain a pole at every point of $F_c^{-1}(0)$. In the present situation

$$u(z) := \frac{1}{2} G_{\mathbb{D}\setminus\{-1/c\}}(F_c(z), 0)$$

is negative harmonic on $D_c \setminus \{0\}$, vanishes at the regular boundary, and satisfies $u(z) = \log |z| + O(1)$ at 0. By uniqueness it is $G_{D_c}(z, 0)$. Removing one point does not change a planar Green function, because a singleton is polar; see, for example, [8, Chapters 4–5]. Thus $G_{\mathbb{D}\setminus\{-1/c\}}(\xi, 0) = \log |\xi|$, and the displayed formula follows. ∎

For $a \le 0$ we use the notation of [4]:

$$(D_c)_a := e^{-a}\big\{G_{D_c}(\cdot,0) < a\big\}.$$

Proposition 2.2. *For $a \le 0$ and $q := ce^{2a}$ we have*

$$(D_c)_a = D_q := \big\{x \in \mathbb{C} :\ |x|^2 < |1 - qx^2|\big\}.$$

Proof. By Proposition 2.1, $G_{D_c}(z,0) < a$ is equivalent to $|F_c(z)| < e^{2a}$. Writing $z = e^a x$, we have

$$F_c(e^a x) = e^{2a}\frac{x^2}{1 - ce^{2a}x^2} = e^{2a}F_q(x),$$

with $q = ce^{2a}$. Hence $|F_c(e^a x)| < e^{2a}$ is equivalent to $|F_q(x)| < 1$, i.e. $x \in D_q$. Since $(D_c)_a = \{x : e^a x \in \{G_{D_c}(\cdot,0) < a\}\}$, the claim follows. ■

Since $\log q = \log c + 2a$ is affine in a, the convexity of $a \mapsto \log K^{z^6}_{(D_c)_a}(0)$ is equivalent to the convexity of $s \mapsto \log K^{z^6}_{D_{e^s}}(0)$ in $s = \log q$. We shall therefore study the family D_q, $q > 1$.

2.2 Reduction to a weighted Bergman kernel on the unit disk

For $q > 1$ we abbreviate $K_q := K^{z^6}_{D_q}(0)$. Since D_q is invariant under $x \mapsto -x$, the sixth derivative at 0 only sees the even part of a holomorphic function, and we may restrict to

$$f(x) = g(x^2), \qquad g \in \mathcal{O}(U_q),$$

where

$$U_q := \big\{y \in \mathbb{C} :\ |y| < |1 - qy|\big\} = \mathbb{C} \setminus \overline{B\Big(\frac{q}{q^2-1}, \frac{1}{q^2-1}\Big)}$$

is the image of D_q under the two-to-one map $y = x^2$.

We record the elementary facts used below.

Lemma 2.3. *Let $p \ge 1$, $\gamma \ge 2$, and let Φ be holomorphic on a punctured disk $0 < |z| < \delta$. If*

$$\int_{|z|<\delta} |\Phi(z)|^p |z|^{-\gamma}\, dA(z) < +\infty,$$

then Φ extends holomorphically across 0 and

$$\operatorname{ord}_0\Phi > \frac{\gamma - 2}{p}.$$

Proof. Put

$$M_p(s) := \frac{1}{2\pi}\int_0^{2\pi} |\Phi(se^{i\theta})|^p\, d\theta.$$

The hypothesis says that $\int_0^\delta M_p(s)s^{1-\gamma}ds < +\infty$. Consequently there is a sequence $s_\nu \to 0$ such that $s_\nu^{2-\gamma}M_p(s_\nu) \to 0$. If $\Phi(z) = \sum_{j\in\mathbb{Z}} a_j z^j$ is its Laurent expansion, Cauchy's formula and Hölder's inequality give, for every $n \ge 1$,

$$|a_{-n}| \le s_\nu^n M_p(s_\nu)^{1/p} = \big(s_\nu^{2-\gamma}M_p(s_\nu)\big)^{1/p} s_\nu^{n+(\gamma-2)/p} \longrightarrow 0.$$

Hence all negative Laurent coefficients vanish. If $m = \operatorname{ord}_0\Phi$, then the integrand is comparable to $|z|^{pm-\gamma}$ near 0, and its area integral is finite precisely when $pm + 2 - \gamma > 0$. ■

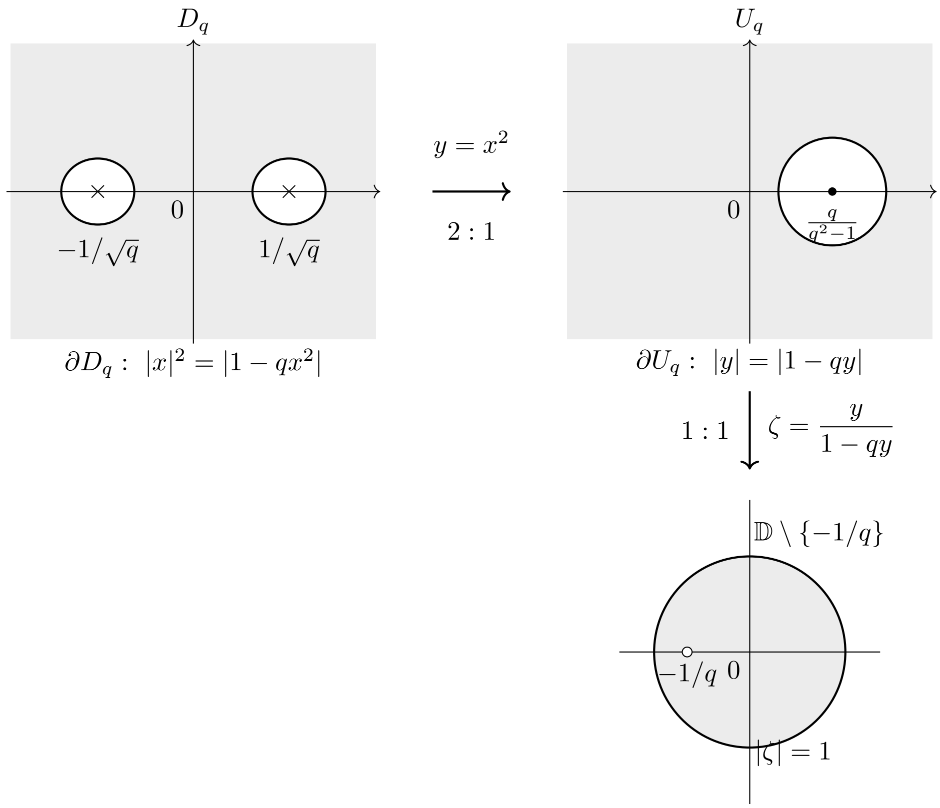

Figure 2: The domain $D_q = \{x \in \mathbb{C} : |x|^2 < |1 - qx^2|\}$ in the x-plane (shaded; the two complementary components surround the poles $\pm 1/\sqrt{q}$), its image $U_q = \{y \in \mathbb{C} : |y| < |1 - qy|\} = \mathbb{C} \setminus \overline{B\big(q/(q^2-1), 1/(q^2-1)\big)}$ in the y-plane under the two-to-one map $y = x^2$, and the unit disk $\mathbb{D} \setminus \{-1/q\}$ in the ζ-plane obtained via the Möbius map $\zeta = y/(1 - qy)$ $(1:1)$.

