Counterexamples to ξ-reduction and minimal-integral duality below one

Shijie Bao
Department of Mathematics, Graduate School of Science
Osaka Metropolitan University
3-3-138 Sugimoto, Sumiyoshi-ku Osaka, Japan 558-8585
shijie.bao@omu.ac.jp

Date: August 24, 2026

Abstract

For $p \geq 1$, higher order p-Bergman kernels admit a reduction to p-Bergman kernels with respect to functionals ξ. We show that this identity fails below one. On the unit disk with $H(z) = z$, the failure occurs for every

$$0 < p < p_* := 2(\sqrt{2} - 1),$$

and the case $p = 1/2$ is computed explicitly. The same extremal model yields counterexamples, in the identical range, to the natural L^p analogue of the minimal-integral duality $B(F, I, D) = C_{F,I}(D)$. In the opposite direction, we prove that the L^p duality holds for every $p \geq 1$, every pseudoconvex domain, every ideal, and every holomorphic germ.

Contents

1 Introduction

For $p > 0$ and a homogeneous polynomial H of degree k, the higher order p-Bergman kernel $K_D^{H,p}(z)$ and, for a functional $\xi \in \ell_1^{(n)}$, the p-Bergman kernel with respect to ξ $K_{\xi,D,p}(z)$ are defined in Section 2. Bao–Guan–Sun [3] proved that for every $p \in [1, +\infty)$,

$$K_D^{H,p}(z) = \inf_{\xi \in S_H} K_{\xi,D,p}(z),$$

where S_H is the affine family of functionals whose degree-k part is prescribed by H; see Theorem 1.7 in [3].

Mathematics Subject Classification (2020): 32A36.
Keywords: Higher order p-Bergman kernel, ξ-Bergman kernel, minimal L^p integral, duality, nonconvex extremal problem

It is natural to ask: whether the identity persists for $0 < p < 1$? This counterexample answers this question in the negative. Already in the simplest one-dimensional case the equality fails.

THEOREM 1.1. *Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk, $p = 1/2$, $H(z) = z$ and $z = 0$. Then*

$$K_{\mathbb{D}}^{H,1/2}(0) = \frac{5}{4\pi},$$

while

$$\inf_{\xi \in S_H} K_{\xi,\mathbb{D},1/2}(0) > \frac{5}{4\pi}.$$

Consequently the identity

$$K_{\mathbb{D}}^{H,1/2}(0) = \inf_{\xi \in S_H} K_{\xi,\mathbb{D},1/2}(0)$$

fails, and the infimum is not attained at the common value.

As shown in Section 3, the same example also answers in the negative the natural L^p analogue of the identity $B(F, I, D) = C_{F,I}(D)$ of Bao–Guan [2].

2 The ξ-reduction fails for $0 < p < 1$

2.1 The counterexample on the unit disk

Let $D \subset \mathbb{C}^n$ be a domain, $p > 0$, and let

$$H(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$$

be a homogeneous polynomial of degree k. Following [3], denote by

$$\ell_1^{(n)} := \Big\{ \xi = (\xi_\alpha)_{\alpha \in \mathbb{N}^n} : \ \xi_\alpha \in \mathbb{C}, \ \sum_{\alpha \in \mathbb{N}^n} |\xi_\alpha| \rho^{|\alpha|} < +\infty \text{ for every } \rho > 0 \Big\}$$

the space of functionals on holomorphic germs. For $\xi \in \ell_1^{(n)}$ and $f \in \mathcal{O}(D)$, the pairing is defined by

$$(\xi \cdot f)(z) := \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha \frac{D^\alpha f(z)}{\alpha!}.$$

