INDEFINITE THETA HESSIANS AT GREEN-FUNCTION PARAMETERS

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Abstract. Yamada proposed the negative definiteness of the logarithmic theta Hessian at the parameter determined by the critical divisor of a planar Green function. We show that this assertion fails for triply connected circle domains. For a family obtained by removing two disks of fixed pseudohyperbolic radius from the unit disk, and for a suitable family of poles, the Hessian has one positive and one negative eigenvalue. Their leading terms have orders ε^4 and $-\varepsilon^6$, respectively, and we compute both coefficients. The proof uses harmonic measures to determine the period matrix and the Abel displacement, a second-order theta expansion to produce the positive direction, and Yamada's extremal characterization to identify the Green-function parameter. The resulting counterexample concerns the matrix strengthening of Saitoh's inequality.

1. Introduction

Let Ω be a bounded planar domain with $g+1$ analytic boundary components, and let $\widehat{\Omega}$ be its Schottky double. For general background on the theta function we refer to Fay [Fay73]. We use the normalization of Yamada [Yam99]: the A-periods of the normalized holomorphic differentials are $2\pi i$, while the B-cycles are boundary curves with the orientations specified in Section 3. The B-period matrix τ is real symmetric and negative definite. Thus

$$\theta_\tau(z) = \sum_{n\in\mathbb{Z}^g} \exp\left(\frac{1}{2}n^T\tau n + n^T z\right). \tag{1}$$

The real torus relevant here is $i\mathbb{R}^g/(2\pi i\mathbb{Z}^g)$. On it, θ_τ is strictly positive. Define the *holomorphic* Hessian

$$H_\tau(e) = \left(\frac{\partial^2 \log\theta_\tau}{\partial z_j\partial z_k}(e)\right)_{j,k=1}^g. \tag{2}$$

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It is a real symmetric matrix for $e \in i\mathbb{R}^g$. In particular, for $L(t) = \log \theta_\tau(it)$,

$$H_\tau(it) = -D_t^2 L(t). \tag{3}$$

For $p \in \Omega$, let D_p be the zero divisor in Ω of the differential of the Green function with pole p, with multiplicities. Its degree is g. In Abel–Jacobi notation, set

$$e_0(p) = D_p - p - \Delta, \tag{4}$$

where Δ is the Riemann divisor class. A representative belongs to $i\mathbb{R}^g$, and the Hessian is independent of the allowed period ambiguity. The theta representation of reproducing kernels leads to a natural definiteness problem for (2). At the Green-function parameter, Yamada [Yam98, p. 41] proposed

$$H_\tau(e_0(p)) < 0 \tag{5}$$

as a strong form of Saitoh's conjecture. The same problem is discussed in the introduction of [Yam99], where a relative comparison between theta Hessians is established. The scalar comparison between the conjugate Hardy kernel and the Bergman kernel was subsequently proved by Guan [Gua19]. The matrix assertion additionally requires the quadratic form (2) to be negative in every nonzero real direction.

The purpose of this paper is to disprove (5). The first case in which a transverse direction is available is $g = 2$, and our examples have precisely this genus.

Fix $0 < \rho < 1$ and put

$$L = \log(1/\rho), \qquad d_* = \frac{2\pi^2}{L}. \tag{6}$$

For small $\varepsilon > 0$, let $c = 1 - \varepsilon$ and define

$$K_\pm(c) = \left\{ z \in \mathbb{D} : \left| \frac{z \mp c}{1 \mp cz} \right| \le \rho \right\}, \tag{7}$$

$$\Omega_\varepsilon = \mathbb{D} \setminus \big(K_+(c) \cup K_-(c)\big), \qquad\qquad p_\varepsilon = i(1 - \varepsilon^5). \tag{8}$$

For $0 < \varepsilon < 1 - \rho$, these are disjoint closed Euclidean disks strictly inside $\mathbb{D}$, so Ω_ε is a regular triply connected domain. The centers and the common radius of the disks are, respectively,

$$\pm \frac{c(1-\rho^2)}{1 - c^2\rho^2}, \qquad \frac{\rho(1-c^2)}{1-c^2\rho^2}. \tag{9}$$

Indeed, the positive center in (9) exceeds the radius exactly when $c > \rho$, since

$$c(1-\rho^2) - \rho(1-c^2) = (c-\rho)(1+c\rho) > 0.$$

Thus neither disk meets the imaginary axis, and $p_\varepsilon \in \Omega_\varepsilon$.

For $d > 0$, write

$$P_d(t) = \sum_{n\in\mathbb{Z}} e^{-dn^2/2 + int}, \qquad \ell_d(t) = \log P_d(t). \tag{10}$$

Lemma 2.1 proves that there is a unique $t_d \in (0,\pi)$ such that $\ell_d''(t_d) = 0$, and that

$$\kappa_d = \ell_d'''(t_d) > 0. \tag{11}$$

Let $\kappa_* := \kappa_{d_*}$.