Lemma 2.4. *With the notation above, the following hold.*

(i) For $f(x) = g(x^2)$,

$$\|f\|^2_{L^2(D_q)} = \frac{1}{2}\int_{U_q} |g(y)|^2 \, \frac{dA(y)}{|y|}.$$

(ii) The conditions $f(0) = f'(0) = \cdots = f^{(5)}(0) = 0$ *and* $f^{(6)}(0) = 1$ *are equivalent to*

$$g(y) = b_3 y^3 + O(y^4) \quad (y \to 0), \qquad b_3 = \frac{1}{6!}.$$

(iii) Under the Möbius transformation

$$\zeta = \frac{y}{1 - qy}, \qquad y = \frac{\zeta}{1 + q\zeta},$$

which maps U_q *biholomorphically onto* $\mathbb{D} \setminus \{-1/q\}$, *the square integrability of* $|g|^2/|y|$ *forces* $g(y(\zeta))$ *to extend across and vanish at* $\zeta = -1/q$. *Hence there exists* $h \in \mathcal{O}(\mathbb{D})$ *with*

$$g(y(\zeta)) = (1 + q\zeta)h(\zeta),$$

and

$$\|f\|^2_{L^2(D_q)} = \frac{1}{2}\int_{\mathbb{D}} |h(\zeta)|^2 \, \frac{dA(\zeta)}{|\zeta|\,|1 + q\zeta|}.$$

(iv) In the variable ζ, *the normalization* $b_3 = 1/6!$ *becomes*

$$h(\zeta) = \frac{1}{6!}\zeta^3 + O(\zeta^4).$$

Proof. (i) The map $y = x^2$ is two-to-one from D_q onto U_q, and $dA(y) = 4|x|^2 dA(x) = 4|y|\, dA(x)$, whence $dA(x) = dA(y)/(4|y|)$.

(ii) Writing $g(y) = \sum_{m\ge 0} g_m y^m$, we have $f(x) = \sum_m g_m x^{2m}$; the first six derivatives vanish at 0 iff $g_0 = g_1 = g_2 = 0$, and $f^{(6)}(0) = 6!\, g_3$.

(iii) The Möbius map is a biholomorphism between U_q and $\mathbb{D} \setminus \{-1/q\}$ because $y = \zeta/(1 + q\zeta)$ and

$$1 - qy = \frac{1}{1 + q\zeta}, \qquad |y| < |1 - qy| \iff |\zeta| < 1.$$

Near $\zeta_0 := -1/q$ one has $y \sim -1/(q^2(\zeta - \zeta_0)) \to \infty$. The function $\Phi(\zeta) := g(y(\zeta))$ is therefore initially holomorphic only on a punctured neighborhood of ζ_0. Before any factorization, the change of variables gives

$$\frac{1}{2}\int |g(y)|^2 \frac{dA(y)}{|y|} = \frac{1}{2}\int \frac{|\Phi(\zeta)|^2}{|\zeta|\,|1 + q\zeta|^3}\, dA(\zeta).$$

Since $|\zeta|$ is bounded above and below near ζ_0, Lemma 2.3 with $(p, \gamma) = (2, 3)$ shows that Φ extends across ζ_0 and vanishes there. Thus $h = \Phi/(1 + q\zeta)$ is holomorphic on $\mathbb{D}$. The change of variables $y = \zeta/(1 + q\zeta)$ has Jacobian $dA(y) = |1 + q\zeta|^{-4} dA(\zeta)$, and $|y|^{-1} = |1 + q\zeta|/|\zeta|$, so

$$\frac{1}{2}\int_{U_q} |g|^2 \frac{dA(y)}{|y|} = \frac{1}{2}\int_{\mathbb{D}} |h|^2 \frac{dA(\zeta)}{|\zeta|\,|1 + q\zeta|}.$$

(iv) From $y = \zeta + O(\zeta^2)$ we get $g(y(\zeta)) = g_3\zeta^3 + O(\zeta^4)$; dividing by $(1 + q\zeta)$ does not change the leading term. ■

We now introduce the infinite Gram matrix. For $i, j \geq 3$ set

$$M_{ij}(q) := \int_{\mathbb{D}} \frac{\zeta^i \bar{\zeta}^j}{|\zeta|\,|1+q\zeta|}\, dA(\zeta),$$

and let $K_3(q) := \big(M(q)^{-1}\big)_{33}$ be the $(3,3)$-entry of the inverse of the infinite matrix $M(q)$ (which is positive definite on the space of holomorphic functions h with $\int |h|^2/(|\zeta||1+q\zeta|)\, dA < \infty$).

Proposition 2.5. *For every $q > 1$,*

$$K_q = K^{z^6}_{D_q}(0) = 2(6!)^2\, K_3(q).$$

Proof. By definition,

$$K_q^{-1} = \min\Big\{ \|f\|^2_{L^2(D_q)} :\ f \in \mathcal{O}(D_q),\ f^{(j)}(0) = 0\ (0 \leq j < 6),\ f^{(6)}(0) = 1 \Big\}.$$

By Lemma 2.4 this equals

$$\min\Big\{ \frac{1}{2} \int_{\mathbb{D}} |h|^2 \frac{dA}{|\zeta|\,|1+q\zeta|} :\ h \in \mathcal{O}(\mathbb{D}),\ h(\zeta) = \frac{1}{6!}\zeta^3 + O(\zeta^4) \Big\}.$$

Writing $h(\zeta) = \frac{1}{6!}\zeta^3 + \sum_{m \geq 4} c_m \zeta^m$, the minimum over the higher coefficients is

$$\frac{1}{2(6!)^2 \big(M(q)^{-1}\big)_{33}},$$

and taking the reciprocal gives the claim. ■

2.3 Asymptotics of the Gram matrix

Set $\varepsilon := 1/q$ and $A(\varepsilon) := qM(q)$, so that

$$A_{ij}(\varepsilon) = \int_{\mathbb{D}} \frac{\zeta^i \bar{\zeta}^j}{|\zeta|\,|\zeta+\varepsilon|}\, dA(\zeta), \qquad i, j \geq 3.$$

We write $D = \operatorname{diag}(\pi/3, \pi/4, \pi/5, \dots)$, and for the off-diagonal terms we use the convention that $B_{i,i+1} = B_{i+1,i}$ for $i \geq 3$.

