

ZHOU WEIGHTS AND VALUATIVE SEPARATION BY GENERALIZED LELONG NUMBERS

SHIJIE BAO

ABSTRACT. We prove that Zhou weights are flat: the generalized Lelong number associated to a Zhou weight, divided by its residual Monge–Ampère mass, equals its relative type on every plurisubharmonic germ. The residual mass is the volume of the corresponding Zhou valuation. The algebraic step computes the asymptotic multiplicity of valuation ideals on a hypersurface from an exact colon identity. We then extract a uniform log-discrepancy bound from the Bao–Guan–Zhou density construction and use multiplier ideals to extend finite-ideal approximation to plurisubharmonic germs. Consequently, generalized Lelong numbers tested only against Zhou weights determine valuative transforms. This implies the remaining implication in Theorem A of Boucksom–Favre–Jonsson. No quasi-monomiality assumption on Zhou valuations is used.

1. INTRODUCTION

Work at the origin $o \in \mathbb{C}^n$, with $n \geq 2$, and put $R = \mathcal{O}_{\mathbb{C}^n, o}$ and $\mathfrak{m} = \mathfrak{m}_o$. For a maximal plurisubharmonic weight Φ with isolated pole at o , write

$$\sigma(u, \Phi) = \sup\{c \geq 0 : u \leq c\Phi + O(1)\},$$

$$\nu_\Phi(u) = (dd^c u \wedge (dd^c \Phi)^{n-1})\{o\}, \quad \tau_\Phi = (dd^c \Phi)^n\{o\}.$$

Demailly’s comparison theorem gives $\tau_\Phi \sigma(u, \Phi) \leq \nu_\Phi(u)$. Our main result establishes equality for Zhou weights. A maximal weight with this equality for every psh germ is called *flat*; see [13, Section 6]. Thus the first conclusion is: *Zhou weights are flat*.

Zhou weights were introduced by Bao–Guan–Mi–Yuan [1] through maximality under a non-integrability constraint. They admit tame maximal representatives, and their holomorphic relative types define centered real valuations

$$w_\Phi(f) = \sigma(\log |f|, \Phi), \quad f \in R \setminus \{0\}.$$

These are the two properties of Zhou weights used in the mass calculation. The additional jumping-number structure enters the density argument. Definitions and the precise results used from the literature are recalled below.

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Theorem 1.1 (Zhou weights are flat). *Let Φ be a local Zhou weight, represented by its tame maximal representative. Set $w = w_\Phi$, $\lambda = w(\mathfrak{m})$, and $v = w/\lambda$. Then*

$$(1.1) \quad \tau_\Phi = \text{vol}(w) > 0, \quad \nu_\Phi(u) = \text{vol}(w)\sigma(u, \Phi) \quad \text{for every psh germ } u.$$

In the normalized valuative space of Boucksom–Favre–Jonsson,

$$(1.2) \quad \widehat{\Phi}(v) = -\lambda^{-1}, \quad \text{MA}(\widehat{\Phi}) = \lambda \text{vol}(w)\delta_v, \quad \sigma(u, \Phi) = \lambda(-\widehat{u}(v)).$$

Here

$$\mathfrak{a}_t(w) = \{f \in R : w(f) \geq t\}, \quad \text{vol}(w) = \lim_{m \rightarrow \infty} \frac{n! \ell(R/\mathfrak{a}_m(w))}{m^n}.$$

The main algebraic input is

$$(1.3) \quad \lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m(w); R/(f))}{m^{n-1}} = w(f) \text{vol}(w), \quad 0 \neq f \in \mathfrak{m}.$$

The numerator is the Hilbert–Samuel multiplicity on the hypersurface $f = 0$. The proof first uses Cutkosky’s theorem [5] to establish the existence of its asymptotic limit. The colon identity $(\mathfrak{a}_t(w) : f) = \mathfrak{a}_{t-w(f)}(w)$ then determines the coefficient by summing exact length differences. Tameness compares these ideal multiplicities with the masses of Φ .

The second result concerns separation. The notation $\widehat{u}$ denotes the valuative transform of a psh germ in [4]; the sign convention is $\widehat{\log |f|}(v) = -v(f)$.

