A four-connected domain with no point of equality in the higher order Suita conjecture

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Abstract

We construct a four-connected planar domain for which equality in the higher order Suita conjecture never holds at any point or at any derivative order. More generally, such a domain exists for every connectivity $n \geq 4$.

1 Introduction

Let Ω be a planar domain in $\mathbb{C}$ bounded by n analytic Jordan curves $\Gamma_1, \ldots, \Gamma_n$, where Γ_n is the boundary of the unbounded component of $\mathbb{C} \setminus \Omega$. For each $j \in \{1, \ldots, n-1\}$, let u_j be the harmonic function on $\overline{\Omega}$ determined by

$$u_j|_{\Gamma_j} \equiv 1, \qquad u_j|_{\Gamma_k} \equiv 0 \quad (k \in \{1, \ldots, n\} \setminus \{j\}).$$

We recall the higher order Suita conjecture in the form used below. If $G_\Omega(\cdot, w)$ is the Green function of Ω with pole at w, set

$$c_\Omega(w) := \exp\left(\lim_{z \to w}\big(G_\Omega(z, w) - \log|z - w|\big)\right).$$

For $m \geq 0$, define the m-th order Bergman kernel by

$$K_\Omega^{(m)}(w) := \sup\left\{|f^{(m)}(w)|^2 : f \in \mathcal{O}(\Omega),\ f^{(j)}(w) = 0\ (0 \leq j < m),\ \int_\Omega |f|^2\, d\lambda \leq 1\right\}.$$

The higher order Suita inequality, proved by Błocki–Zwonek [1, 2], is

$$K_\Omega^{(m)}(w) \geq \frac{m!(m+1)!\, c_\Omega(w)^{2m+2}}{\pi}.$$

Guan–Sun–Yuan [4] characterized its equality locus: equality at $z \in \Omega$ holds if and only if

$$(m+1)u_j(z) \in \mathbb{Z}, \qquad j = 1, \ldots, n-1.$$

Consequently, equality holds at some order at z precisely when all the numbers $u_1(z), \ldots, u_{n-1}(z)$ are rational.

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Recently, Xu–Zhou [7] constructed, for every $n \geq 3$ and every sufficiently large integer M, smoothly bounded n-connected planar domains on which equality fails at every point for each order $m = 0, 1, \dots, M$. The following result gives one domain for which equality fails simultaneously at all orders.

THEOREM 1.1. *There exists a planar domain Ω in $\mathbb{C}$ bounded by four analytic Jordan curves such that, for the corresponding harmonic functions u_1, u_2, u_3, there is no $z \in \Omega$ with*

$$u_j(z) \in \mathbb{Q}, \qquad j = 1, 2, 3.$$

In particular, at no point of Ω does equality in the higher order Suita conjecture hold.

The proof is a generic-parameter construction. We vary the radii of three circular holes and show that the corresponding harmonic-measure map is a submersion. A dimension argument based on Sard's theorem then excludes all rational harmonic-measure vectors for almost every parameter. The same construction works for every connectivity $n \geq 4$; see Section 3.

2 The counterexample and the proof

We prove Theorem 1.1 by a generic-parameter construction.

Fix three pairwise distinct points $c_1, c_2, c_3 \in \mathbb{D}$ and choose $\rho_0 > 0$ so small that the closed disks $\overline{\mathbb{D}(c_j, 2\rho_0)}$ are pairwise disjoint and contained in $\mathbb{D}$. Let

$$B := \left(\tfrac{1}{2}\rho_0, \rho_0\right) \times \left(\tfrac{1}{2}\rho_0, \rho_0\right) \times \left(\tfrac{1}{2}\rho_0, \rho_0\right) \subset \mathbb{R}^3,$$

where $(\frac{1}{2}\rho_0, \rho_0)$ denotes the open interval in $\mathbb{R}$. For $\mathbf{r} = (r_1, r_2, r_3) \in B$ put

$$D_j(\mathbf{r}) := \overline{\mathbb{D}(c_j, r_j)}, \qquad \Gamma_j(\mathbf{r}) := \partial D_j(\mathbf{r}), \quad j = 1, 2, 3, \qquad \Gamma_4 := \partial \mathbb{D},$$

and

$$\Omega(\mathbf{r}) := \mathbb{D} \setminus \Big(D_1(\mathbf{r}) \cup D_2(\mathbf{r}) \cup D_3(\mathbf{r})\Big).$$

Then $\Omega(\mathbf{r})$ is bounded by the four pairwise disjoint analytic Jordan curves $\Gamma_1(\mathbf{r}), \Gamma_2(\mathbf{r}), \Gamma_3(\mathbf{r})$ and Γ_4; see Figure 1.