Theorem 1.1. *Fix $0 < \rho < 1$ and let $\Omega_\varepsilon, p_\varepsilon$ be given by (8). In the normalized basis associated with the two inner boundary components, the matrix*

$$H_{\tau_\varepsilon}\big(e_0(p_\varepsilon)\big)$$

has eigenvectors $(1,1)$ and $(1,-1)$. Their respective eigenvalues satisfy

$$\lambda_\parallel(\varepsilon) = -\frac{\pi\kappa_*}{L}\varepsilon^6 + O(\varepsilon^7), \tag{12}$$

$$\lambda_\perp(\varepsilon) = \frac{\pi^4\kappa_*^2}{L^4}\varepsilon^4 + O(\varepsilon^5). \tag{13}$$

Consequently, for all sufficiently small positive ε, the matrix has one strictly positive and one strictly negative eigenvalue.

For each sufficiently small positive ε, all three boundary components are nondegenerate and the double is a smooth genus-two surface. In particular, the indefiniteness occurs on genuine Jacobians of planar doubles. The theorem answers the proposed universal negative-definiteness assertion; it does not classify the signature of $H_\tau(e)$ for arbitrary parameters e.

The mechanism can be described in terms of two small parameters. The off-diagonal coupling b in the period matrix tends to zero, so the theta function tends to a product of elliptic theta functions. At an auxiliary extremum, stationarity forces one Hessian eigenvalue to vanish. A second-order expansion gives the other eigenvalue as $b^2\kappa^2 + O(b^3) > 0$. The Abel displacement δ associated with an interior pole then moves the extremum and makes the vanishing eigenvalue negative at order δ. The geometry is chosen so that $\delta = o(b^2)$, preserving the positive eigenvalue.

The proof has three parts. Section 2 isolates the positive transverse term in a theta expansion. Sections 3 and 4 realize the theta parameters by the circle domains and estimate their sizes. Section 5 identifies the maximizer with (4) and proves the theorem. Appendix A gives numerical illustrations and formulas for reproducing them.

2. A perturbation of a product theta function

2.1. **A nondegenerate one-dimensional maximum.**

Lemma 2.1. *For every $d > 0$, the function $-\ell_d'$ on $\mathbb{R}/(2\pi\mathbb{Z})$ has a unique maximum at a point $t_d \in (\pi/2, \pi)$. This maximum is nondegenerate. More precisely, $\ell_d''(t_d) = 0$ and $\kappa_d > 0$. The functions $d \mapsto t_d$ and $d \mapsto \kappa_d$ are smooth.*

Proof. Set $h_d = -\ell_d''$ and $q = e^{-d/2}$. The Jacobi product formula is

$$P_d(t) = \prod_{m\geq 1}(1-q^{2m})(1+2q^{2m-1}\cos t + q^{4m-2}).$$

Every factor is positive for real t. With $r_m = q^{2m-1}$, logarithmic differentiation gives

$$h_d(t) = 2\sum_{m\geq 1}\frac{r_m((1+r_m^2)\cos t + 2r_m)}{(1+2r_m\cos t + r_m^2)^2}. \tag{14}$$

The product and its real derivatives converge uniformly. In particular,

$$h_d(0) = 2\sum_{m\geq 1}\frac{r_m}{(1+r_m)^2} > 0,$$

$$h_d(\pi) = -2\sum_{m\geq 1}\frac{r_m}{(1-r_m)^2} < 0,$$

$$h_d(\pi/2) = 4\sum_{m\geq 1}\frac{r_m^2}{(1+r_m^2)^2} > 0.$$

For completeness, strict monotonicity on $(0,\pi)$ can be read off from the elliptic extension. The entire function P_d has simple zeros at

$$\pi + id/2 + 2\pi\mathbb{Z} + id\mathbb{Z},$$

and satisfies $P_d(t+id) = e^{d/2-it}P_d(t)$. Hence h_d, extended meromorphically, has periods $2\pi, id$ and one double pole per fundamental parallelogram, with principal part $(t-\pi-id/2)^{-2}$. Therefore

$$h_d(t) = \wp(t-\pi-id/2;\, 2\pi\mathbb{Z} + id\mathbb{Z}) + C_d. \tag{15}$$

This also follows from the standard theta–Weierstrass identities [DLMF, §23.6]. The zeros of $\wp'$ are exactly the three nonzero half-periods. The segment $t-\pi-id/2$, $0<t<\pi$, contains no half-period and no pole. Thus h_d' has no zero there. It is real and continuous, and the two endpoint values show that it is strictly negative.

It follows that h_d has exactly one zero t_d in $(0,\pi)$, necessarily in $(\pi/2,\pi)$. Evenness and periodicity give its only other zero on the real period circle, namely $2\pi - t_d$. Since $(-\ell_d')' = h_d$, the first zero is the unique maximum of $-\ell_d'$ and the other is its unique minimum. Moreover $\ell_d'''(t_d) = -h_d'(t_d) > 0$. Smooth dependence follows from the implicit function theorem, since this derivative does not vanish. □

2.2. **Uniform analytic expansion in the coupling.** For $|b| < d$ define

$$\Theta_{d,b}(z_1,z_2) = \sum_{m,n\in\mathbb{Z}} e^{-\frac{d}{2}(m^2+n^2)+bmn+mz_1+nz_2}, \tag{16}$$

$$\mathcal{L}_{d,b}(t_1,t_2) = \log\Theta_{d,b}(it_1,it_2), \qquad \mathcal{H}_{d,b}(it) = -D_t^2\mathcal{L}_{d,b}(t). \tag{17}$$

All variables t_j are understood modulo 2π. If

$$A = \begin{pmatrix} d & -b \\ -b & d \end{pmatrix} > 0,$$

Poisson summation gives

$$\Theta_{d,b}(it) = \frac{2\pi}{\sqrt{\det A}} \sum_{k\in\mathbb{Z}^2} \exp\left(-\frac{1}{2}(t-2\pi k)^T A^{-1}(t-2\pi k)\right) > 0. \tag{18}$$

Thus the real logarithm in (17) is defined everywhere on the real torus.