For the homogeneous polynomial H of degree k above, let S_H denote the affine family of all $\xi \in \ell_1^{(n)}$ such that

$$\xi_\alpha = \begin{cases} a_\alpha \alpha!, & |\alpha| = k, \\ 0, & |\alpha| > k, \end{cases}$$

the entries with $|\alpha| < k$ being free. With this notation, the higher order p-Bergman kernel and the p-Bergman kernel with respect to ξ are

$$K_D^{H,p}(z) := \sup\Big\{ |P_H(f)(z)|^p : \ f \in A^p(D), \ f^{(j)}(z) = 0 \ (0 \le j < k), \ \|f\|_{L^p(D)} \le 1 \Big\},$$

$$K_{\xi,D,p}(z) := \sup\Big\{ |(\xi \cdot f)(z)|^p : \ f \in A^p(D), \ \|f\|_{L^p(D)} \le 1 \Big\}.$$

For a positive homogeneous functional, equivalently,

$$K_{\xi,D,p}(z) = \frac{1}{\inf\bigl\{\|f\|^p_{L^p(D)} :\ f \in A^p(D),\ (\xi \cdot f)(z) = 1\bigr\}}.$$

We answer the question whether the identity in Theorem 1.7 of [3] extends to $0 < p < 1$ in the negative. Already the one-dimensional model $D = \mathbb{D}$, $H(z) = z$, $p = 1/2$ gives a counterexample. For $H(z) = z$ the set S_H consists of all sequences

$$\xi = (\xi_0, 1, 0, 0, \ldots), \qquad \xi_0 \in \mathbb{C}.$$

2.1.1 Proof of Theorem 1.1

Proof of Theorem 1.1. We first compute $K^{H,1/2}_{\mathbb{D}}(0)$. Let $f \in \mathcal{O}(\mathbb{D})$ satisfy $f(0) = 0$ and $f'(0) = 1$, and write $f(z) = zg(z)$ with $g(0) = 1$. Since $|g|^{1/2}$ is subharmonic, for every $r \in (0, 1)$,

$$\frac{1}{2\pi}\int_0^{2\pi} |g(re^{i\theta})|^{1/2}\, d\theta \geq |g(0)|^{1/2} = 1.$$

Hence

$$\int_{\mathbb{D}} |f|^{1/2}\, dA = \int_0^1 r^{1/2} r\Bigl(\int_0^{2\pi} |g(re^{i\theta})|^{1/2}\, d\theta\Bigr) dr \geq 2\pi \int_0^1 r^{3/2}\, dr = \frac{4\pi}{5},$$

with equality for $g \equiv 1$, i.e. $f(z) = z$. Therefore

$$K^{H,1/2}_{\mathbb{D}}(0) = \frac{1}{4\pi/5} = \frac{5}{4\pi}.$$

Next, fix $\alpha = 3/2$ and set

$$M(\alpha) := \frac{1}{\pi}\int_{\mathbb{D}} |1 + \alpha z|\, dA(z).$$

We need the elementary estimate

$$\frac{M(\alpha)}{\sqrt{2\alpha}} < \frac{4}{5}. \tag{2.1}$$

For $0 \leq x \leq 1$ the fourth Taylor polynomial of $\sqrt{1-x}$ gives

$$\sqrt{1-x} \leq 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128}, \tag{2.2}$$

because all remaining Taylor coefficients of $\sqrt{1-x}$ are negative. Write $t = \alpha r$ and

$$\kappa := \frac{4t}{(1+t)^2}.$$

Then

$$|1 + te^{i\theta}| = (1+t)\sqrt{1 - \kappa \sin^2 \frac{\theta}{2}}.$$

Using (2.2) and the averages

$$\frac{1}{2\pi}\int_0^{2\pi}\sin^2\frac{\theta}{2}\,d\theta = \frac{1}{2}, \qquad \frac{1}{2\pi}\int_0^{2\pi}\sin^4\frac{\theta}{2}\,d\theta = \frac{3}{8},$$
$$\frac{1}{2\pi}\int_0^{2\pi}\sin^6\frac{\theta}{2}\,d\theta = \frac{5}{16}, \qquad \frac{1}{2\pi}\int_0^{2\pi}\sin^8\frac{\theta}{2}\,d\theta = \frac{35}{128},$$

we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}|1+te^{i\theta}|\,d\theta \le (1+t)\Big(1-\frac{\kappa}{4}-\frac{3\kappa^2}{64}-\frac{5\kappa^3}{256}-\frac{175\kappa^4}{16384}\Big).$$