For $j = 1, 2, 3$ let $u_j(\cdot\,; \mathbf{r})$ be the harmonic function on $\Omega(\mathbf{r})$ with boundary values 1 on $\Gamma_j(\mathbf{r})$ and 0 on the other three curves, and set $u_4 := 1 - u_1 - u_2 - u_3$. Basic facts on harmonic measure used below can be found in [3].

LEMMA 2.1. *For every $z \in \Omega(\mathbf{r})$ and $j, k \in \{1, 2, 3\}$,*

$$\frac{\partial u_j}{\partial r_j}(z; \mathbf{r}) > 0, \qquad \frac{\partial u_j}{\partial r_k}(z; \mathbf{r}) < 0 \quad (j \neq k), \qquad \frac{\partial u_4}{\partial r_k}(z; \mathbf{r}) < 0. \tag{2.1}$$

Proof. The functions u_j are real-analytic in $(z, \mathbf{r})$; this follows from the standard analytic dependence of solutions of the Dirichlet problem on analytic boundary perturbations; see [5, 6]. The first inequality in (2.1) is the usual monotonicity of harmonic measure under enlargement of the corresponding boundary component (Figure 2); it follows from the maximum principle, since on the new, smaller domain the old solution is < 1 on the new boundary and $= 0$ on all other boundary components.

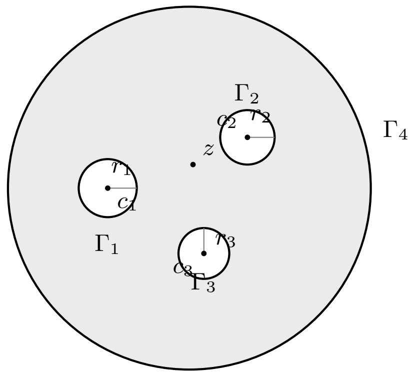

Figure 1: The four-connected domain $\Omega(\mathbf{r}) = \mathbb{D}\setminus\big(D_1(\mathbf{r})\cup D_2(\mathbf{r})\cup D_3(\mathbf{r})\big)$ with boundary components $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.

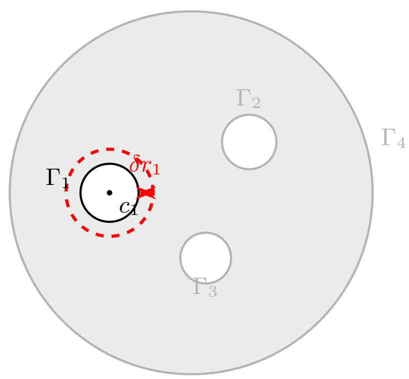

Figure 2: Enlargement of the inner disk bounded by $\Gamma_1(\mathbf{r})$ by $\delta r_1 > 0$: the dashed circle is the new boundary, and the red arrow indicates the radial displacement corresponding to the increase of r_1 in the variational formula (2.2).

For the strict derivative signs we use Hadamard's variational formula [6]. Let $G_{\mathbf{r}}(w,z)$ be the Green function of $\Omega(\mathbf{r})$ and let n denote the outward unit normal. Up to a fixed positive multiplicative constant,

$$\frac{\partial u_j}{\partial r_k}(z;\mathbf{r}) = \int_{\Gamma_k(\mathbf{r})} \frac{\partial u_j}{\partial n}(w)\,\frac{\partial G_{\mathbf{r}}}{\partial n}(w,z)\,ds(w). \tag{2.2}$$

For an inner boundary $\Gamma_k(\mathbf{r})$ one has $\partial G_{\mathbf{r}}/\partial n > 0$ on $\Gamma_k(\mathbf{r})$. By Hopf's lemma, $\partial u_j/\partial n$ on $\Gamma_k(\mathbf{r})$ is strictly positive if $j=k$ and strictly negative if $j\neq k$; similarly $\partial u_4/\partial n < 0$ on $\Gamma_k(\mathbf{r})$. Hence (2.1) follows from (2.2). ■

For $z \in \Omega(\mathbf{r})$ put

$$A(z;\mathbf{r}) := \Big(\tfrac{\partial u_j}{\partial r_k}(z;\mathbf{r})\Big)_{1\le j,k\le 3}.$$