In the remainder of this section d lies in a sufficiently small compact neighborhood $I \Subset (0,\infty)$ of an arbitrary fixed $d_* > 0$. Constants in all remainder estimates can be chosen uniformly for $d \in I$.

Lemma 2.2. *Put $\ell = \ell_d$, and abbreviate $p_j = \ell'(t_j)$, $q_j = \ell''(t_j)$. Then, as $b \to 0$,*

$$\begin{aligned}\mathcal{L}_{d,b}(t_1,t_2) &= \ell(t_1) + \ell(t_2) - bp_1p_2 \\ &\qquad + \frac{b^2}{2}\left(q_1q_2 + q_1p_2^2 + p_1^2q_2\right) + O(b^3).\end{aligned} \tag{19}$$

The expansion holds in every fixed C^r norm on the real torus, uniformly for $d \in I$.

Proof. For $|b| \le b_0 < \min I$, the quadratic exponent is bounded above by $-(\min I - b_0)(m^2+n^2)/2$. Differentiating a summand in b, in d, or in t only introduces polynomial factors. All such differentiated series converge uniformly by Gaussian summability. This justifies Taylor expansion with a uniform remainder, also after any prescribed finite number of derivatives.

Let $P = P_d$ and $U(t_1,t_2) = P(t_1)P(t_2)$. Since differentiating twice, once in each real variable, multiplies a Fourier mode by $-mn$, expansion of e^{bmn} gives

$$\Theta_{d,b}(it) = U - b\,\partial_1\partial_2 U + \frac{b^2}{2}\partial_1^2\partial_2^2 U + O(b^3).$$

Dividing by U and using $P'/P = \ell'$ and $P''/P = \ell'' + (\ell')^2$, we obtain

$$\frac{\Theta_{d,b}(it)}{U} = 1 - bp_1p_2 + \frac{b^2}{2}(q_1+p_1^2)(q_2+p_2^2) + O(b^3).$$

The term $-\frac{1}{2}b^2p_1^2p_2^2$ in the logarithm cancels the corresponding fourth-degree product. This proves (19). A uniform positive lower bound for U, and then for $\Theta_{d,b}(it)$ for small b, follows from positivity and compactness. Thus taking logarithms and derivatives preserves the uniform remainder bounds. □

2.3. **A positive transverse eigenvalue.** Let $D = \partial_1 + \partial_2$ and define

$$S_{d,b}(t) = -D\mathcal{L}_{d,b}(t). \tag{20}$$

Proposition 2.3. *For sufficiently small $|b|$, $S_{d,b}$ has a unique global maximum on the real torus. It is nondegenerate and has the form $(u_{d,b}, u_{d,b})$,* *where* $u_{d,b} = t_d + O(b)$. *At this point the eigenvalues of* $\mathcal{H}_{d,b}$ *in the directions* $(1,1)$ *and* $(1,-1)$ *are, respectively,*

$$0, \qquad b^2\kappa_d^2 + O(b^3). \tag{21}$$

Moreover, if $\Lambda_{\parallel}(u)$ *is the* $(1,1)$ *eigenvalue of* $\mathcal{H}_{d,b}(iu, iu)$*, then*

$$\Lambda_{\parallel}'(u_{d,b}) = -\kappa_d + O(b). \tag{22}$$

Proof. At $b = 0$ the function in (20) is $-\ell_d'(t_1) - \ell_d'(t_2)$. By Lemma 2.1, its unique global maximum is (t_d, t_d) and its Hessian there is $-\kappa_d I_2$. The implicit function theorem gives a unique nearby critical point for small b, with negative definite Hessian. To verify global uniqueness, choose a small convex coordinate neighborhood on which the Hessian at $b = 0$ is uniformly negative definite. Outside this neighborhood the maximum has a strictly positive value gap, by compactness. Both the gap and strict concavity persist under a sufficiently small C^2 perturbation. These choices can be made uniformly for $d \in I$, after shrinking I if necessary.

The maximum is fixed by interchanging the two variables. In the chosen local coordinates it is therefore (u, u), with $u = t_d + O(b)$. Write $\mathcal{L} = \mathcal{L}_{d,b}$. Its stationarity equation is

$$\mathcal{L}_{11}(u, u) + \mathcal{L}_{12}(u, u) = 0. \tag{23}$$

Set $p = \ell_d'(u)$, $q = \ell_d''(u)$ and $r = \ell_d'''(u)$. Then $q = O(b)$ and $r = \kappa_d + O(b)$. Taking the mixed derivative in (19) gives the full second-order expression

$$\mathcal{L}_{12}(u, u) = -bq^2 + \frac{b^2}{2}(r^2 + 4pqr) + O(b^3) = \frac{b^2}{2}\kappa_d^2 + O(b^3). \tag{24}$$

Since $q = O(b)$ at the perturbed maximum, the contribution $-bq^2$ is of order b^3; the strictly positive term $b^2\kappa_d^2/2$ determines the leading sign.