Consequently

$$M(\alpha) \le \int_0^1 2r(1+\alpha r)\Big(1-\frac{\kappa}{4}-\frac{3\kappa^2}{64}-\frac{5\kappa^3}{256}-\frac{175\kappa^4}{16384}\Big)\,dr.$$

For $\alpha = 3/2$ the integral on the right is

$$\frac{13771}{10000}.$$

Since

$$\Big(\frac{13771}{10000}\Big)^2 = \frac{189640441}{100000000} < \frac{48}{25} = \Big(\frac{4}{5}\sqrt{3}\Big)^2,$$

we have

$$\frac{13771}{10000} < \frac{4}{5}\sqrt{3} = \frac{4}{5}\sqrt{2\alpha},$$

which proves (2.1).

Now fix any $c \in \mathbb{C}$. Choose a unit complex number $e^{i\theta}$ such that c and $2\alpha e^{i\theta}$ have the same argument (take $e^{i\theta} = 1$ if $c = 0$), so that $c + 2\alpha e^{i\theta}$ has modulus $|c| + 2\alpha$. Choose a unit complex number $e^{i\psi}$ with $e^{i\psi}(c + 2\alpha e^{i\theta}) = |c| + 2\alpha$, and define

$$f_{c,\theta}(z) := e^{i\psi}\frac{(1+\alpha e^{i\theta}z)^2}{|c|+2\alpha}.$$

Then $f_{c,\theta}$ is a polynomial, and

$$f_{c,\theta}(0) = \frac{e^{i\psi}}{|c|+2\alpha}, \qquad f'_{c,\theta}(0) = \frac{2\alpha e^{i\theta}e^{i\psi}}{|c|+2\alpha},$$

so that for $\xi_c = (c, 1, 0, 0, \ldots) \in S_H$,

$$(\xi_c \cdot f_{c,\theta})(0) = cf_{c,\theta}(0) + f'_{c,\theta}(0) = \frac{e^{i\psi}(c+2\alpha e^{i\theta})}{|c|+2\alpha} = 1.$$

Moreover, by the rotation invariance of dA,

$$\int_{\mathbb{D}}|f_{c,\theta}|^{1/2}\,dA = \frac{1}{\sqrt{|c|+2\alpha}}\int_{\mathbb{D}}|1+\alpha e^{i\theta}z|\,dA = \frac{\pi M(\alpha)}{\sqrt{|c|+2\alpha}} \le \frac{\pi M(\alpha)}{\sqrt{2\alpha}} < \frac{4\pi}{5},$$

since $|c| + 2\alpha \ge 2\alpha$. By the equivalent form of $K_{\xi,D,p}$, the same estimate gives the uniform lower bound

$$K_{\xi_c,\mathbb{D},1/2}(0) \ge \frac{1}{\int_{\mathbb{D}}|f_{c,\theta}|^{1/2}\,dA} \ge \frac{\sqrt{2\alpha}}{\pi M(\alpha)} > \frac{5}{4\pi} = K_{\mathbb{D}}^{H,1/2}(0).$$

Since the middle lower bound is independent of c, we may take the infimum and obtain

$$\inf_{\xi\in S_H} K_{\xi,\mathbb{D},1/2}(0) \ge \frac{\sqrt{2\alpha}}{\pi M(\alpha)} > \frac{5}{4\pi} = K_{\mathbb{D}}^{H,1/2}(0),$$

which proves the theorem. ■

2.2 The threshold $p_* = 2(\sqrt{2}-1)$ for the counterexample

The counterexample in Theorem 1.1 is not an isolated phenomenon: the second variation at the extremal $f(z)=z$ shows that the identity

$$K_{\mathbb{D}}^{H,p}(0) = \inf_{\xi\in S_H} K_{\xi,\mathbb{D},p}(0) \tag{2.3}$$

fails for every $0<p<p_*$, where

$$p_* := 2(\sqrt{2}-1).$$

For $p>p_*$ the same computation shows that z is a strict local minimizer in the class $f'(0)=1$, and at $p=p_*$ the second variation degenerates along a single explicit direction; numerically, z remains the global minimizer for every $p\ge p_*$ in this example.