Theorem 1.2 (Separation by Zhou weights). *For psh germs u_1, u_2 at o , the following are equivalent:*

- (i) $\nu_\Phi(u_1) = \nu_\Phi(u_2)$ for every local Zhou weight Φ ;
- (ii) $\widehat{u}_1 = \widehat{u}_2$ on the normalized valuative space.

In particular, equality of generalized Lelong numbers against all tame weights implies equality of Lelong numbers on all modifications, giving (4) $\Rightarrow$ (1) in Theorem A of [4].

Bao–Guan–Zhou proved that the cone of algebraic Zhou valuations is weakly dense and established its analytic correspondence [3, Theorem 9.2 and Theorem 11.2]. Weak density alone does not justify taking limits of valuative transforms of arbitrary psh germs. We retain an extra bound in their construction: the approximating Zhou valuations w can be chosen with $A(w) \leq A(v)$ when the target v has finite log discrepancy. The valuative Demailly estimate then provides a uniform error bound. This gives a density proof of [theorem 1.2](#) in which every limit is controlled.

The result of [1] that Zhou relative types determine multiplier ideals gives another deduction of the separation theorem once [theorem 1.1](#) is known. Our proof instead spells out the density argument. We do not assert that every quasi-monomial valuation is a Zhou valuation, or that every Zhou valuation is quasi-monomial.

1.1. What the two results say. The relative type asks a pointwise question: how large a multiple of Φ can still lie above u , up to a bounded error? The generalized Lelong number asks a mass question: how much of the mixed positive measure $dd^c u \wedge (dd^c \Phi)^{n-1}$ is concentrated at the origin? These questions have different definitions. Flatness says that, after division by the mass of Φ itself, their answers coincide for every u . It does not say that u is a multiple of Φ modulo a bounded function.

The separation statement has a different quantifier. A single weight gives one measurement of u . The collection of measurements obtained by allowing *all* Zhou weights determines the valuative transform, which records vanishing information in every birational direction. The conclusion is equality of these measurements, not necessarily $u_1 = u_2 + O(1)$.

1.2. Reading guide and proof dependencies. The argument is organized so that the main computations can be read without prior familiarity with valuative spaces. Section 2 introduces the analytic and algebraic vocabulary. The proof then has four stages:

1. On a hypersurface, calculate the leading growth of valuation ideals by an exact sequence and a telescoping sum (proposition 3.2).
2. Transfer that calculation to analytic masses by the two-sided tame approximation (lemma 4.1 and proposition 4.2).
3. Show that equality on all holomorphic tests forces the BFJ measure to be supported at one point, giving equality on all psh germs (lemma 4.3).
4. Approximate any finite-discrepancy valuation on a finite list of psh germs by Zhou valuations, and apply this to the pair being compared (theorem 5.7 and Section 6).

We recall, but do not reprove, the general theories of the Monge–Ampère operator, resolution of singularities, multiplier ideals, Cutkosky’s asymptotic multiplicity theorem, and the BFJ representation theorems. Their exact roles are stated where used. Thus this is a detailed proof for a reader with basic graduate mathematics, not a construction of pluripotential theory from first principles.

The route to the BFJ implication is analytic in its use of weights, comparison of masses, and multiplier-ideal approximation. It also uses the algebraic hypersurface calculation and the Bao–Guan–Zhou density theorem. In particular, “analytic proof” here does not mean a proof free of algebraic or birational inputs. No new intersection-theoretic nondegeneracy assertion is required.

2. ZHOU WEIGHTS AND ANALYTIC INPUTS

2.1. Conventions and Zhou weights.

Germ and singularity comparisons. A germ at o is a function considered only on a sufficiently small neighborhood of o ; shrinking that neighborhood does not change the germ. The local ring $R = \mathbb{C}\{z_1, \dots, z_n\}$ consists of

convergent power series. Its maximal ideal $\mathfrak{m} = (z_1, \dots, z_n)$ consists of germs vanishing at o ; a unit is a germ nonzero at o . A function is plurisubharmonic, abbreviated psh, if it is upper semicontinuous and its restriction to each complex line is subharmonic where defined. Examples are $\log|f|$, $\log|I|$, and maxima of finitely many psh functions. Singularities are allowed: $\log|z|$ tends to $-\infty$ at the origin.