By symmetry the diagonal entries of $\mathcal{H} = -D^2\mathcal{L}$ are equal. Equation (23) implies that its $(1,1)$ eigenvalue vanishes, while its $(1,-1)$ eigenvalue is $2\mathcal{L}_{12}$. This proves (21). Finally,

$$\Lambda_{\parallel}(u) = -(\mathcal{L}_{11} + \mathcal{L}_{12})(u, u).$$

At $b = 0$ this equals $-\ell_d''(u)$, whose derivative at t_d is $-\kappa_d$. Smooth dependence and $u_{d,b} = t_d + O(b)$ prove (22). □

2.4. **Moving the maximum into a finite theta ratio.** For $\delta > 0$ set

$$R_{d,b,\delta}(t) = \mathcal{L}_{d,b}(t - \delta\mathbf{1}) - \mathcal{L}_{d,b}(t), \qquad \mathbf{1} = (1,1). \tag{25}$$

Proposition 2.4. *For* $d \in I$ *and sufficiently small* $|b|, \delta > 0$, (25) *has a unique global maximum, at a point* $v_{d,b,\delta}\mathbf{1}$. *It satisfies*

$$v_{d,b,\delta} = u_{d,b} + \frac{\delta}{2} + O(\delta^2). \tag{26}$$

At this maximum the two eigenvalues of $\mathcal{H}_{d,b}$ *satisfy*

$$\lambda_{\|}(d,b,\delta) = -\frac{\kappa_d}{2}\delta + O(|b|\delta + \delta^2), \tag{27}$$

$$\lambda_{\perp}(d,b,\delta) = b^2\kappa_d^2 + O(|b|^3 + \delta). \tag{28}$$

All remainders are uniform for the indicated parameters.

Proof. A centered Taylor expansion, in C^r for any fixed finite r, gives

$$\frac{R_{d,b,\delta}(w + \frac{\delta}{2}\mathbf{1})}{\delta} = -D\mathcal{L}_{d,b}(w) - \frac{\delta^2}{24}D^3\mathcal{L}_{d,b}(w) + O(\delta^4). \tag{29}$$

Proposition 2.3 provides a unique nondegenerate maximum for the first term. The same compactness and strict-concavity argument used there gives a unique global maximum of (29). The gradient perturbation is $O(\delta^2)$ and the inverse Hessian stays bounded, so its location is $u_{d,b}\mathbf{1} + O(\delta^2)$. Exchange symmetry makes the two coordinates equal. This proves (26).

Taylor expansion of $\Lambda_{\|}$ at $u_{d,b}$, using its exact zero there and (22), yields (27). The transverse eigenvalue changes by $O(\delta)$ when the point moves by $O(\delta)$; combining this with (21) gives (28). □

Remark 2.5. The negative eigenvalue has no error term independent of δ. This is a consequence of the exact row-null relation (23). It permits the choice $\delta \ll b^2$. A bound of the form $-\kappa_d\delta/2 + O(b^2+\delta^2)$ would be insufficient for the construction.

3. Period normalization from harmonic measures

We record the relation between periods and the planar Dirichlet problem with orientations, in order to fix the signs and factors in the construction.

Let Γ_0 be the outer boundary of a regular $(g+1)$-connected planar domain, and let $\Gamma_1, \ldots, \Gamma_g$ be the inner boundaries. Let ω_j be the harmonic function with boundary values 1 on Γ_j and 0 on the other components. Define

$$C_{jk} = \int_\Omega \nabla\omega_j \cdot \nabla\omega_k \, dA = \int_{\Gamma_j} \partial_\nu \omega_k \, ds, \tag{30}$$

where ν is the outward unit normal of Ω, pointing into a hole on an inner boundary. The equality follows from Green's identity. The matrix C is symmetric positive definite: the energy of a linear combination can vanish only if that combination is constant, and its value on Γ_0 is zero.

Use the Hodge star $*dx = dy$, $*dy = -dx$, and set

$$u_k = -i\pi(d\omega_k + i * d\omega_k). \tag{31}$$

The expression $d\omega_k + i{*}d\omega_k = 2\partial\omega_k$ is a holomorphic differential. Its multiple u_k is real on tangent vectors to every boundary, because $d\omega_k$ vanishes on those vectors. It therefore extends by Schwarz reflection to a holomorphic differential on the double, with $\phi^* u_k = \overline{u_k}$.

Orient B_j counterclockwise around the jth hole. Let A_j run in the front copy from Γ_j to Γ_0 and return along the reflected path in the back copy.