Proposition 2.6. *The matrix function $A(\varepsilon)$ admits the expansion*

$$A(\varepsilon) = D + \varepsilon B + \varepsilon^2 C + \varepsilon^3 E + O(\varepsilon^4)$$

in the sense of bounded quadratic forms on the Hilbert space $\{h = \zeta^3 v :\ v \in L^2_h(\mathbb{D})\}$ with norm $\int |h|^2 |\zeta|^{-2} dA$. The nonzero leading terms are

$$D_{ii} = \frac{\pi}{i}, \qquad B_{i,i+1} = B_{i+1,i} = -\frac{\pi}{2i}, \qquad C_{ii} = \frac{\pi}{4(i-1)}, \qquad i \geq 3.$$

Consequently, for $\varepsilon > 0$ sufficiently small,

$$\big(A(\varepsilon)^{-1}\big)_{33} = \frac{3}{\pi} - \frac{1}{8\pi}\varepsilon^2 + O(\varepsilon^4).$$

Proof. Write $h(\zeta) = \zeta^3 v(\zeta)$ and denote the Hilbert norm in the statement by $\|h\|_{\mathcal{H}}$. We first record the elementary estimates

$$\int_{\mathbb{D}} |v|^2 dA \le 3 \int_{\mathbb{D}} |v|^2 |\zeta|^4 dA = 3\|h\|_{\mathcal{H}}^2, \qquad \sup_{|\zeta|\le 1/2} |v(\zeta)|^2 \le C\|h\|_{\mathcal{H}}^2. \tag{2.1}$$

Indeed, both follow by writing $v = \sum_{j\ge 0} v_j \zeta^j$ and using

$$\int_{\mathbb{D}} |v|^2 dA = \pi \sum_{j\ge 0} \frac{|v_j|^2}{j+1}, \qquad \|h\|_{\mathcal{H}}^2 = \pi \sum_{j\ge 0} \frac{|v_j|^2}{j+3},$$

together with Cauchy–Schwarz for the supremum estimate.

On the outer region $|\zeta| \ge 2\varepsilon$, Taylor's formula gives, through third order,

$$\begin{aligned}\frac{1}{|\zeta+\varepsilon|} &= \frac{1}{|\zeta|}\Big(1 + \frac{2\varepsilon \operatorname{Re}\zeta}{|\zeta|^2} + \frac{\varepsilon^2}{|\zeta|^2}\Big)^{-1/2} \\ &= \frac{1}{|\zeta|} - \frac{\varepsilon \operatorname{Re}\zeta}{|\zeta|^3} + \varepsilon^2 \frac{3(\operatorname{Re}\zeta)^2 - |\zeta|^2}{2|\zeta|^5} + \varepsilon^3 R_3(\zeta) + O\Big(\frac{\varepsilon^4}{|\zeta|^5}\Big),\end{aligned}$$

where $|R_3(\zeta)| \le C|\zeta|^{-4}$. After multiplication by the additional factor $|\zeta|^{-1}$ in $A(\varepsilon)$, the fourth-order remainder is bounded by $C\varepsilon^4|\zeta|^{-6}$. Hence (2.1) gives the uniform quadratic-form estimate

$$C\varepsilon^4 \int_{\mathbb{D}} |v|^2\, dA \le C'\varepsilon^4 \|h\|_{\mathcal{H}}^2.$$

It remains to justify that the inner disk does not alter these coefficients. On $|\zeta| < 2\varepsilon$, the substitution $\zeta = \varepsilon s$ and the second estimate in (2.1) give

$$\int_{|\zeta|<2\varepsilon} \frac{|h(\zeta)|^2}{|\zeta|\,|\zeta+\varepsilon|}\, dA(\zeta) \le C\varepsilon^6 \|h\|_{\mathcal{H}}^2 \int_{|s|<2} \frac{|s|^5}{|s+1|}\, dA(s).$$

Each of the Taylor terms of orders $0,1,2,3$, integrated over the same disk, is likewise $O(\varepsilon^6)\|h\|_{\mathcal{H}}^2$. This proves the asserted expansion in bounded quadratic-form norm, with a bounded third-order form E whose entries are not needed below.

The angular Fourier integrals give

$$\int_0^{2\pi} e^{id\theta}\, d\theta = 2\pi\delta_{d0}, \qquad \int_0^{2\pi} \cos\theta\, e^{i\theta} d\theta = \pi, \qquad \int_0^{2\pi} \big(3\cos^2\theta - 1\big) d\theta = \pi,$$

which yield, after the radial integration,

$$D_{ii} = \int_0^1 \rho^{2i} \frac{2\pi}{\rho}\, d\rho = \frac{\pi}{i},$$

$$B_{i,i+1} = -\int_0^1 \rho^{2i+1} \frac{\pi}{\rho^2}\, d\rho = -\frac{\pi}{2i},$$

$$C_{ii} = \int_0^1 \rho^{2i} \frac{\pi}{2\rho^3}\, d\rho = \frac{\pi}{4(i-1)}.$$

The first-order off-diagonal entries of B with $|i-j| \neq 1$ vanish by angular orthogonality.

The form D is exactly $\|\cdot\|_{\mathcal{H}}^2$, so it is coercive on $\mathcal{H}$. The bounded-form expansion therefore implies, for small $|\varepsilon|$, a convergent Neumann expansion (see, for example, [6, Chapter IV])

$$A^{-1} = D^{-1} - \varepsilon D^{-1}BD^{-1} + \varepsilon^2\big(D^{-1}BD^{-1}BD^{-1} - D^{-1}CD^{-1}\big) + \varepsilon^3 T + O(\varepsilon^4)$$

for some bounded operator T. Since D is diagonal, the second-order contribution to the $(3,3)$-entry receives only the diagonal term C_{33} and the unique off-diagonal path $3 \to 4 \to 3$, so that

$$\begin{aligned}
\big(A^{-1}\big)_{33} &= \frac{3}{\pi} + \varepsilon^2\Big(\frac{B_{34}^2}{D_{33}^2 D_{44}} - \frac{C_{33}}{D_{33}^2}\Big) + O(\varepsilon^4)\\
&= \frac{3}{\pi} + \varepsilon^2\Big(\frac{(\pi/6)^2}{(\pi/3)^2(\pi/4)} - \frac{\pi/8}{(\pi/3)^2}\Big) + O(\varepsilon^4)\\
&= \frac{3}{\pi} - \frac{1}{8\pi}\varepsilon^2 + O(\varepsilon^4).
\end{aligned}$$

There is no third-order term: the transformation $\zeta \mapsto -\zeta$ unitarily identifies $A(\varepsilon)$ with $A(-\varepsilon)$, while the coefficient functional is unchanged, so the expansion of $\big(A^{-1}\big)_{33}$ contains only even powers. ■

Combining Proposition 2.5 with Proposition 2.6 gives the asymptotic law for the kernel.