Since $H(z)=z$, every $\xi\in S_H$ has the form $\xi_c=(c,1,0,0,\ldots)$, $c\in\mathbb{C}$, and

$$K_{\xi_c,\mathbb{D},p}(0) = \frac{1}{m_{\xi_c}(p)}, \qquad m_{\xi_c}(p) := \inf\Big\{\int_{\mathbb{D}} |f|^p\,dA:\ f\in\mathcal{O}(\mathbb{D}),\ cf(0)+f'(0)=1\Big\}.$$

The function $f(z)=z$ is admissible for every ξ_c, and $\int_{\mathbb{D}}|z|^p\,dA = 2\pi/(p+2)$; hence $m_{\xi_c}(p)\le 2\pi/(p+2)$ for all c.

THEOREM 2.1. *Let $p_* := 2(\sqrt{2}-1)$. For every $0<p<p_*$, with $D=\mathbb{D}$, $H(z)=z$ and $z=0$,*

$$\inf_{\xi\in S_H} K_{\xi,\mathbb{D},p}(0) > K_{\mathbb{D}}^{H,p}(0);$$

that is, the identity (2.3) fails for every $p\in(0,p_*)$. In particular, Theorem 2.1 contains the case $p=1/2$ of Theorem 1.1.

Proof. By the same argument as in the proof of Theorem 1.1, with p in place of $1/2$,

$$K_{\mathbb{D}}^{H,p}(0) = \frac{p+2}{2\pi}.$$

It suffices to show that $m_{\xi_0}(p) < 2\pi/(p+2)$. For $\lambda>0$ and $\varepsilon\in\mathbb{R}$ consider

$$f_\varepsilon(z) := z+\varepsilon(1+\lambda z^2),$$

so that $f_\varepsilon'(0)=1$ and $f_0=z$. Write $J_p(\varepsilon) := \int_{\mathbb{D}}|f_\varepsilon|^p\,dA$. The first derivative at 0 vanishes by symmetry, and a direct expansion (given after the proof) yields

$$J_p''(0) = \pi\Big(p+\frac{p^2\lambda^2}{p+4}+\frac{2p(p-2)\lambda}{p+2}\Big). \tag{2.4}$$

The quadratic polynomial in λ on the right-hand side has minimum

$$p - \frac{(2-p)^2(p+4)}{(p+2)^2} = \frac{4(p^2+4p-4)}{(p+2)^2} = \frac{4(p-p_*)(p+2+2\sqrt{2})}{(p+2)^2},$$

attained at

$$\lambda_p := \frac{(2-p)(p+4)}{p(p+2)}.$$

Since $p+2+2\sqrt{2} > 0$ for $p > 0$, the minimum is negative precisely for $p < p_*$. Fixing such a p and taking $\lambda = \lambda_p$, we have $J_p''(0) < 0$; as $J_p'(0) = 0$ and $J_p(0) = 2\pi/(p+2)$, for all sufficiently small $\varepsilon > 0$,

$$\int_{\mathbb{D}} |f_\varepsilon|^p \, dA < \frac{2\pi}{p+2}.$$

Thus $m_{\xi_0}(p) < 2\pi/(p+2)$, and Proposition 2.2 yields $\sup_{c\in\mathbb{C}} m_{\xi_c}(p) < 2\pi/(p+2)$. Consequently

$$\inf_{\xi\in S_H} K_{\xi,\mathbb{D},p}(0) = \frac{1}{\sup_{c\in\mathbb{C}} m_{\xi_c}(p)} > \frac{p+2}{2\pi} = K_{\mathbb{D}}^{H,p}(0),$$