These paths can be chosen as the usual disjoint crosscuts; with these orientations they form a canonical basis. If I is the integral of u_k along the front path, the full A_j-integral is $I - \overline{I}$. Since the imaginary part of a local primitive of u_k is $-\pi\omega_k$ plus a constant, this gives

$$\int_{A_j} u_k = 2\pi i \delta_{jk}. \tag{32}$$

Furthermore, $\int_{B_j} d\omega_k = 0$, and $\int_{B_j} *d\omega_k$ is the flux with normal pointing out of the hole, the negative of the normal in (30). Consequently,

$$\tau_{jk} = \int_{B_j} u_k = -\pi C_{jk}. \tag{33}$$

Lemma 3.1. *Evaluate $p - \phi(p)$ along a symmetric path from $\phi(p)$ to p passing through Γ_0. Then, in the above normalized coordinates,*

$$p - \phi(p) = -2\pi i \big(\omega_1(p), \dots, \omega_g(p)\big). \tag{34}$$

Proof. Let J_k be the integral of u_k on the front half of the path, from Γ_0 to p. The full integral is $J_k - \overline{J_k}$, and $\operatorname{Im} J_k = -\pi\omega_k(p)$. This is (34). Changing the admissible paths by A-periods only adds $2\pi i \mathbb{Z}^g$, which does not affect theta values or their Hessians. $\square$

Remark 3.2 (An annulus check). For the annulus $\rho < |z| < 1$, the harmonic measure is $\log|z|/\log\rho$ and its energy is $2\pi/\log(1/\rho)$. Formula (33) gives the scalar period $-2\pi^2/\log(1/\rho)$. This verifies the same factor d_* as in (6).

4. Explicit estimates for the circle domains

In this section $c \to 1^-$ and ρ is fixed. Let Ω_c be the domain in (8), and label the inner harmonic measures by ω_+, ω_-. We suppress c from some notation. Reflection $z \mapsto -\overline{z}$ interchanges the holes and preserves the outer circle, so

$$\pi C = \begin{pmatrix} d & -b \\ -b & d \end{pmatrix}. \tag{35}$$

Here $b > 0$: on either inner boundary the other harmonic measure vanishes, is strictly positive in the interior, and has strictly negative outward normal derivative by the Hopf lemma. Hence $C_{+-} < 0$.

4.1. Comparison with the exact single-hole solutions.

Put

$$W_\pm(z) = -\frac{1}{L} \log \left| \frac{z \mp c}{1 \mp cz} \right|. \tag{36}$$

This is the harmonic measure of the hole $K_\pm$ in the domain with that hole alone. It is 1 on its own inner circle, 0 on the unit circle, and has single-hole flux

$$C_* = \frac{2\pi}{L}, \qquad d_* = \pi C_*. \tag{37}$$

For the interaction parameter define

$$a = \frac{2c}{1+c^2}, \qquad \gamma = -\log a. \tag{38}$$

For c sufficiently close to 1, $a > \rho$. The automorphism $u = (z+c)/(1+cz)$ maps K_- to $\{|u| \le \rho\}$, and in this coordinate

$$W_+(z(u)) = -\frac{1}{L}\log\left|\frac{u-a}{1-au}\right|. \tag{39}$$

It follows that the minimum and maximum of W_+ on ∂K_- are

$$\alpha = \frac{1}{L}\log\frac{1+a\rho}{a+\rho}, \qquad \beta = \frac{1}{L}\log\frac{1-a\rho}{a-\rho}. \tag{40}$$

The same numbers apply with $+$ and $-$ interchanged. Both are strictly positive, and

$$\alpha = \frac{1-\rho}{L(1+\rho)}\gamma + O(\gamma^2), \qquad \beta = \frac{1+\rho}{L(1-\rho)}\gamma + O(\gamma^2). \tag{41}$$

Lemma 4.1. *Let $V_+ = W_+ - \omega_+$. Then*

$$\alpha\omega_- \le V_+ \le \beta\omega_- \quad \text{on } \Omega_c. \tag{42}$$

The analogous inequalities hold with the labels interchanged. Moreover,

$$\alpha d \le b \le \beta d, \tag{43}$$

$$\alpha b \le d - d_* \le \beta b. \tag{44}$$

In particular, when $\beta < 1$,

$$\frac{d_*}{1-\alpha^2} \le d \le \frac{d_*}{1-\beta^2}. \tag{45}$$

Thus $d = d_* + O(\gamma^2)$ and $b \asymp \gamma$.

Proof. The function V_+ is harmonic, vanishes on Γ_0 and ∂K_+, and equals $W_+ \in [\alpha, \beta]$ on ∂K_-. The maximum principle proves (42).

Green's second identity for V_+ and ω_- gives

$$\int_{\partial K_-} \partial_\nu V_+ \, ds = \int_{\partial K_-} W_+ \partial_\nu \omega_- \, ds. \tag{46}$$

The flux of W_+ around K_- is zero because W_+ is harmonic through that disk. Therefore the left side is $-C_{-+}$. On its own inner boundary $\partial_\nu\omega_- > 0$, with total flux C_{--}. Bounding W_+ between α and β in (46), and multiplying by π, proves (43).

On ∂K_+ all three functions in (42) vanish. Differentiation in the outward normal of Ω_c reverses the inequalities:

$$\beta\partial_\nu\omega_- \le \partial_\nu V_+ \le \alpha\partial_\nu\omega_-.$$

Their integrated forms are

$$\beta C_{+-} \le C_* - C_{++} \le \alpha C_{+-}.$$

Multiplication by $-\pi$ yields (44). Combining (43) and (44) gives $\alpha^2 d \le d - d_* \le \beta^2 d$, which is exactly (45). Finally (41) proves the stated orders and the positive lower bound implicit in $b \asymp \gamma$. $\square$

4.2. **The leading interaction coefficient.** The preceding estimates already suffice to prove opposite signs. The next refinement gives the explicit leading coefficient in Theorem 1.1.