Lemma 2.2. *For every $\mathbf{r}\in B$ and every $z\in\Omega(\mathbf{r})$ the matrix $A(z;\mathbf{r})$ is invertible.*

Proof. By (2.1), $A_{jk}<0$ for $j\neq k$, and

$$(A^T\mathbf{1})_j = \sum_{i=1}^{3} \frac{\partial u_i}{\partial r_j} = \frac{\partial}{\partial r_j}\big(u_1+u_2+u_3\big) = -\frac{\partial u_4}{\partial r_j} > 0.$$

Thus A^T has positive diagonal, nonpositive off-diagonal entries and positive row sums. If $A^T\mathbf{x}=0$ for some $\mathbf{x}\neq 0$, choose an index i at which $x_i=\max_j x_j>0$; if all $x_j\le 0$, choose instead an index i at which $x_i=\min_j x_j<0$. In the first case, since $A^T_{ik}\le 0$ for $k\neq i$,

$$0=(A^T\mathbf{x})_i = A^T_{ii}x_i + \sum_{k\neq i} A^T_{ik}x_k \ge A^T_{ii}x_i + \sum_{k\neq i} A^T_{ik}x_i = (A^T\mathbf{1})_i x_i > 0,$$

a contradiction. The second case is analogous with all inequalities reversed. Hence A^T, and therefore A, is nonsingular. ■

Proof of Theorem 1.1. Let

$$X := \big\{(z,\mathbf{r})\in\mathbb{C}\times B : z\in\Omega(\mathbf{r})\big\},$$

an open subset of $\mathbb{R}^2\times\mathbb{R}^3\cong\mathbb{R}^5$, and define

$$F : X\to\mathbb{R}^3, \qquad F(z,\mathbf{r}) := \big(u_1(z;\mathbf{r}),u_2(z;\mathbf{r}),u_3(z;\mathbf{r})\big).$$

By Theorem 2.1, F is real-analytic. Its derivative contains the invertible 3×3 block $D_{\mathbf{r}}F = A(z;\mathbf{r})$; hence F is a submersion at every point of X.

For $\mathbf{q}=(q_1,q_2,q_3)\in\mathbb{Q}^3$ set $S_{\mathbf{q}} := F^{-1}(\mathbf{q})$. By the implicit function theorem, $S_{\mathbf{q}}$ is a real-analytic submanifold of X of dimension $\dim_{\mathbb{R}}X-3=2$ (possibly empty). Let $\pi_{\mathbf{q}}: S_{\mathbf{q}}\to B$ be the projection onto the parameter box. Each connected component of $S_{\mathbf{q}}$ is a two-dimensional C^1 manifold. Since $\dim_{\mathbb{R}}B=3$, Sard's theorem shows that its image under $\pi_{\mathbf{q}}$ has empty interior (indeed Lebesgue measure zero). A second countable manifold has at most countably many connected components, so $\pi_{\mathbf{q}}(S_{\mathbf{q}})$ is a countable union of null sets, hence a null set. Consequently

$$E := \bigcup_{\mathbf{q}\in\mathbb{Q}^3} \pi_{\mathbf{q}}(S_{\mathbf{q}})$$

is a countable union of null sets, and $B \setminus E$ is nonempty (in fact of full measure).

Choose $\mathbf{r} \in B \setminus E$. We claim that $\Omega := \Omega(\mathbf{r})$ has the required property. Indeed, if for some $z \in \Omega$ all three numbers $u_j(z;\mathbf{r})$ were rational, then with $\mathbf{q} := \big(u_1(z;\mathbf{r}), u_2(z;\mathbf{r}), u_3(z;\mathbf{r})\big) \in \mathbb{Q}^3$ we would have $(z,\mathbf{r}) \in S_{\mathbf{q}}$ and hence $\mathbf{r} \in E$, a contradiction. This completes the proof. ∎

3 Generalization to higher connectivity

The proof of Theorem 1.1 extends verbatim to every $n \geq 4$.

THEOREM 3.1. *For every integer $n \geq 4$ there exists a planar domain $\Omega \subset \mathbb{C}$ bounded by n analytic Jordan curves such that, for the corresponding harmonic functions $u_1, \dots, u_{n-1}$, there does not exist $z \in \Omega$ with*

$$u_j(z) \in \mathbb{Q}, \qquad j = 1, \dots, n-1.$$

In particular, at no point of Ω does equality in the higher order Suita conjecture hold.