■

PROPOSITION 2.2. *The identity (2.3) holds if and only if $m_{\xi_0}(p) = 2\pi/(p+2)$, i.e. if and only if z minimizes $\int_{\mathbb{D}} |f|^p \, dA$ among all holomorphic f with $f'(0) = 1$.*

Proof. Since

$$\inf_{\xi\in S_H} K_{\xi,\mathbb{D},p}(0) = \inf_{c\in\mathbb{C}} \frac{1}{m_{\xi_c}(p)} = \frac{1}{\sup_{c\in\mathbb{C}} m_{\xi_c}(p)},$$

it suffices to show that $\sup_{c\in\mathbb{C}} m_{\xi_c}(p)$ equals $2\pi/(p+2)$ if and only if it is attained at $c = 0$.

We first note that $c \mapsto m_{\xi_c}(p)$ is continuous. Write $J_p(f) := \int_{\mathbb{D}} |f|^p \, dA$ and

$$N(c) := \sup_{f\neq 0} \frac{|cf(0) + f'(0)|}{J_p(f)^{1/p}},$$

so that $m_{\xi_c}(p) = N(c)^{-p}$. Since $|f(0)| \le \pi^{-1/p} J_p(f)^{1/p}$ by subharmonicity,

$$|N(c) - N(d)| \le |c-d| \, \pi^{-1/p},$$

so N, and hence $c \mapsto m_{\xi_c}(p)$, is continuous. Taking $f \equiv 1$ shows $N(c) \ge |c| \, \pi^{-1/p}$, whence $m_{\xi_c}(p) \le \pi \, |c|^{-p} \to 0$ as $|c| \to \infty$.

It remains to check that $m_{\xi_c}(p) < 2\pi/(p+2)$ for every $c \neq 0$. Set $\eta = \bar{c}/|c|$ and $f_t(z) = z + t\eta(1-cz)$. Then $cf_t(0) + f_t'(0) = 1$ for every t, and the local factorization argument used for the second variation below justifies differentiation at $t = 0$ for every $p > 0$. Hence

$$\frac{d}{dt}\Big|_{t=0} \int_{\mathbb{D}} |f_t|^p \, dA = p \, \Re \int_{\mathbb{D}} |z|^{p-2} \bar{z} \, \eta(1-cz) \, dA = -p \, \Re(\eta c) \int_{\mathbb{D}} |z|^p \, dA = -p|c| \, \frac{2\pi}{p+2} < 0,$$

because $\int_{\mathbb{D}} |z|^{p-2} \bar{z} \, dA = 0$ by rotation invariance. Thus $J_p(f_t) < 2\pi/(p+2)$ for all sufficiently small $t > 0$, which gives $m_{\xi_c}(p) < 2\pi/(p+2)$.

Consequently, if $m_{\xi_0}(p) = 2\pi/(p+2)$ then the supremum is attained at $c = 0$; if instead $m_{\xi_0}(p) < 2\pi/(p+2)$, then $m_{\xi_c}(p) < 2\pi/(p+2)$ for every $c \in \mathbb{C}$, and continuity together with the decay at infinity yields $\sup_{c\in\mathbb{C}} m_{\xi_c}(p) < 2\pi/(p+2)$. ■

For completeness we record and justify the second variation used above. Let $h(z) = \sum_{m\neq 1} h_m z^m$ be holomorphic on a neighborhood of $\overline{\mathbb{D}}$ with $h'(0) = 0$. Although the pointwise second derivative is singular at $z = 0$ when $p < 2$, differentiation under the integral is valid for every $p > 0$. Indeed, choose a cutoff supported in a small disk about 0. By the holomorphic implicit function theorem, for small real ε the function $z + \varepsilon h(z)$ has there a unique simple zero $a(\varepsilon)$ and factors as

$$z + \varepsilon h(z) = (z - a(\varepsilon))u_\varepsilon(z),$$

where u_ε is nonvanishing and depends smoothly on ε. After the translation $w = z - a(\varepsilon)$, two derivatives fall only on smooth factors multiplying $|w|^p$, which is locally integrable. Away from 0 the usual dominated-convergence argument applies. Thus the integral is twice differentiable at 0, and we may expand $|z + \varepsilon h|^p = |z|^p|1 + \varepsilon h/z|^p$ for $z \neq 0$. Using the elementary averages $\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta}\, d\theta = \delta_{k,0}$, we obtain