Corollary 2.7. *As $q \to \infty$,*

$$K_3(q) = q\big(A(q^{-1})^{-1}\big)_{33} = \frac{3q}{\pi}\Big(1 - \frac{1}{24q^2} + O(q^{-4})\Big),$$

and therefore, with $C_0 := \log\big(2(6!)^2 \cdot 3/\pi\big)$,

$$\log K_{D_q}^{z^6}(0) = C_0 + \log q - \frac{1}{24q^2} + O(q^{-4}).$$

2.4 Proof of the main theorem

Proof of Theorem 1.2. Let $Q > 1$ and $c = 2Q$. By Proposition 2.2, the three values

$$a_- = -\log 2, \qquad a_0 = -\tfrac{1}{2}\log 2, \qquad a_+ = 0$$

correspond to

$$q_- = \frac{Q}{2}, \qquad q_0 = Q, \qquad q_+ = 2Q.$$

Since $a_0 = (a_- + a_+)/2$ and $\log q = \log c + 2a$ is affine in a, it suffices to prove

$$\log K_{D_Q}^{z^6}(0) - \frac{1}{2}\Big(\log K_{D_{Q/2}}^{z^6}(0) + \log K_{D_{2Q}}^{z^6}(0)\Big) > 0$$

for Q sufficiently large. By Corollary 2.7, the terms C_0 cancel, and so do the linear terms $\log q$:

$$\log Q - \frac{1}{2}\big(\log(Q/2) + \log(2Q)\big) = 0.$$

The remaining contribution is

$$-\frac{1}{24}\Big(\frac{1}{Q^2} - \frac{1}{2}\Big(\frac{4}{Q^2} + \frac{1}{4Q^2}\Big)\Big) + O(Q^{-4}) = \frac{3}{64Q^2} + O(Q^{-4}) > 0$$

for Q large enough. This proves the strict midpoint inequality, hence the failure of convexity. ■

Remark 2.8 (A different convexity question). *We recall that Błocki and Zwonek also conjectured in [3] the convexity of the* <u>*area*</u> *function* $t \mapsto \log \lambda(\{G_{\Omega,w} < t\})$*; counterexamples on annuli were found numerically in [1] and proved in full generality in [3]. That conjecture concerns the volume of sublevel sets and is unrelated to Question 1.1, which concerns the Bergman kernel.*

3 Monotonicity of higher order p-Bergman kernels fails for $p > 2$

In Sections 3.1 to 3.5 we treat the family D_q of Theorem 1.2, which yields the counterexample for $p > 3$; the full range $p > 2$ is then reached in Section 3.6 by the r-fold symmetric families.

Throughout Sections 3.1 to 3.5, $p > 3$ is fixed, $q > 1$, and we use freely the notation of Section 2. By Proposition 2.1 and Proposition 2.2, $(D_c)_a = D_q$ with $q = ce^{2a}$, so the monotonicity of $a \mapsto K^{z^6,p}_{(D_c)_a}(0)$ is equivalent to that of $q \mapsto K^{z^6,p}_{D_q}(0)$.

3.1 Reduction to a weighted A^p problem on the disk

For $f \in \mathcal{O}(D_q)$ write

$$f(x) = g_e(x^2) + x g_o(x^2), \qquad g_e, g_o \in \mathcal{O}(U_q),$$

with U_q as in Subsection 2.2. Since $q > 1$, the domain D_q is unbounded and contains a neighborhood of ∞. It is important not to assume that an arbitrary function holomorphic on D_q is holomorphic at ∞. Instead, suppose that $f \in L^p(D_q)$, as in the extremal problem. Because $p > 3 > 1$, its even and odd parts are separately in $L^p(D_q)$. With $\zeta_0 = -1/q$ and $\Phi_e(\zeta) := g_e(y(\zeta))$, the changes of variables $x^2 = y$ and $y = \zeta/(1 + q\zeta)$ show, near ζ_0, that

$$\int |\Phi_e(\zeta)|^p |\zeta - \zeta_0|^{-3}\, dA(\zeta) < +\infty.$$

For $\Phi_o(\zeta) := g_o(y(\zeta))$, the extra factor $|x|^p = |y|^{p/2}$ gives instead

$$\int |\Phi_o(\zeta)|^p |\zeta - \zeta_0|^{-(p/2+3)}\, dA(\zeta) < +\infty.$$

Lemma 2.3 gives

$$\operatorname{ord}_{\zeta_0} \Phi_e > \frac{1}{p}, \qquad \operatorname{ord}_{\zeta_0} \Phi_o > \frac{1}{2} + \frac{1}{p}.$$

Both orders are therefore at least one. Consequently, in the variable $\zeta = y/(1 - qy)$ of Lemma 2.4,

$$g_e(y(\zeta)) = (1 + q\zeta) h_e(\zeta), \qquad g_o(y(\zeta)) = (1 + q\zeta) h_o(\zeta),$$

for some $h_e, h_o \in \mathcal{O}(\mathbb{D})$, because $y = \infty$ corresponds to $\zeta = -1/q$.

Lemma 3.1. *With the notation above, the following hold.*

(i) The jet conditions $f^{(j)}(0) = 0$ for $0 \le j < 6$ and $f^{(6)}(0) = 1$ are equivalent to

$$h_e(\zeta) = \frac{\zeta^3}{6!} + O(\zeta^4), \qquad h_o(\zeta) = O(\zeta^3).$$

(ii) For $b := \sqrt{\zeta/(1 + q\zeta)}$, one has

$$\int_{D_q} |f|^p\, dA = \frac{1}{4} \int_{\mathbb{D}} \big(|h_e + b h_o|^p + |h_e - b h_o|^p\big) \frac{|1 + q\zeta|^{p-3}}{|\zeta|}\, dA(\zeta).$$