Proposition 4.2. *As $c \to 1^-$,*

$$b = \frac{2\pi^2}{L^2}\gamma + O(\gamma^3). \tag{47}$$

Consequently, with $c = 1 - \varepsilon$,

$$d = d_* + O(\varepsilon^4), \qquad b = \frac{\pi^2}{L^2}\varepsilon^2 + O(\varepsilon^3). \tag{48}$$

Proof. Work in the u coordinate of (39). Choose a fixed r_0 with $\rho < r_0 < 1$. For c near 1 the other hole is outside $|u| \le r_0$. On this fixed disk the cross potential in (39), and each fixed finite number of its derivatives, is $O(\gamma)$: the logarithm is smooth in (u, a) for $|u| \le r_0$ and a near 1, and it vanishes at $a = 1$.

Use (42) with the labels interchanged:

$$0 \le V_- := W_- - \omega_- \le \beta\omega_+ \le \beta W_+.$$

Thus $V_- = O(\gamma^2)$ on the fixed collar $\rho \le |u| \le r_0$, and $V_- = 0$ on $|u| = \rho$. Comparing with a constant multiple of

$$\gamma^2 \frac{\log(|u|/\rho)}{\log(r_0/\rho)}$$

bounds its normal derivative on $|u| = \rho$ by $O(\gamma^2)$, uniformly along that circle. Since $W_-(z(u)) = -\log|u|/L$, the transformed harmonic measure $\widetilde{\omega}_-(u) = \omega_-(z(u))$ satisfies

$$\partial_{\nu_u}\widetilde{\omega}_-(u) = \frac{1}{\rho L} + O(\gamma^2), \qquad |u| = \rho. \tag{49}$$

Flux measure is conformally invariant, so (46) becomes

$$\frac{b}{\pi} = \int_{|u|=\rho} \left(-\frac{1}{L}\log\left|\frac{u-a}{1-au}\right|\right) \partial_{\nu_u}\widetilde{\omega}_-(u)\, ds_u.$$

The logarithmic factor is $O(\gamma)$ on this circle. Its mean is exactly γ/L: it is harmonic on $|u| \le \rho$ and its value at 0 is $-\log a/L$. Inserting (49) gives

$$\frac{b}{\pi} = \frac{1}{\rho L}\frac{2\pi\rho\gamma}{L} + O(\gamma^3),$$

which proves (47). Lastly,

$$\gamma = \log\frac{1+c^2}{2c} = \log\left(1 + \frac{\varepsilon^2}{2(1-\varepsilon)}\right) = \frac{1}{2}\varepsilon^2 + O(\varepsilon^3).$$

This and Lemma 4.1 prove (48). $\square$

4.3. **A pole whose displacement is smaller than the squared coupling.**

Proposition 4.3. *At* $p_\varepsilon = i(1-\varepsilon^5)$*,*

$$\omega_+(p_\varepsilon) = \omega_-(p_\varepsilon) =: h_\varepsilon, \tag{50}$$

and

$$\delta_\varepsilon := 2\pi h_\varepsilon = \frac{2\pi}{L}\varepsilon^6 + O(\varepsilon^7). \tag{51}$$

In particular $\delta_\varepsilon / b_\varepsilon^2 \to 0$*.*

Proof. The reflection $z \mapsto -\overline{z}$ fixes p_ε, so the two measures are equal. The single-hole potentials are also equal there; denote their common value by w_ε. Evaluating (42) at this point gives

$$\alpha h_\varepsilon \le w_\varepsilon - h_\varepsilon \le \beta h_\varepsilon, \qquad \frac{w_\varepsilon}{1+\beta} \le h_\varepsilon \le \frac{w_\varepsilon}{1+\alpha}.$$

Since $\alpha, \beta = O(\varepsilon^2)$, it follows that $h_\varepsilon = w_\varepsilon(1 + O(\varepsilon^2))$.

For $y = 1-\varepsilon^5$ and $c = 1-\varepsilon$, the value is explicit:

$$\begin{aligned} w_\varepsilon &= -\frac{1}{2L}\log\frac{y^2+c^2}{1+c^2y^2} \\ &= -\frac{1}{2L}\log\left(1 - \frac{(1-c^2)(1-y^2)}{1+c^2y^2}\right). \end{aligned} \tag{52}$$

The fraction in (52) equals $2\varepsilon^6 + O(\varepsilon^7)$. Taylor expansion of $-\log(1-x)$ therefore yields $w_\varepsilon = \varepsilon^6/L + O(\varepsilon^7)$. This proves (51). Proposition 4.2 gives $b_\varepsilon^2 \asymp \varepsilon^4$, proving the last assertion. $\square$

5. Identification with the Green-function parameter

The following is the specific external extremal theorem used in the proof.