The notation $u \leq v + O(1)$ means $u(z) \leq v(z) + K$ near o for some constant K . It says that u is at least as singular as v . The sign is important: $2 \log|z| \leq \log|z|$ when $|z| < 1$. Writing $u = v + O(1)$ means that their difference is bounded. Adding a bounded function does not affect any singularity invariant used below.

We exclude the identically $-\infty$ germ from PSH_o . Use $d^c = (\partial - \bar{\partial})/(2\pi i)$, so that $dd^c \log|f| = [\text{div}(f)]$. For a nonzero ideal $I = (f_1, \dots, f_r)$, set

$$\log|I| = \frac{1}{2} \log(|f_1|^2 + \dots + |f_r|^2).$$

Changing generators changes this germ by a bounded function. All comparisons of singularities in this paper are modulo bounded functions. The multiplier ideal is

$$\mathcal{J}(u)_o = \{f \in R : |f|^2 e^{-2u} \text{ is locally integrable at } o\}.$$

We omit the stalk subscript below.

Weights, masses, and flatness. A weight has an isolated pole: it is locally bounded away from o , tends to $-\infty$ at o , and has continuous exponential. It is maximal outside o when it cannot be increased on a relatively compact region away from o without increasing its boundary values. For the weights here this is equivalent to $(dd^c \Phi)^n = 0$ away from o . For a smooth function, dd^c is a constant multiple of its complex Hessian. For psh functions it is interpreted as a positive current; the products used here are the classical products for isolated-pole weights, as developed in [7]. They are not the non-pluripolar products, which discard mass on the pole set. The notation $\mu\{o\}$ means the mass that a measure μ assigns to the singleton $\{o\}$.

In particular τ_Φ is a number, whereas $(dd^c \Phi)^n = \tau_\Phi \delta_o$ is a measure identity for a maximal weight. When $0 < \tau_\Phi < \infty$, flatness is precisely

$$\frac{\nu_\Phi(u)}{\tau_\Phi} = \sigma(u, \Phi) \quad \text{for every } u \in \text{PSH}_o.$$

This is the terminology of [13, Section 6]; it is unrelated to flatness of a module in commutative algebra.

Why the Zhou definition is an extremality condition. The following definition starts with an integrable density. Multiplying by $e^{-2\Phi}$ puts it beyond integrability. The last condition says that one cannot make the singularity of Φ weaker, while staying non-integrable, except by a bounded change. The first condition ensures that a sufficiently high vanishing factor repairs integrability.

Definition 2.1 ([1]). Let F_0 be a nonzero tuple of holomorphic germs and let ψ_0 be psh, with $|F_0|^2 e^{-2\psi_0}$ locally integrable. A local Zhou weight for this datum is a psh germ Φ satisfying:

- (i) $|F_0|^2 e^{-2\psi_0} |z|^{2N} e^{-2\Phi}$ is integrable for some integer $N > 0$;
- (ii) $|F_0|^2 e^{-2\psi_0} e^{-2\Phi}$ is not integrable;
- (iii) if $\Psi \geq \Phi + O(1)$ is psh and $|F_0|^2 e^{-2\psi_0} e^{-2\Psi}$ is not integrable, then $\Psi = \Phi + O(1)$.

For orientation, in one variable with datum $|F_0|^2 = |z|^{2p}$ and $\psi_0 = 0$, the candidate $\Phi = (p+1) \log |z|$ gives the radial integral $\int_0^\varepsilon r^{-1} dr$, which diverges. Replacing $p+1$ by any smaller logarithmic coefficient makes it converge. This computation illustrates the critical scale, although the full Zhou maximality condition tests all psh competitors, not only logarithmic ones.