Proof. (i) follows exactly as in Lemma 2.4, since $y = \zeta + O(\zeta^2)$. For (ii), the map $x \mapsto y = x^2$ is two-to-one from D_q onto U_q with $dA(y) = 4|x|^2 dA(x) = 4|y| dA(x)$, and the two preimages of y contribute $|g_e(y) \pm w g_o(y)|^p$, $w^2 = y$. Since $w = \sqrt{\zeta/(1+q\zeta)}$ and $dA(y) = |1+q\zeta|^{-4} dA(\zeta)$, $|y|^{-1} = |1+q\zeta|/|\zeta|$, we obtain

$$\int_{D_q} |f|^p \, dA = \frac{1}{4} \int_{\mathbb{D}} |1+q\zeta|^p \big(|h_e + b h_o|^p + |h_e - b h_o|^p\big) |1+q\zeta|^{-3} |\zeta|^{-1} \, dA(\zeta),$$

with $b := \sqrt{\zeta/(1+q\zeta)}$, which is the claimed formula. ■

For $p \ge 1$ the function $\mathbb{C} \ni z \mapsto |z|^p$ is convex, so Jensen's inequality gives pointwise

$$\frac{1}{2}\big(|h_e + b h_o|^p + |h_e - b h_o|^p\big) \ge |h_e|^p,$$

and the odd part of f can only increase the L^p norm. Hence the infimum defining $K_{D_q}^{z^6,p}(0)$ may be taken among even functions, and by Lemma 3.1,

$$K_{D_q}^{z^6,p}(0) = 2(6!)^p M(q)^{-1},$$

where

$$M(q) := \inf\Big\{ \int_{\mathbb{D}} |h|^p \, \frac{|1+q\zeta|^{p-3}}{|\zeta|} \, dA(\zeta) : \ h \in \mathcal{O}(\mathbb{D}), \ h_3 = 1 \Big\},$$

and h_3 denotes the coefficient of ζ^3 in the Taylor expansion of h at 0. For $p = 2$ this is exactly the reduction of Lemma 2.4 and Proposition 2.5: the weight $|1+q\zeta|^{p-3}|\zeta|^{-1}$ becomes $|1+q\zeta|^{-1}|\zeta|^{-1}$.

3.2 Lower bound

Take $h_0(\zeta) = \zeta^3/6!$, which corresponds to the even admissible function

$$f_q(x) = \frac{x^6(1-qx^2)^{-4}}{6!},$$

because $1 + q\zeta = (1-qy)^{-1}$ and $g_e(y) = y^3(1-qy)^{-4}/6!$. By Lemma 3.1(ii),

$$\int_{D_q} |f_q|^p \, dA = \frac{1}{2(6!)^p} \int_{\mathbb{D}} |\zeta|^{3p-1} |1+q\zeta|^{p-3} \, dA(\zeta) \le \frac{\pi(1+q)^{p-3}}{(6!)^p(3p+1)},$$

since $p > 3$ gives $|1+q\zeta|^{p-3} \le (1+q)^{p-3}$ and $\int_{\mathbb{D}} |\zeta|^{3p-1} dA = 2\pi/(3p+1)$. Therefore

$$K_{D_q}^{z^6,p}(0) \ge \frac{(6!)^p(3p+1)}{\pi(1+q)^{p-3}} \ge c_p \, q^{-(p-3)}, \qquad c_p := \frac{(6!)^p(3p+1)}{\pi \, 2^{p-3}},$$

for $q \ge 1$.

3.3 Upper bound by duality

By homogeneity,

$$M(q)^{-1} = \sup\Big\{ |h_3|^p : \int_{\mathbb{D}} |h|^p w_q \, dA \le 1 \Big\} = \|\ell\|^p,$$

where $w_q(\zeta) := |1 + q\zeta|^{p-3}|\zeta|^{-1}$ and $\ell(h) := h_3$ on the weighted space $A^p(\mathbb{D}, w_q)$. Fix ζ_0 with $|\zeta_0| = 1/2$. Since $|h|^p$ is subharmonic for $p \ge 1$,

$$|h(\zeta_0)|^p \le \frac{16}{\pi} \int_{B(\zeta_0,1/4)} |h|^p \, dA.$$

On $B(\zeta_0, 1/4) \subset \mathbb{D}$ we have $|\zeta| \le 3/4$, and for $q \ge 8$,

$$|1 + q\zeta| \ge q|\zeta| - 1 \ge \frac{q}{8}, \qquad w_q(\zeta) \ge \frac{4}{3}\Big(\frac{q}{8}\Big)^{p-3}.$$

Hence

$$|h(\zeta_0)|^p \le \frac{12}{\pi}\Big(\frac{q}{8}\Big)^{-(p-3)} \int_{\mathbb{D}} |h|^p w_q \, dA.$$

Since $h_3 = (2\pi i)^{-1} \int_{|\zeta|=1/2} h(\zeta)\zeta^{-4} d\zeta$, we have $|h_3| \le 8 \max_{|\zeta|=1/2} |h(\zeta)|$. Thus

$$\|\ell\| \le 8\Big(\frac{12}{\pi}\Big)^{1/p}\Big(\frac{q}{8}\Big)^{-(p-3)/p},$$

and consequently

$$K^{z^6,p}_{D_q}(0) \le C_p \, q^{-(p-3)}, \qquad C_p := \frac{24(6!)^p 8^{2p-3}}{\pi}.$$

3.4 The limit as $q \to 0$

For $0 < q < 1$ we compare D_q with disks. If $|x|^2 < 1/(1 + q)$, then $(1 + q)|x|^2 < 1$, whence $|x|^2 < 1 - q|x|^2 \le |1 - qx^2|$; conversely, if $x \in D_q$, then $|x|^2 < |1 - qx^2| \le 1 + q|x|^2$, whence $|x|^2 < 1/(1 - q)$. Hence

$$D\Big(0, (1 + q)^{-1/2}\Big) \subset D_q \subset D\Big(0, (1 - q)^{-1/2}\Big).$$

Since the kernel is non-increasing under domain inclusion and $K^{z^6,p}_{D(0,R)}(0) = R^{-(6p+2)} K^{z^6,p}_{\mathbb{D}}(0)$ by rescaling, with

$$K^{z^6,p}_{\mathbb{D}}(0) = \frac{(6!)^p(6p + 2)}{2\pi},$$

we get

$$(1 - q)^{3p+1}\frac{(6!)^p(6p + 2)}{2\pi} \le K^{z^6,p}_{D_q}(0) \le (1 + q)^{3p+1}\frac{(6!)^p(6p + 2)}{2\pi},$$

and $K^{z^6,p}_{D_q}(0) \to (6!)^p(6p + 2)/(2\pi)$ as $q \to 0^+$.