Theorem 5.1 (Yamada, [Yam99, Theorem 2.1])**.** *Let* Ω *be a regular planar domain, and use the normalized symmetric homology basis and theta convention of Section 3. For* $p \in \Omega$*, evaluate* $p - \phi(p)$ *along a symmetric path. On* $i\mathbb{R}^g/(2\pi i\mathbb{Z}^g)$ *the positive function*

$$e \longmapsto \frac{\theta_\tau(p - \phi(p) + e)}{\theta_\tau(e)} \tag{53}$$

has its unique maximum at $e = e_0(p)$*. Here* $e_0(p)$ *is the Green critical-divisor parameter in (4), with multiplicities. In particular this characterization does not require the critical points to be simple.*

We use this theorem with its uniqueness assertion, not a converse inference from the scalar Saitoh inequality. The sign question for the individual matrix $H_\tau(e_0(p))$ is not part of the quoted theorem.

Proof of Theorem 1.1. For Ω_ε, equations (33) and (35) identify its theta function with $\Theta_{d_\varepsilon,b_\varepsilon}$. The parameters satisfy

$$d_\varepsilon = d_* + O(\varepsilon^4), \qquad b_\varepsilon = \frac{\pi^2}{L^2}\varepsilon^2 + O(\varepsilon^3) > 0.$$

By Lemma 3.1 and Proposition 4.3,

$$p_\varepsilon - \phi(p_\varepsilon) = -i\delta_\varepsilon(1,1), \qquad \delta_\varepsilon = \frac{2\pi}{L}\varepsilon^6 + O(\varepsilon^7) > 0.$$

Taking the logarithm of (53), with $e = it$, gives exactly $R_{d_\varepsilon,b_\varepsilon,\delta_\varepsilon}(t)$ from (25). Both Theorem 5.1 and Proposition 2.4 characterize its unique global maximum. Hence

$$e_0(p_\varepsilon) = iv_{d_\varepsilon,b_\varepsilon,\delta_\varepsilon}(1,1) \pmod{2\pi i\mathbb{Z}^2}. \tag{54}$$

The Hessian is $2\pi i\mathbb{Z}^2$-periodic, so this ambiguity is irrelevant.

Smooth dependence in Lemma 2.1 yields $\kappa_{d_\varepsilon} = \kappa_* + O(\varepsilon^4)$. Substitution into (27) gives

$$\lambda_\parallel = -\frac{\kappa_*}{2}\frac{2\pi}{L}\varepsilon^6 + O(\varepsilon^7) + O(\varepsilon^8) + O(\varepsilon^{12}),$$

which is (12). Likewise (28) gives

$$\lambda_\perp = \kappa_*^2\left(\frac{\pi^2}{L^2}\varepsilon^2 + O(\varepsilon^3)\right)^2 + O(\varepsilon^6) = \frac{\pi^4\kappa_*^2}{L^4}\varepsilon^4 + O(\varepsilon^5).$$

This is (13). The constants in both leading terms are strictly positive because $L > 0$ and $\kappa_* > 0$. □

Corollary 5.2. *For c sufficiently close to 1, fix the domain Ω_c with the two holes (7). There exists $y_0(c) < 1$ such that $H_{\tau_c}(e_0(iy))$ is indefinite for every $y_0(c) < y < 1$.*

Proof. For fixed such c, the transverse eigenvalue at $\delta = 0$ is positive by Proposition 2.3, since $b_c > 0$ is sufficiently small. Proposition 2.4 makes the longitudinal eigenvalue negative for all sufficiently small $\delta > 0$, while the transverse one stays positive. Along the imaginary axis the common harmonic measure is positive inside and tends to zero at i. Theorem 5.1 identifies these maxima with $e_0(iy)$ just as above. □

6. The definiteness problem and the scalar inequality

The example disproves the universal negative-definiteness assertion (5) already in genus two. It does not classify the signature of $H_\tau(e)$ at every point of every Jacobian. In particular, it does not contradict positivity at $e = 0$: directly from (1), $H_\tau(0)$ is the covariance matrix of a strictly positive distribution on $\mathbb{Z}^g$, and is positive definite.

The distinction between a scalar kernel inequality and matrix definiteness is essential here. The extremal characterization used in Theorem 5.1 controls the variation of a *ratio* of theta functions. The second derivative of its logarithm compares the Hessians at two points; it does not prescribe the signature at either point separately.

Yamada's relative comparison [Yam99, Theorem 3.2] concerns the difference of Hessians at two translated points. Such a difference can be positive definite even when one of the individual Hessians is indefinite. Likewise, the scalar Saitoh inequality proved in [Gua19] does not imply that every real direction in the theta parameter space is negative. The counterexample addresses precisely this distinction.

Appendix A. Numerical illustrations

We give two formulas for evaluating the Hessian and illustrate the asymptotic regimes in Theorem 1.1 and Proposition 2.3. The accompanying NumPy script implements these formulas and a circular-multipole approximation of harmonic measures. These floating-point computations are supplementary: the existence assertion in Theorem 1.1 follows from the analytic estimates in Sections 2–5. Boundary collocation residuals are not used as rigorous error bounds.