The non-integrability condition fixes a scale relative to the datum. It does not impose the normalization $w_\Phi(\mathfrak{m}) = 1$. A maximal representative has isolated pole and satisfies $(dd^c \Phi)^n = \tau_\Phi \delta_o$, with $\tau_\Phi > 0$. A tame representative additionally has continuous e^Φ and a constant $C > 0$ such that

$$(2.1) \quad f \in \mathcal{J}(t\Phi) \implies \log |f| \leq (t - C)\Phi + O(1), \quad t > 0.$$

We use the following results of [1, Section 1.1, Proposition 1.7, and Section 1.2]: every local Zhou weight has such a tame maximal representative; there are constants $0 < c < L$ with

$$(2.2) \quad L \log |z| + O(1) \leq \Phi \leq c \log |z| + O(1);$$

and w_Φ is a real valuation, finite on every nonzero germ, positive on $\mathfrak{m} \setminus \{0\}$, and zero on units. For weights related to holomorphic ideals, an algebraic treatment of the correspondence and tameness is given in [3, Theorem 11.2 and Corollary 11.8]. The paper [1] also proves additivity of $\sigma(\cdot, \Phi)$ under addition of psh germs and its minimum rule under their maximum. Our proof of [theorem 1.1](#) only needs the valuation property on holomorphic germs, not this stronger additivity result.

The logarithmic bounds also check finiteness and positivity of the mass. Comparison with $\log |z|$, whose residual mass is 1, gives

$$0 < c^n \leq \tau_\Phi \leq L^n < \infty.$$

Thus divisions by τ_Φ later are legitimate; positivity will not be inferred merely from the existence of a volume limit.

2.2. Comparison, approximation, and multiplicity. We list the analytic statements needed later, with their normalizations. The comparison results are due to Demailly [6, 7]; the relative-type attainment statement is discussed in [12] and [4, Section 5.4].

Proposition 2.2. *The following statements hold.*

- (i) If Φ is maximal, the relative-type inequality is attained: $u \leq \sigma(u, \Phi)\Phi + O(1)$. Its value is finite for $u \in \text{PSH}_o$ when (2.2) holds.
- (ii) For $t > 0$, Demailly approximation gives $u \leq t^{-1} \log |\mathcal{J}(tu)| + O(1)$.
- (iii) If weights $\alpha \leq \beta + O(1)$ have isolated poles, then $\nu_\alpha(u) \geq \nu_\beta(u)$. Residual Monge–Ampère masses obey the same monotonicity. Scaling a weight by $b > 0$ multiplies these two quantities by b^{n-1} and b^n , respectively.
- (iv) If I is $\mathfrak{m}$ -primary, then

$$(dd^c \log |I|)^n \{o\} = e(I; R), \quad \nu_{\log |I|}(\log |f|) = e(I; R/(f)) \quad (0 \neq f \in \mathfrak{m}).$$

For clarity, the second formula in (iv) uses the integration current of the hypersurface, not an ambient n -dimensional multiplicity. If f is irreducible, Poincaré–Lelong identifies it with $(dd^c \log |I|)^{n-1} \wedge [f = 0] \{o\}$. The analytic multiplicity formula on the possibly singular reduced hypersurface is also a case of [8, Theorem 1.1], with the same ideal repeated $n - 1$ times. For a general f , both sides are additive over its irreducible factors, with their multiplicities. Thus no smoothness assumption on $f = 0$ is used.

How to read the comparison inequalities. More negative weights have larger residual masses. Replacing Φ by $b\Phi$ multiplies each of the $n - 1$ weight factors in $\nu_\Phi(u)$ by b , and hence multiplies the result by b^{n-1} . In τ_Φ there are n factors, so the multiplier is b^n . On the other hand $\sigma(u, b\Phi) = b^{-1}\sigma(u, \Phi)$. These scaling rules agree with the flatness identity. The inequality $\nu_\Phi(u) \geq \tau_\Phi \sigma(u, \Phi)$ follows by comparison in the u variable from $u \leq \sigma(u, \Phi)\Phi + O(1)$. There is no general reverse comparison.

Example (Why flatness needs a proof). The following example is from [13, Example 1]. In $\mathbb{C}^2$, put $I = (x^3 - y^3, xy)$ and $\phi = \log |I|$. This is an isolated-pole maximal weight. The two generators form a system of parameters. Their intersection multiplicity is 6: the divisor $xy = 0$ has two components and $x^3 - y^3$ has order 3 on each. Thus $\tau_\phi = e(I) = 6$. On $x = 0$ the ideal becomes (y^3) , so $\nu_\phi(\log |x|) = 3$. The normalized mass is $1/2$. But on $y = 0$ the inequality $\log |x| \leq c\phi + O(1)$ would give $\log |x| \leq 3c \log |x| + O(1)$, which forces $c \leq 1/3$. Hence $\sigma(\log |x|, \phi) \leq 1/3 < 1/2$ and this weight is not flat.