3.5 Proof of Theorem 1.3

By Subsections 3.2 and 3.3,

$$c_p q^{-(p-3)} \le K_{D_q}^{z^6,p}(0) \le C_p q^{-(p-3)}, \qquad q \ge 8,$$

so $K_{D_q}^{z^6,p}(0) \to 0$ as $q \to \infty$, while Subsection 3.4 gives $K_{D_q}^{z^6,p}(0) \to (6!)^p(6p+2)/(2\pi) > 0$ as $q \to 0^+$. Choose $c > 1$ so large that $C_p c^{-(p-3)} < (1-c^{-1})^{3p+1}(6!)^p(6p+2)/(2\pi)$. Then

$$K_{D_c}^{z^6,p}(0) \le C_p c^{-(p-3)} < (1-c^{-1})^{3p+1}\frac{(6!)^p(6p+2)}{2\pi} \le K_{D_{1/c}}^{z^6,p}(0).$$

By Proposition 2.2, $D_{1/c} = D_{ce^{2a}}$ with $a = -\log c$, i.e. $D_{1/c} = (D_c)_{-\log c}$, while $D_c = (D_c)_0$. Thus $-\log c < 0$ but $K_{(D_c)_{-\log c}}^{z^6,p}(0) > K_{(D_c)_0}^{z^6,p}(0)$, and $a \mapsto K_{(D_c)_a}^{z^6,p}(0)$ is not non-decreasing. This proves the theorem.

Finally, the Green function formula of Proposition 2.1 gives $G_{D_c}(z,0) = \log|z| + O(1)$ near 0, so the Azukawa indicatrix of D_c at 0 is the unit disk $\mathbb{D}$, and the limit

$$\lim_{q\to 0^+} K_{D_q}^{z^6,p}(0) = \frac{(6!)^p(6p+2)}{2\pi} = K_{I_{D_c}(0)}^{z^6,p}(0)$$

holds with equality.

3.6 Extension to every $p > 2$ by r-fold symmetry

Fix integers $r \ge 2$ and $m \ge 1$, set $H(z) = z^{rm}$, and consider

$$D_q^{(r)} := \{x \in \mathbb{C} : \ |x|^r < |1 - qx^r|\}, \qquad q > 1.$$

The map

$$F_q(x) := \frac{x^r}{1 - qx^r}$$

maps $D_q^{(r)}$ properly, r-to-one, onto $\mathbb{D} \setminus \{-1/q\}$ and vanishes to order r at 0, so the Green function of $D_q^{(r)}$ with pole at 0 is

$$G_{D_q^{(r)}}(x,0) = \frac{1}{r}\log|F_q(x)| = \log|x| - \frac{1}{r}\log|1 - qx^r|,$$

and, exactly as in Proposition 2.2,

$$(D_c^{(r)})_a = D_q^{(r)}, \qquad q = ce^{ra}.$$

From now on assume $p > 2 + 2/r$, which is exactly the regime used in the counterexample below. Let $f \in \mathcal{O}(D_q^{(r)}) \cap L^p(D_q^{(r)})$ satisfy $f^{(j)}(0) = 0$ for $0 \le j < rm$ and $f^{(rm)}(0) = 1$, and let $\omega := e^{2\pi i/r}$. Averaging f over the r-th roots of unity,

$$f_0(x) := \frac{1}{r}\sum_{j=0}^{r-1} f(\omega^j x),$$

we get $f_0(\omega x) = f_0(x)$, hence $f_0(x) = g(x^r)$ with $g \in \mathcal{O}(U_q)$. Convexity gives $\|f_0\|_{L^p} \le \|f\|_{L^p}$, and since $\omega^{jrm} = 1$, the average f_0 satisfies the same jet conditions as f. Thus the extremal problem for $K^{z^{rm},p}_{D_q^{(r)}}(0)$ may be restricted to r-fold symmetric functions. Put $\Phi(\zeta) := g(y(\zeta))$ and $\zeta_0 = -1/q$. The first integral in (3.2), derived just below, shows near ζ_0 that

$$\int |\Phi(\zeta)|^p |\zeta - \zeta_0|^{-(2+2/r)}\, dA(\zeta) < +\infty.$$

Since $p > 2 + 2/r$, Lemma 2.3 implies $\operatorname{ord}_{\zeta_0}\Phi > 2/(rp)$, hence $\operatorname{ord}_{\zeta_0}\Phi \ge 1$. Therefore $g(y(\zeta)) = (1 + q\zeta)h(\zeta)$ with $h \in \mathcal{O}(\mathbb{D})$ and $h(\zeta) = \zeta^m/(rm)! + O(\zeta^{m+1})$.

Since $dA(y) = r^2|x|^{2r-2}dA(x) = r^2|y|^{2-2/r}dA(x)$ and every y has exactly r preimages, each contributing $|g(y)|^p$, we obtain

$$\int_{D_q^{(r)}} |f|^p\, dA = \frac{1}{r}\int_{U_q} |g(y)|^p\, |y|^{2/r-2}\, dA(y) = \frac{1}{r}\int_{\mathbb{D}} |h(\zeta)|^p\, |\zeta|^{2/r-2}\, |1 + q\zeta|^{p-2-2/r}\, dA(\zeta), \qquad (3.2)$$

using $dA(y) = |1 + q\zeta|^{-4}dA(\zeta)$ and $|y|^{2/r-2} = |\zeta|^{2/r-2}|1 + q\zeta|^{2-2/r}$. Therefore

$$K^{z^{rm},p}_{D_q^{(r)}}(0) = r(rm)!^p\, M(q)^{-1}, \qquad M(q) := \inf\Big\{\int_{\mathbb{D}} |h|^p w_q\, dA :\ h \in \mathcal{O}(\mathbb{D}),\ h_m = 1\Big\},$$

where $w_q(\zeta) := |\zeta|^{2/r-2}|1 + q\zeta|^{p-2-2/r}$ and h_m is the coefficient of ζ^m. This is the natural r-fold analogue of the reduction of Section 3.1; for $r = 2$, $m = 3$ it recovers Theorem 3.1 with $H(z) = z^6$.

<u>Lower bound.</u> Set $\beta := p - 2 - 2/r > 0$. Take $h_0(\zeta) = \zeta^m$, which corresponds to the symmetric admissible function

$$f_q(x) = \frac{x^{rm}(1 - qx^r)^{-(m+1)}}{(rm)!}.$$

Since $|1 + q\zeta| \le 1 + q$ on $\mathbb{D}$ and $\int_{\mathbb{D}} |\zeta|^{pm+2/r-2}dA = 2\pi/(pm + 2/r)$,

$$M(q) \le \int_{\mathbb{D}} |\zeta|^{pm+2/r-2}|1 + q\zeta|^{\beta}dA \le \frac{2\pi(1 + q)^{\beta}}{pm + 2/r},$$

hence

$$K^{z^{rm},p}_{D_q^{(r)}}(0) \ge \frac{r(rm)!^p(pm + 2/r)}{2\pi(1 + q)^{\beta}} \ge c_{p,r,m}\, q^{-\beta}$$

for $q \ge 1$.