$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} \int_{\mathbb{D}} |z + \varepsilon h|^p \, dA = \pi\left(p^2 \sum_{m\neq 1} \frac{|h_m|^2}{p + 2m} + \frac{2p(p-2)}{p+2}\Re(h_0 h_2)\right), \tag{2.5}$$

which for $h(z) = 1 + \lambda z^2$ reduces to (2.4). The only potentially negative term is the cross term $\Re(h_0 h_2)$, which arises from the pairing of the modes $h_0 z^{-1}$ and $h_2 z$ in $(h/z)^2$. The threshold p_* is exactly the value at which the two-mode quadratic form $(x, y) \mapsto px^2 + p^2y^2/(p + 4) + 2p(p - 2)xy/(p + 2)$ becomes positive semidefinite.

Remark 2.3. *It remains open whether (2.3) holds for every $p \geq p_*$. For $p \geq 1$ this follows from [3, Theorem 1.7], and the numerical evidence above suggests that z remains the global minimizer for every $p \geq p_*$ in this example; a rigorous proof of (2.3) in the remaining range $p_* \leq p < 1$ is not known.*

3 The L^p analogue of $B = C$

Bao–Guan [2] proved that for $p = 2$,

$$B(F, I, D) := \sup_{\xi\in\ell_I} \frac{|(\xi \cdot F)(0)|^2}{K_{\xi,D,2}(0)} = C_{F,I}(D) := \inf\Big\{\int_D |\tilde{F}|^2 : \ (\tilde{F} - F, 0) \in I, \ \tilde{F} \in \mathcal{O}(D)\Big\},$$

where

$$\ell_I := \big\{\xi \in \ell_1^{(n)} \setminus \{0\} : \ (\xi \cdot f)(0) = 0 \ \ \forall\, (f, 0) \in I\big\}.$$

It is natural to ask whether the L^p analogue

$$B_p(F, I, D) := \sup_{\xi\in\ell_I} \frac{|(\xi \cdot F)(0)|^p}{K_{\xi,D,p}(0)}, \qquad C_{F,I,p}(D) := \inf\Big\{\int_D |\tilde{F}|^p : \ (\tilde{F} - F, 0) \in I, \ \tilde{F} \in \mathcal{O}(D)\Big\}$$

holds for all $p \in (0, +\infty)$. The counterexample above answers this in the negative for every $0 < p < p_*$, while for $p \geq 1$ the identity holds for every ideal I.

THEOREM 3.1. *Let $p_* := 2(\sqrt{2} - 1)$. For every $0 < p < p_*$, with $D = \mathbb{D}$, $I = \mathfrak{m}_0^2$ and $F(z) = z$,*

$$B_p(z, \mathfrak{m}_0^2, \mathbb{D}) < C_{z,\mathfrak{m}_0^2,p}(\mathbb{D}) = \frac{2\pi}{p+2}.$$

Proof. Since $(\widetilde{F} - z, 0) \in \mathfrak{m}_0^2$ means $\widetilde{F}(0) = 0$ and $\widetilde{F}'(0) = 1$, the computation of $K_{\mathbb{D}}^{z,p}(0)$ above gives

$$C_{z,\mathfrak{m}_0^2,p}(\mathbb{D}) = \frac{1}{K_{\mathbb{D}}^{z,p}(0)} = \frac{2\pi}{p+2}.$$

