LOG DISCREPANCY FROM JUMPING NUMBERS

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ABSTRACT. We show that the invariant $\mathscr{A}$ introduced by Bao–Guan–Yuan equals log discrepancy for every nontrivial real valuation on an excellent regular local domain containing $\mathbb{Q}$, including valuations of infinite discrepancy. The proof is a short compactness argument; principal auxiliary ideals suffice.

1. THE STATEMENT

Let R be an excellent regular local domain containing $\mathbb{Q}$, and let Val_R^* be the nontrivial real valuations of $\mathrm{Frac}(R)$ nonnegative on R. Write $A(v)$ for log discrepancy. This includes $R = \mathcal{O}_{\mathbb{C}^n,o}$; the center of a valuation need not be closed. For a nonzero graded sequence $\mathfrak{b}_\bullet$ and a nonzero ideal $\mathfrak{q}$, put

$$v(\mathfrak{b}_\bullet) = \inf_{m\geq 1} \frac{v(\mathfrak{b}_m)}{m}, \qquad \mathrm{lct}^{\mathfrak{q}}(\mathfrak{b}_\bullet) = \sup_{m\geq 1} m\,\mathrm{lct}^{\mathfrak{q}}(\mathfrak{b}_m),$$

with the usual conventions at zero terms. The invariant introduced in Bao–Guan–Yuan [BGY], written here over R, is

$$\mathscr{A}(v) = \sup_{\substack{\mathfrak{q}\neq 0,\, \mathfrak{b}_\bullet\neq 0\\ \mathrm{lct}^{\mathfrak{q}}(\mathfrak{b}_\bullet)<\infty}} \big(v(\mathfrak{b}_\bullet)\,\mathrm{lct}^{\mathfrak{q}}(\mathfrak{b}_\bullet) - v(\mathfrak{q})\big). \tag{1}$$

The unit ideal is allowed. The valuative formula of Jonsson–Mustaţă [JM, Corollary 6.9] immediately gives $\mathscr{A}(v) \leq A(v)$.

Bao–Guan–Yuan [BGY] proved jumping-number formulas involving $\mathscr{A}$, with $\mathscr{A} = A$ for quasimonomial valuations. Bao–Guan–Zhou [BGZ] obtained many corresponding results with A in place of $\mathscr{A}$, including $A(v) = 1 - v(\mathfrak{q})$ for Zhou valuations related to $\mathfrak{q}$. This suggested to the author the question of equality for all valuations. We give a short proof using [JM].

Theorem 1.1. *For every $v \in \mathrm{Val}_R^*$, with $\mathfrak{a}_m^v = \{f \in R : v(f) \geq m\}$, one has*

$$A(v) = \mathscr{A}(v) = \sup_{0\neq f\in R} \big(\mathrm{lct}^{(f)}(\mathfrak{a}_\bullet^v) - v(f)\big). \tag{2}$$

The equalities hold in $[0, +\infty]$. In (1), it suffices to use principal auxiliary ideals and sequences of ordinary powers of a single nonzero proper ideal.

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2. The proof

Order valuations by $w \geq v$ if $w(f) \geq v(f)$ for all $0 \neq f \in R$, and put $D_v = \{w \in \mathrm{Val}_R^* : w \geq v\}$. All topological statements refer to pointwise convergence on R.

Lemma 2.1. *For every finite $L > 0$, the set $K_L = \{w \in D_v : A(w) \leq L\}$ is compact.*

Proof. For each nonzero nonunit f, the single-ideal valuative inequality gives

$$0 \leq w(f) \leq \frac{L}{\mathrm{lct}((f))} \qquad (w \in K_L),$$

where $0 < \mathrm{lct}((f)) < \infty$. Units have value zero. Thus K_L lies in a product of compact intervals with finite endpoints. In its closure, multiplicativity and the valuation inequality persist, since they are closed conditions on finitely many coordinates. Every limit is finite on $R \setminus \{0\}$ and extends to a valuation of $\mathrm{Frac}(R)$ by the quotient rule. Domination of v persists as well and excludes the trivial valuation. Finally, lower semicontinuity of A [JM, Lemma 5.7] preserves $A \leq L$. Hence K_L is closed in that compact product. This is the same compactness argument as in [JM, Proposition 5.9]. □