<u>Upper bound.</u> Let $\ell(h) := h_m$ on $A^p(\mathbb{D}, w_q)$. By homogeneity $M(q)^{-1} = \|\ell\|^p$. Fix $|\zeta_0| = 1/2$. Subharmonicity gives $|h(\zeta_0)|^p \le \frac{16}{\pi}\int_{B(\zeta_0,1/4)} |h|^p dA$. On $B(\zeta_0, 1/4)$ we have $|\zeta| \le 3/4$; since $2/r - 2 \le 0$, $|\zeta|^{2/r-2} \ge (3/4)^{2/r-2}$, and for $q \ge 8$, $|1 + q\zeta| \ge q/8$. Hence

$$w_q(\zeta) \ge \Big(\frac{3}{4}\Big)^{2/r-2}\Big(\frac{q}{8}\Big)^{\beta},$$

and

$$|h(\zeta_0)|^p \le \frac{16}{\pi}\Big(\frac{3}{4}\Big)^{2-2/r}\Big(\frac{q}{8}\Big)^{-\beta}\int_{\mathbb{D}} |h|^p w_q\, dA.$$

Since $h_m = (2\pi i)^{-1} \int_{|\zeta|=1/2} h(\zeta)\zeta^{-m-1} d\zeta$, one has $|h_m| \le 2^m \max_{|\zeta|=1/2} |h(\zeta)|$, and consequently

$$K^{z^{rm},p}_{D^{(r)}_q}(0) \le C_{p,r,m}\, q^{-\beta}, \qquad q \ge 8.$$

The limit as $q \to 0$. For $0 < q < 1$, exactly as in Section 3.4,

$$D\Big(0, (1+q)^{-1/r}\Big) \subset D^{(r)}_q \subset D\Big(0, (1-q)^{-1/r}\Big),$$

and rescaling gives $K^{z^{rm},p}_{D(0,R)}(0) = R^{-(2+prm)} K^{z^{rm},p}_{\mathbb{D}}(0)$ with $K^{z^{rm},p}_{\mathbb{D}}(0) = (rm)!^p(rmp+2)/(2\pi)$. Therefore

$$\lim_{q \to 0^+} K^{z^{rm},p}_{D^{(r)}_q}(0) = \frac{(rm)!^p(rmp+2)}{2\pi},$$

and the Green function formula above gives $I_{D^{(r)}_c}(0) = \mathbb{D}$, so the comparison with the Azukawa indicatrix holds with equality.

Proof of Theorem 1.4. For $p > 2$ choose an integer $r > 2/(p-2)$, so that $\beta = p - 2 - 2/r > 0$, and take any $m \ge 1$. The two bounds above give

$$c_{p,r,m}\, q^{-\beta} \le K^{z^{rm},p}_{D^{(r)}_q}(0) \le C_{p,r,m}\, q^{-\beta}, \qquad q \ge 8,$$

so the kernel tends to 0 as $q \to \infty$, while its limit at $q \to 0^+$ is the positive constant $(rm)!^p(rmp+2)/(2\pi)$. Choose $c > 1$ so large that $C_{p,r,m}c^{-\beta} < (1-c^{-1})^{(2+prm)/r}(rm)!^p(rmp+2)/(2\pi)$; then

$$K^{z^{rm},p}_{D^{(r)}_c}(0) \le C_{p,r,m}c^{-\beta} < (1-c^{-1})^{(2+prm)/r}\frac{(rm)!^p(rmp+2)}{2\pi} \le K^{z^{rm},p}_{D^{(r)}_{1/c}}(0).$$

Since $q = ce^{ra}$, the identity $q = 1/c$ corresponds to $a = -\frac{2}{r}\log c$, i.e. $D^{(r)}_{1/c} = (D^{(r)}_c)_{-\frac{2}{r}\log c}$, while $D^{(r)}_c = (D^{(r)}_c)_0$. Thus the kernel at the smaller value $a = -\frac{2}{r}\log c < 0$ exceeds its value at $a = 0$, and $a \mapsto K^{z^{rm},p}_{(D^{(r)}_c)_a}(0)$ is not non-decreasing. ∎

4 The L^p concavity of minimal integrals fails for $p > 2$

Throughout this subsection we fix $p > 2$ and integers $r \geq 1$, $m \geq 1$, and set $k := rm$. We fix $c \in (0, 1)$ and write

$$D := D_c^{(r)} = \{x \in \mathbb{C} : \ |x|^r < |1 - cx^r|\}, \qquad F_c(x) := \frac{x^r}{1 - cx^r}.$$

Since $c < 1$, the domain D is bounded: for $x \in D$,

$$|x|^r < |1 - cx^r| \leq 1 + c|x|^r,$$

so $|x| < (1 - c)^{-1/r}$.

4.1 The multiplier ideal of the Green function

Exactly as in Theorem 2.1 (whose proof applies verbatim to the map F_c), the map $F_c : D \to \mathbb{D} \setminus \{-1/c\}$ is proper of degree r, it vanishes to order r at 0, and consequently

$$G_D(x, 0) = \frac{1}{r} \log |F_c(x)| = \log |x| - \frac{1}{r} \log |1 - cx^r|$$

is the Green function of D with pole at 0. Set

$$\varphi := 2(k + 1)\, G_D(\cdot, 0), \qquad F(x) := \frac{x^k}{k!}.$$

Then φ is a negative plurisubharmonic function on D with $\varphi(0) = -\infty$, and near 0,

$$e^{-\varphi} = |x|^{-2(k+1)}\, |1 - cx^r|^{2(k+1)/r} = |x|^{-2(k+1)}\big(1 + O(|x|^r)\big).$$

Lemma 4.1. *One has $\mathcal{I}(\varphi)_0 = \mathfrak{m}_0^{k+1}$.*

Proof. For $f \in \mathcal{O}_0$ with $\operatorname{ord}_0 f = j$, one has $|f|^2 e^{-\varphi} \asymp |x|^{2j-2(k+1)}$ near 0, and in polar coordinates

$$\int_{|x|<\delta} |f|^2 e^{-\varphi}\, dA \asymp \int_0^\delta s^{2j-2k-1}\, ds,$$

which is finite exactly when $j \geq k + 1$. ∎

4.2 Reduction to a higher order p-Bergman kernel

Let $\Omega \subset \mathbb{C}$ be a domain containing 0. Since $(\tilde{F} - F, 0) \in \mathfrak{m}_0^{k+1}$ is equivalent to

$$\tilde{F}^{(j)}(0) = 0 \ \ (0 \leq j < k), \qquad \tilde{F}^{(k)}(0) = 1,$$

Lemma 4.1 gives

$$C_{F,\mathcal{I}(\varphi)_0,p}(\Omega) = \inf\Big\{\int_\Omega |\tilde{F}|^p : \ \tilde{F} \in \mathcal{O}(\Omega),\ \tilde{F}^{(j)}(0) = 0 \ (0 \leq j < k),\ \tilde{F}^{(k)}(0) = 1\Big\} = \frac{1}{K_\Omega^{z^k,p}(0)},$$

where $K_\Omega^{z^k,p}(0)$ denotes the higher order p-Bergman kernel introduced above. Indeed, every admissible $\tilde{F}$ satisfies $\|\tilde{F}\|_{L^p(\Omega)}^p \geq 1/K_\Omega^{z^k,p}(0)$ by the definition of the kernel, and rescaling functions with $f^{(j)}(0) = 0$ $(0 \leq j < k)$ and $f^{(k)}(0) \neq 0$ by $1/f^{(k)}(0)$ produces admissible functions with norms arbitrarily close to $1/K_\Omega^{z^k,p}(0)$.