2.3. Valuations and birational directions.

A valuation is a generalized order of vanishing. A centered real valuation on R assigns $w(f) \in \mathbb{R}_{\geq 0}$ to each nonzero germ and satisfies

$$\begin{aligned} w(fh) &= w(f) + w(h), & w(f+h) &\geq \min\{w(f), w(h)\}, \\ w(c) &= 0 & (c \in \mathbb{C}^*). \end{aligned}$$

We set $w(0) = +\infty$. Centered at o means that $w(f) > 0$ for $0 \neq f \in \mathfrak{m}$ and that units have value zero. For a finitely generated nonzero ideal $I = (f_1, \dots, f_r)$,

$$w(I) = \min_i w(f_i) = \min_{0 \neq f \in I} w(f).$$

Every $f = \sum a_i f_i$ has value at least $\min_i w(f_i)$, and one generator realizes that minimum. This proves independence of generators, and also gives

$$w(IJ) = w(I) + w(J), \quad w(I + J) = \min\{w(I), w(J)\}, \\ I \subset J \implies w(I) \geq w(J).$$

For $w_\Phi(f) = \sigma(\log |f|, \Phi)$, the valuation axioms are an input from Zhou-weight theory, not a consequence of the definition of relative type for arbitrary weights.

Monomial and quasi-monomial valuations. For positive real $\alpha_1, \dots, \alpha_n$, the monomial valuation is

$$w_\alpha \left(\sum_{\beta} c_{\beta} z^{\beta} \right) = \min_{c_{\beta} \neq 0} \sum_{i=1}^n \alpha_i \beta_i.$$

Ordinary order of vanishing is the case $\alpha_i = 1$. A modification $\pi : Y \rightarrow \mathbb{C}^n$ is a proper holomorphic map which is an isomorphism outside a smaller analytic set; in the BFJ setup the modifications are above o . A blowup is the basic example. In the chart $(x, t) \mapsto (x, xt)$ of the blowup of $\mathbb{C}^2$ at o ,

$$(x^2 + y^3) \circ \pi = x^2(1 + xt^3).$$

The order along the exceptional divisor $x = 0$ is 2. Further blowups can distinguish germs having the same ordinary order.

A divisorial valuation is a positive multiple of the order along a prime divisor on a smooth model. A quasi-monomial valuation is a monomial valuation in local equations of divisors meeting with simple normal crossings on a smooth model. “Simple normal crossings” means that the divisors are coordinate hyperplanes locally. Resolution of singularities provides such models; we use their valuation and retraction formalism from [11, Sections 3–4]. We do not need a quasi-monomial description of the Zhou valuations. Retractions retain finitely many birational coordinates. If $E_1, \dots, E_r$ are the divisors through the center of v on a model, its retraction keeps the numbers $v(E_i)$ and uses the monomial rule in their local equations. Extra cancellation or vanishing in a germ is not recorded by these numbers; hence $r_\pi(v)(f) \leq v(f)$. The models are directed by domination: a sufficiently high model dominates any two fixed ones. Infima and suprema over π below refer to this directed family, not to one selected blowup.

2.4. The BFJ valuative formulas. Let $\mathcal{V}$ be the compact space of semi-valuations on the completed local ring, centered at o and normalized by $v(\mathfrak{m}) = 1$. Evaluations of nonzero elements may equal $+\infty$ at a semi-valuation. An actual centered real valuation on R extends to this space after normalization: truncating a power series modulo $\mathfrak{m}^N$ determines its value whenever that value is less than N . Write $\mathcal{V}_{\text{qm}}$ for the quasi-monomial points.