Proof. Fix $n \geq 4$ and set $m := n-1$. Choose m pairwise distinct points $c_1, \dots, c_m \in \mathbb{D}$ and $\rho_0 > 0$ so small that the closed disks $\overline{\mathbb{D}(c_j, 2\rho_0)}$ are pairwise disjoint and contained in $\mathbb{D}$. Let $B := (\rho_0/2, \rho_0)^m$. For $\mathbf{r} = (r_1, \dots, r_m) \in B$ define

$$D_j(\mathbf{r}) := \overline{\mathbb{D}(c_j, r_j)}, \qquad \Gamma_j(\mathbf{r}) := \partial D_j(\mathbf{r}) \quad (1 \leq j \leq m), \qquad \Gamma_{m+1} := \partial\mathbb{D},$$

and

$$\Omega(\mathbf{r}) := \mathbb{D} \setminus \bigcup_{j=1}^{m} D_j(\mathbf{r}).$$

Let $u_j(\cdot\,;\mathbf{r})$ be the harmonic measure of $\Gamma_j(\mathbf{r})$ with respect to $\Omega(\mathbf{r})$, and put $u_{m+1} := 1 - \sum_{j=1}^{m} u_j$, which is the harmonic measure of the outer boundary Γ_{m+1}.

The variational argument in Lemma 2.1 applies unchanged with m in place of 3: for all $j,k \in \{1,\dots,m\}$,

$$\frac{\partial u_j}{\partial r_j}(z;\mathbf{r}) > 0, \qquad \frac{\partial u_j}{\partial r_k}(z;\mathbf{r}) < 0 \quad (j \neq k), \qquad \frac{\partial u_{m+1}}{\partial r_k}(z;\mathbf{r}) < 0. \tag{3.3}$$

Indeed, the off-diagonal and u_{m+1} estimates follow from Hadamard's formula (2.2) and Hopf's lemma exactly as in Theorem 2.1.

Let $A(z;\mathbf{r}) := (\partial u_j/\partial r_k)_{1\leq j,k\leq m}$. From (3.3),

$$(A^T\mathbf{1})_j = \sum_{i=1}^{m} \frac{\partial u_i}{\partial r_j} = \frac{\partial}{\partial r_j}\Big(\sum_{i=1}^{m} u_i\Big) = -\frac{\partial u_{m+1}}{\partial r_j} > 0.$$

Thus A^T has positive diagonal, nonpositive off-diagonal entries and positive row sums; by the same maximum-entry argument as in Theorem 2.2, A^T, and hence A, is nonsingular.

Define

$$X := \big\{(z,\mathbf{r}) \in \mathbb{C} \times B : z \in \Omega(\mathbf{r})\big\}, \qquad F : X \to \mathbb{R}^m, \quad F(z,\mathbf{r}) := \big(u_1(z;\mathbf{r}), \dots, u_m(z;\mathbf{r})\big).$$

Then F is real-analytic and a submersion, since $D_{\mathbf{r}}F = A$ is invertible. For every $\mathbf{q} \in \mathbb{Q}^m$, the set $S_{\mathbf{q}} := F^{-1}(\mathbf{q})$ is a real-analytic submanifold of X of dimension

$$\dim X - m = (2+m) - m = 2.$$

Let $\pi_{\mathbf{q}} : S_{\mathbf{q}} \to B$ be the projection onto the parameter box. Since $\dim B = m = n-1 \geq 3$, Sard's theorem shows that $\pi_{\mathbf{q}}(S_{\mathbf{q}})$ is a null set: every connected component is a smooth image of a two-dimensional manifold, and there are at most countably many components. Therefore

$$E := \bigcup_{\mathbf{q}\in\mathbb{Q}^m} \pi_{\mathbf{q}}(S_{\mathbf{q}})$$

is a countable union of null sets, and $B \setminus E$ is nonempty.

Choose $\mathbf{r} \in B \setminus E$. If for some $z \in \Omega(\mathbf{r})$ all $u_j(z;\mathbf{r})$ were rational, then $(z,\mathbf{r}) \in S_{\mathbf{q}}$ for $\mathbf{q} := F(z,\mathbf{r}) \in \mathbb{Q}^m$, so $\mathbf{r} \in E$, a contradiction. Hence $\Omega := \Omega(\mathbf{r})$ satisfies the conclusion of Theorem 3.1. ■

Declaration of generative AI assistance

The counterexample presented in this paper was found with the assistance of DeepSeek V4 flash. The author has independently verified all statements and computations and takes full responsibility for the content of the paper.

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