On the other hand $\ell_{\mathfrak{m}_0^2} = \big\{(\xi_0, \xi_1, 0, 0, \dots) : (\xi_0, \xi_1) \neq (0,0)\big\}$, and for $\xi_c := (c, 1, 0, 0, \dots)$ the homogeneity identity $K_{\xi_c,\mathbb{D},p}(0) = 1/m_{\xi_c}(p)$ yields

$$B_p(z, \mathfrak{m}_0^2, \mathbb{D}) = \sup_{c\in\mathbb{C}} m_{\xi_c}(p) < \frac{2\pi}{p+2}$$

by Theorem 2.1 and Proposition 2.2. ■

THEOREM 3.2. *Let $p \geq 1$, let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing o, let I be an ideal of $\mathcal{O}_o$, and let $(F, o) \in \mathcal{O}_o$. Then*

$$B_p(F, I, D) = C_{F,I,p}(D).$$

Here both sides are allowed to be $+\infty$, with the conventions $a/0 = +\infty$ for $a > 0$, $0/0 = 0$, and $\sup\varnothing = 0$ *in the definition of B_p.*

Proof. The inequality $B_p \leq C_{F,I,p}$ is elementary: for every admissible $\widetilde{F}$ and every $\xi \in \ell_I$ one has $(\xi \cdot F)(o) = (\xi \cdot \widetilde{F})(o)$, hence $|(\xi \cdot F)(o)|^p \leq K_{\xi,D,p}(o) \int_D |\widetilde{F}|^p$.

Assume first that D is bounded. For the reverse inequality fix $k \geq 1$ and write $V_{I+\mathfrak{m}^k} := \{f \in A^p(D) : (f, o) \in I + \mathfrak{m}^k\}$. Since $\mathfrak{m}^k \subseteq I + \mathfrak{m}^k$, membership in $I + \mathfrak{m}^k$ is detected by the $(k-1)$-jet, so the quotient $Q_k := A^p(D)/V_{I+\mathfrak{m}^k}$ is finite-dimensional. As $p \geq 1$, $A^p(D)$ is a normed space, and $C_{F,I+\mathfrak{m}^k,p}(D)^{1/p} = \|[F]\|_{Q_k}$ (the class of F being represented by the polynomial $j_{k-1}F$). By the Hahn–Banach theorem,

$$C_{F,I+\mathfrak{m}^k,p}(D)^{1/p} = \|[F]\|_{Q_k} = \sup\Big\{|\varphi(F)| : \ \varphi \in V_{I+\mathfrak{m}^k}^{\perp}, \ \|\varphi\| \leq 1\Big\}.$$

Every $\varphi \in V_{I+\mathfrak{m}^k}^{\perp}$ depends only on the $(k-1)$-jet, i.e. $\varphi(f) = (\xi \cdot f)(o)$ for some $\xi \in \ell_{I+\mathfrak{m}^k}$, and $\|\varphi\| = \|\xi\|_*$ with $\|\xi\|_* := K_{\xi,D,p}(o)^{1/p}$. Therefore

$$B_p(F, I + \mathfrak{m}^k, D) = C_{F,I+\mathfrak{m}^k,p}(D).$$

Since $I \subseteq I + \mathfrak{m}^k$, one has $\ell_{I+\mathfrak{m}^k} \subseteq \ell_I$, hence

$$B_p(F, I, D) \ \geq \ B_p(F, I + \mathfrak{m}^k, D) = C_{F,I+\mathfrak{m}^k,p}(D), \qquad k \geq 1.$$

It remains to note that $\lim_{k\to\infty} C_{F,I+\mathfrak{m}^k,p}(D) = C_{F,I,p}(D)$. Indeed $C_k := C_{F,I+\mathfrak{m}^k,p}(D)$ is increasing in k with $C_k \leq C_{F,I,p}(D)$; if $C_\infty := \lim_k C_k < +\infty$, choose $G_k \in A^p(D)$ with $(G_k - F, o) \in I + \mathfrak{m}^k$ and $\int_D |G_k|^p \leq C_k + 1/k$. By Montel's theorem (through the submean-value property of $|f|^p$) a subsequence converges locally uniformly to some $G \in \mathcal{O}(D)$; Fatou's lemma gives $\int_D |G|^p \leq C_\infty$; and the jet conditions pass to the limit, so $(G - F, o) \in I + \mathfrak{m}^k$ for every k, whence $(G - F, o) \in I$ by the Krull intersection theorem [1, Chapter 10], $\bigcap_{k\geq 1}(I + \mathfrak{m}^k) = I$. Therefore $C_{F,I,p}(D) \leq C_\infty$, and combined with $C_\infty \leq C_{F,I,p}(D)$ this gives the limit. Consequently