4.3 Sublevel sets and the scaling law

For $t \geq 0$ set
$$a := -\frac{t}{2(k+1)}, \qquad q := ce^{ra}.$$
As in Theorem 2.2 (applied to the r-fold map F_c, exactly as in Subsection 3.6),
$$\{G_D(\cdot,0) < a\} = e^a D_q^{(r)}, \qquad D_q^{(r)} := \{x \in \mathbb{C} : \ |x|^r < |1 - qx^r|\} :$$
indeed $G_D(z,0) < a$ is equivalent to $|F_c(z)| < e^{ra}$, and writing $z = e^a x$ gives $F_c(e^a x) = e^{ra} F_q(x)$.

Lemma 4.2. *For every $t \geq 0$,*
$$G_p(t) = \frac{e^{(pk+2)a}}{K_{D_q^{(r)}}^{z^k,p}(0)}, \qquad a = -\frac{t}{2(k+1)}, \quad q = ce^{ra}.$$

Proof. Since $G_D(\cdot,0) < 0$ on D,
$$\{\varphi < -t\} \cap D = \{G_D(\cdot,0) < a\} = e^a D_q^{(r)}.$$
For $\tilde{F} \in \mathcal{O}(e^a D_q^{(r)})$ set $g(x) := \tilde{F}(e^a x) \in \mathcal{O}(D_q^{(r)})$. Then $g^{(j)}(0) = e^{aj} \tilde{F}^{(j)}(0)$, so the admissibility of $\tilde{F}$ is equivalent to $g^{(j)}(0) = 0$ $(0 \leq j < k)$ and $g^{(k)}(0) = e^{ak}$, while
$$\int_{e^a D_q^{(r)}} |\tilde{F}|^p \, dA = e^{2a} \int_{D_q^{(r)}} |g|^p \, dA.$$
By the homogeneity of the previous subsection,
$$G_p(t) = e^{2a} \cdot \frac{e^{pak}}{K_{D_q^{(r)}}^{z^k,p}(0)} = \frac{e^{(pk+2)a}}{K_{D_q^{(r)}}^{z^k,p}(0)}.$$
■

4.4 The limit as $t \to +\infty$ and the failure of concavity

As computed in Subsection 3.6 — the comparison
$$D\big(0, (1+q)^{-1/r}\big) \subset D_q^{(r)} \subset D\big(0, (1-q)^{-1/r}\big), \qquad 0 < q < 1,$$
together with the rescaling $K_{D(0,R)}^{z^k,p}(0) = R^{-(pk+2)} K_{\mathbb{D}}^{z^k,p}(0)$ and the disk value $K_{\mathbb{D}}^{z^k,p}(0) = (k!)^p(kp+2)/(2\pi)$ — one has, for every $p \geq 1$,
$$\lim_{q \to 0^+} K_{D_q^{(r)}}^{z^k,p}(0) = \frac{(k!)^p(kp+2)}{2\pi} =: L.$$

Proof of Theorem 1.5. First, $G_p(0) < +\infty$: since $(F - F, 0) \in \mathfrak{m}_0^{k+1}$ and $F \in L^p(D)$ (D is bounded), we have $G_p(0) \leq \|F\|_{L^p(D)}^p < +\infty$. Moreover $G_p(0) > 0$: every admissible $\tilde{F}$ has

$\tilde{F}^{(k)}(0) = 1$, hence positive L^p norm on a neighborhood of 0; equivalently $G_p(0) = 1/K_D^{z^k,p}(0) \in (0, +\infty)$.

As $t \to +\infty$ we have $a \to -\infty$ and $q = ce^{ra} \to 0$, so Lemma 4.2 and the limit above give

$$G_p(t) \sim \frac{1}{L}\, e^{(pk+2)a} = \frac{1}{L}\, \exp\Bigl(-\frac{pk+2}{2(k+1)}\, t\Bigr).$$

Let $g(\rho) := G_p(-\log \rho)$ for $\rho \in (0, 1]$. Then

$$g(\rho) \sim \frac{1}{L}\, \rho^{\mu}, \qquad \mu := \frac{pk+2}{2(k+1)} = 1 + \frac{k(p-2)}{2(k+1)} > 1,$$

as $\rho \to 0^+$.

The function g is non-decreasing, because G_p is non-increasing in t. Moreover $\lim_{t\to+\infty} G_p(t) = 0$, hence $g(0+) = 0$: indeed,

$$G_p(t) \le \int_{\{\varphi < -t\}} |F|^p \, dA \to 0$$

by dominated convergence as the sublevel sets shrink to $\{0\}$. Also $g(1) = G_p(0) > 0$.

If g were concave on $(0, 1]$, then for $0 < \varepsilon < \rho < 1$,

$$g(\rho) \;\ge\; \frac{\rho - \varepsilon}{1 - \varepsilon}\, g(1) + \frac{1 - \rho}{1 - \varepsilon}\, g(\varepsilon),$$

and letting $\varepsilon \to 0^+$ gives $g(\rho) \ge \rho\, g(1)$ for every $\rho \in (0, 1]$. But $\mu > 1$ and $g(\rho) \sim \rho^{\mu}/L$ imply $g(\rho) < \rho\, g(1)$ for all sufficiently small ρ — a contradiction. Hence $g = G_p(-\log \rho)$ is not concave in $\rho \in (0, 1]$, which proves the theorem. ■

AI declaration

The first counterexample, concerning the non-convexity of the logarithm of a higher order Bergman kernel, was obtained with GPT-5.6-sol (xhigh). The second counterexample, concerning the failure of monotonicity of higher order p-Bergman kernels for $p > 2$, and the third counterexample, concerning the failure of the L^p analogue of Guan's concavity theorem for $p > 2$, were obtained with DeepSeek V4 flash and pro. The author independently verified all statements and computations and takes full responsibility for the content.

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