The valuative transform $\widehat{u}$ is a nonpositive formal psh function. It satisfies

$$\widehat{\log |I|}(v) = -v(I), \quad \widehat{bu} = b\widehat{u} \quad (b > 0), \quad \max(\widehat{u_1}, \widehat{u_2}) = \max(\widehat{u_1}, \widehat{u_2}).$$

For an unnormalized real valuation w , put

$$(2.3) \quad w[u] = w(\mathfrak{m})(-\widehat{u}(w/w(\mathfrak{m}))).$$

This is notation for evaluation of a psh germ; $w(f)$ continues to denote evaluation of a holomorphic germ. In particular, $w[\log |I|] = w(I)$.

At a divisorial valuation cord_E , the number $w[u]$ is c times the generic Lelong number of $u \circ \pi$ along E . At a quasi-monomial valuation it is a directional Lelong number on the model. The formal psh extension supplies the remaining values. The minus sign in $\widehat{u}$ keeps the transform nonpositive; $w[u]$ uses the nonnegative sign of orders of vanishing. Neither notation denotes the pointwise value $u(o)$.

Theorem 2.3 ([4]). *Let Φ be a tame maximal weight and $g = \widehat{\Phi}$. Then g is bounded, continuous, and strictly negative on $\mathcal{V}$, and*

$$(2.4) \quad \nu_\Phi(u) = \int_{\mathcal{V}} -\widehat{u} d\text{MA}(g),$$

$$(2.5) \quad \sigma(u, \Phi) = \inf_{v \in \mathcal{V}} \frac{-\widehat{u}(v)}{-g(v)}.$$

Every formal psh function h admits a sequence $h_j = c_j \log |I_j|$ decreasing to h at every point of $\mathcal{V}$, where $c_j > 0$ and I_j are primary ideals. Finally, its values on all of $\mathcal{V}$ are determined by its values on $\mathcal{V}_{\text{qm}}$:

$$(2.6) \quad h(v) = \inf_{\pi} h(r_{\pi}(v)).$$

Here $r_{\pi}(v)$ is the quasi-monomial retraction associated to a sufficiently high simple modification π .

The sources are Theorem B, Theorem 5.12, Proposition 3.13, and Definition 2.8 of [4], respectively. Boundedness and strict negativity follow from (2.2). Continuity follows from the tame multiplier-ideal approximation [4, Lemma 5.9]: the continuous transforms of primary ideal weights converge uniformly, since their errors are bounded by a constant times $(-g)/m$. Primary ideals in the formal approximation may be regarded as analytic ideals: if $\mathfrak{m}^N \subset I$, truncating generators to polynomials modulo $\mathfrak{m}^N$ does not change the completed ideal or its valuation function.

Here $\text{MA}(g)$ is a positive Radon measure on the compact space $\mathcal{V}$, not the analytic measure $(dd^c \Phi)^n$ on $\mathbb{C}^n$. The first formula is an integral over birational directions; the second is the smallest ratio over those directions. Comparing them is an “average versus minimum” problem. In lemma 4.3, the average for a Zhou weight will be shown to be concentrated at one direction. The retraction formula does not assert continuity of arbitrary $\widehat{u}$ on all of $\mathcal{V}$; such transforms may take the value $-\infty$. The bounded continuity available for a tame Φ is a special property of the weight, not of every test germ.

3. VALUATION IDEALS ON HYPERSURFACES

3.1. What length and multiplicity count. If $(S, \mathfrak{n})$ is a local analytic ring with residue field $\mathbb{C}$ and I contains a power of $\mathfrak{n}$, then S/I is finite-dimensional over $\mathbb{C}$. Its length $\ell_S(S/I)$ is that dimension. The condition $\sqrt{I} = \mathfrak{n}$ is called $\mathfrak{n}$ -primary; geometrically the zero set is just the closed point. It makes all quotient lengths below finite.

For a fixed primary ideal I in a ring of dimension d , the Hilbert–Samuel theorem states that $\ell_S(S/I^q)$ is a polynomial in q for large integers q , with leading term

$$\ell_S(S/I^q) = \frac{e(I; S)}{d!} q^d + O(q^{d-1}).$$

This defines the multiplicity $e(I; S)$; see [10, Chapter 11]. Two indices must be kept separate: q forms powers of one fixed ideal, whereas m below changes the ideal itself. Cutkosky’s theorem relates the resulting asymptotics; the Hilbert–Samuel theorem alone does not do so.