Proof of Theorem 1.1. We first record the useful form of the valuative formula:

$$\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet^v) = \inf_{w \in D_v} \big(A(w) + w(\mathfrak{q})\big). \tag{3}$$

Indeed, [JM, Lemma 2.4] gives

$$u(\mathfrak{a}_\bullet^v) = \inf_{\substack{0 \neq f \in R \\ v(f) > 0}} \frac{u(f)}{v(f)}.$$

Thus $u(\mathfrak{a}_\bullet^v) \geq 1$ if and only if $u \geq v$, and $v(\mathfrak{a}_\bullet^v) = 1$. Every candidate u with $b = u(\mathfrak{a}_\bullet^v) > 0$ in the valuative formula can be rescaled to $w = u/b \in D_v$, with $A(w) + w(\mathfrak{q}) = (A(u) + u(\mathfrak{q}))/b$. Conversely, for $w \in D_v$ the latter quotient is at most $A(w) + w(\mathfrak{q})$, since $w(\mathfrak{a}_\bullet^v) \geq 1$. These two inequalities prove (3), also when the infimum is infinite.

Let $S(v)$ be the principal-ideal supremum in (2). We have $S(v) \leq \mathscr{A}(v) \leq A(v)$. For the first inequality, fix $\mathfrak{q} = (f)$ and $m \geq 1$, and test (1) with the powers of $I = \mathfrak{a}_m^v$. This ideal is nonzero and proper, so $\mathrm{lct}^{\mathfrak{q}}(I) < \infty$. Since $v(I) \geq m$, the test gives

$$\mathscr{A}(v) \geq v(I)\,\mathrm{lct}^{\mathfrak{q}}(I) - v(f) \geq m\,\mathrm{lct}^{\mathfrak{q}}(\mathfrak{a}_m^v) - v(f).$$

Taking suprema over m and f gives the assertion, even if $S(v) = +\infty$. The same argument applies to the restricted supremum in the last sentence of the theorem.

It remains to prove $S(v) \geq A(v)$. Fix $0 < L < A(v)$ with L finite. If $K_L = \varnothing$, (3) with $\mathfrak{q} = R$ already gives $S(v) \geq L$. Otherwise, each $w \in K_L$

differs from v, so $w(f) > v(f)$ for some $0 \neq f \in R$. The open sets

$$\{w \in K_L : w(f) > v(f)\}, \qquad 0 \neq f \in R,$$

cover K_L. By Lemma 2.1, a finite subcover $f_1, \dots, f_r$ suffices. Set $g = f_1 \cdots f_r \neq 0$. Every summand in

$$w(g) - v(g) = \sum_{i=1}^{r} \big(w(f_i) - v(f_i)\big)$$

is nonnegative, and at least one is positive. Compactness and continuity therefore give $\delta = \min_{w \in K_L}(w(g) - v(g)) > 0$. Choose an integer $k > L/\delta$. For $w \in K_L$, $A(w) + k(w(g) - v(g)) \geq k\delta > L$; for $w \in D_v \setminus K_L$, the same expression exceeds L because $A(w) > L$ and $w \geq v$. Applying (3) to $\mathfrak{q} = (g^k)$ yields

$$\operatorname{lct}^{(g^k)}(\mathfrak{a}^v_\bullet) - kv(g) = \inf_{w \in D_v} \big(A(w) + k(w(g) - v(g))\big) \geq L.$$

Consequently $S(v) \geq L$ for every finite $0 < L < A(v)$, proving $S(v) \geq A(v)$, including $A(v) = +\infty$. If $A(v) = 0$, the same conclusion follows from the nonnegative test $f = 1$. □

Thus $A(v) < \infty$ is equivalent to a uniform upper bound on $\operatorname{lct}^{\mathfrak{q}}(\mathfrak{a}^v_\bullet) - v(\mathfrak{q})$ over all nonzero $\mathfrak{q}$. The theorem does not assert attainment by a single ideal or that every valuation computes a jumping number.

References

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Declaration on the use of artificial intelligence

The question and research idea originate with Shijie Bao, motivated by [BGY, BGZ]. The proof and exposition were developed with GPT (Astra). This note has not undergone peer review.

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