$$B_p(F, I, D) \ \geq \ \lim_{k\to\infty} C_{F,I+\mathfrak{m}^k,p}(D) = C_{F,I,p}(D) \ \geq \ B_p(F, I, D),$$

which proves the theorem when D is bounded.

For a general pseudoconvex domain, choose a bounded pseudoconvex exhaustion

$$D_1 \Subset D_2 \Subset \cdots \Subset D, \qquad o \in D_1, \qquad \bigcup_{j \geq 1} D_j = D.$$

The same Montel–Fatou argument used above gives

$$C_{F,I,p}(D_j) \nearrow C_{F,I,p}(D). \tag{3.6}$$

Indeed, the left-hand side is increasing and bounded above by the right-hand side; if its limit is finite, a diagonal limit of almost minimizers on D_j is an admissible function on D with no larger L^p integral.

For a fixed $\xi \in \ell_I$, apply the same argument to

$$m_{\xi,p}(D_j) := \inf\Big\{ \int_{D_j} |f|^p : f \in A^p(D_j),\ (\xi \cdot f)(o) = 1 \Big\} = \frac{1}{K_{\xi,D_j,p}(o)}.$$

It follows that

$$m_{\xi,p}(D_j) \nearrow m_{\xi,p}(D), \qquad K_{\xi,D_j,p}(o) \searrow K_{\xi,D,p}(o), \tag{3.7}$$

where the extended values are permitted. Hence, using the monotonicity in j to interchange the two suprema,

$$\begin{aligned} B_p(F,I,D) &= \sup_{\xi \in \ell_I} \lim_{j \to \infty} \frac{|(\xi \cdot F)(o)|^p}{K_{\xi,D_j,p}(o)} \\ &= \lim_{j \to \infty} \sup_{\xi \in \ell_I} \frac{|(\xi \cdot F)(o)|^p}{K_{\xi,D_j,p}(o)} = \lim_{j \to \infty} B_p(F,I,D_j) \\ &= \lim_{j \to \infty} C_{F,I,p}(D_j) = C_{F,I,p}(D), \end{aligned}$$

where the bounded case was used on every D_j. This completes the proof. ■

Remark 3.3. *For $p = 2$, Theorem 3.2 is the result of [2]; the argument above is the same chain with the L^2 orthogonal projection replaced by the Hahn–Banach duality. In the example of Theorem 3.1, the identity $B_p(z, \mathfrak{m}_0^2, \mathbb{D}) = C_{z,\mathfrak{m}_0^2,p}(\mathbb{D})$ for $p \geq 1$ is equivalent to the ξ-reduction $K_{\mathbb{D}}^{z,p}(0) = \inf_{\xi \in S_z} K_{\xi,\mathbb{D},p}(0)$ of [3, Theorem 1.7]. Thus the failure established here occurs in the subadditive range $0 < p < p_*$; the status of this example for $p_* \leq p < 1$ remains open, as stated in Remark 2.3.*

AI declaration

The counterexamples and proofs presented in this paper were obtained with DeepSeek V4 flash and pro. The author independently verified all statements and computations and takes full responsibility for the content.

References

[1] M. F. Atiyah and I. G. Macdonald. Introduction to Commutative Algebra. Addison–Wesley, Reading, MA, 1969.

[2] S. J. Bao and Q. A. Guan. Equivalence of the sharp effectiveness results of strong openness property. arXiv:2408.16372, 2024.

[3] S. J. Bao, Q. A. Guan and X. Sun, On the p-Bergman kernel with respect to a functional ξ, arXiv:2510.13144, 2025.