A.1. **Two independent formulas for the theta Hessian.** Write

$$A = \begin{pmatrix} d & -b \\ -b & d \end{pmatrix}, \qquad w_n(t) = e^{-n^T An/2+in\cdot t}, \qquad Z(t) = \sum_{n\in\mathbb{Z}^2} w_n(t).$$

Direct differentiation of the Fourier series gives

$$H_{jk}(it) = \frac{\sum_n n_j n_k w_n(t)}{Z(t)} - \frac{\left(\sum_n n_j w_n(t)\right)\left(\sum_n n_k w_n(t)\right)}{Z(t)^2}. \tag{55}$$

Here the products do not use complex conjugation. A second formula follows from (18). Give $X = 2\pi k$ probability proportional to $\exp[-(t - 2\pi k)^T A^{-1}(t-2\pi k)/2]$. Then

$$H(it) = A^{-1} - A^{-1}\operatorname{Cov}(X)A^{-1}. \tag{56}$$

The script compares (55) and (56), rather than inferring signs from finite differences of a logarithm.

For $d = 4$, the scalar constants are approximately

$$t_d = 1.8354437625, \qquad \kappa_d = 0.30050166, \qquad \kappa_d^2 = 0.09030125.$$

At the maximum of $S_{d,b}$, the values in Table 1 illustrate the convergence $\lambda_\perp/b^2 \to \kappa_d^2$.

Table 1. The transverse eigenvalue at the auxiliary maximum, with $d = 4$.

b	$\lambda_\perp$	$\lambda_\perp/b^2$
0.100	8.8096501×10^{-4}	0.08809650
0.050	2.2312796×10^{-4}	0.08925118
0.020	3.5957394×10^{-5}	0.08989348
0.010	9.0099383×10^{-6}	0.09009938
0.005	2.2550204×10^{-6}	0.09020082

A.2. **Harmonic measure by circular multipoles.** For numerical evaluation only, write the two Euclidean holes as $|z - c_j| \le r_j$. Harmonic basis functions vanishing on the unit circle are

$$B_{j,0}(z) = \log\left|\frac{z - c_j}{1 - \overline{c_j}z}\right|,$$
$$B^{\mathrm{R}}_{j,n}(z) = \mathrm{Re}\left[\left(\frac{r_j}{z - c_j}\right)^n - \left(\frac{r_j z}{1 - \overline{c_j}z}\right)^n\right],$$
$$B^{\mathrm{I}}_{j,n}(z) = \mathrm{Im}\left[\left(\frac{r_j}{z - c_j}\right)^n + \left(\frac{r_j z}{1 - \overline{c_j}z}\right)^n\right], \qquad n \ge 1.$$

Boundary collocation determines coefficients approximating each ω_k. The logarithmic coefficient a_{jk} determines the flux exactly for that finite harmonic approximation, since the multipoles and image terms have zero net flux:

$$C^{\mathrm{approx}}_{jk} = -2\pi a_{jk}, \qquad A^{\mathrm{approx}}_{jk} = -2\pi^2 a_{jk}.$$

Increasing both the multipole order and the number of boundary sample points tests numerical stability. The exact bounds (43) and (45) provide additional checks independent of collocation.

For the illustrative choice $\rho = e^{-\pi^2/2}$, $c = 0.85$, and $p = 0.99999i$, representative numerical values are

$$A \approx \begin{pmatrix} 4.0000284261 & -0.0106578265 \\ -0.0106578265 & 4.0000284261 \end{pmatrix},$$

$$\omega_+(p) = \omega_-(p) \approx 3.25596257 \times 10^{-7}.$$

The eigenvalues at the numerically located maximum are approximately

$$\lambda_\| \approx -3.074 \times 10^{-7}, \qquad \lambda_\perp \approx 9.925 \times 10^{-6}.$$

These particular decimal parameters are a numerical illustration, not a separately certified explicit finite-parameter counterexample. Theorem 1.1 establishes the counterexample family by analytic asymptotics.

A.3. **Avoiding cancellation for small displacements.** On the diagonal let $J(u) = \mathcal{L}_{d,b}(u,u)$. The critical-point equation for the finite ratio is $J'(v-\delta) = J'(v)$. Direct subtraction loses precision when δ is tiny. Since $J'' = -2\Lambda_{\|}$, the same equation is

$$\int_{-1/2}^{1/2} \Lambda_{\|}(w+\delta s)\,ds = 0, \qquad v = w + \delta/2.$$

This equation can be solved using Gaussian quadrature and a bracketed root finder near $u_{d,b}$. At an approximate root w, the expression

$$\Lambda_{\|}(w+\delta/2) - \int_{-1/2}^{1/2} \Lambda_{\|}(w+\delta s)\,ds$$

reduces the effect of a residual common to the averaged and endpoint curvatures. At the exact root it equals the longitudinal eigenvalue. The accompanying calculation records both this value and the uncorrected endpoint curvature, compares two quadrature orders, and compares (55) with (56).

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Declaration on the use of artificial intelligence

The author selected Yamada's definiteness problem, suggested searching for a counterexample, and supplied a related earlier counterexample as context. The new domain construction, theta expansions, harmonic-measure estimates, proofs, numerical calculations, and exposition in this paper were developed with OpenAI Codex (GPT-6). The use of AI therefore included mathematical research and argumentation, as well as writing. Previously established results used in the proof are identified by citations.

This manuscript is released for mathematical examination. Arguments generated with AI require the author's mathematical verification before submission. The numerical checks reported here do not constitute independent verification or peer review.

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