For example, $R/\mathfrak{m}^m$ has as a basis the monomials of total degree less than m . Thus

$$\ell_R(R/\mathfrak{m}^m) = \binom{m+n-1}{n}, \quad e(\mathfrak{m}^m; R) = m^n.$$

Both become 1 after division by m^n and the length normalization by $n!$. On $R/(z_1)$ the analogous power has multiplicity m^{n-1} . This explains the dimensions and factorials in the formulas below.

3.2. The existence theorem and its hypotheses. This section is algebraic. Let w be any centered real valuation of $R = \mathbb{C}\{z_1, \dots, z_n\}$. Set $\lambda = w(\mathfrak{m}) > 0$ and

$$\mathfrak{a}_t = \{f : w(f) \geq t\} \quad (t > 0), \quad \mathfrak{a}_t = R \quad (t \leq 0), \quad H(t) = \ell_R(R/\mathfrak{a}_t).$$

The family is decreasing and graded: $\mathfrak{a}_s \mathfrak{a}_t \subset \mathfrak{a}_{s+t}$. Moreover $\mathfrak{m}^{\lceil t/\lambda \rceil} \subset \mathfrak{a}_t \subset \mathfrak{m}$ for $t > 0$, so its positive-index ideals are primary.

To verify the containment, each monomial of total degree $N = \lceil t/\lambda \rceil$ has value at least $N\lambda \geq t$. These monomials generate $\mathfrak{m}^N$. Since units have value zero, $\mathfrak{a}_t \subset \mathfrak{m}$ for $t > 0$. No finite generation of the *whole graded algebra* $\bigoplus_m \mathfrak{a}_m$ is asserted; only each individual ideal is finitely generated, because R is Noetherian.

Theorem 3.1 (Cutkosky). *Let $(S, \mathfrak{n})$ be an analytically unramified equicharacteristic local ring of dimension $d > 0$, with perfect residue field. For a graded family $I_\bullet$ of $\mathfrak{n}$ -primary ideals, both limits below exist and*

$$\lim_{m \rightarrow \infty} \frac{d! \ell_S(S/I_m)}{m^d} = \lim_{m \rightarrow \infty} \frac{e(I_m; S)}{m^d}.$$

This is [5, Theorems 5.8 and 6.5]. Analytically unramified means that the completion is reduced. Reduced complex analytic local rings are excellent and

analytically unramified. Hence the theorem applies to R and, for irreducible f , to $R/(f)$ even when this hypersurface is singular. In particular,

$$(3.1) \quad \text{vol}(w) = \lim_{m \rightarrow \infty} \frac{n!H(m)}{m^n} = \lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m; R)}{m^n}.$$

The limit is finite; positivity is not assumed here. Monotonicity of H extends its asymptotic formula from integer to real indices by comparison with the floor and ceiling.

More explicitly, if $m = \lfloor t \rfloor$, then $H(m) \leq H(t) \leq H(m+1)$, and $m/t, (m+1)/t \rightarrow 1$. Hence

$$H(t) = \frac{\text{vol}(w)}{n!} t^n + o(t^n) \quad (t \rightarrow \infty, t \in \mathbb{R}).$$

The same reasoning applies to the increasing hypersurface lengths. “Analytically unramified” concerns reducedness of the completion, not smoothness. For irreducible f , excellence of the reduced analytic ring ensures that its completion is reduced too. A repeated factor would destroy reducedness. We treat repeated factors only after establishing the reduced case.

3.3. An exact sequence computes the hypersurface coefficient.

Proposition 3.2 (Hypersurface-volume formula). *For every nonzero nonunit $f \in R$,*

$$\lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m(w); R/(f))}{m^{n-1}} = w(f) \text{vol}(w).$$

No finite-generation or quasi-monomiality hypothesis is required.

Proof. Step 1: shift the valuation cutoff by multiplication. First suppose f is irreducible. Put $s = w(f) > 0$ and $S = R/(f)$. For every real t and $h \in R$,

$$h \in (\mathfrak{a}_t : f) \iff w(fh) = s + w(h) \geq t \iff h \in \mathfrak{a}_{t-s}.$$

Thus multiplication by f gives the exact sequence

$$(3.2) \quad 0 \longrightarrow R/\mathfrak{a}_{t-s} \xrightarrow{\cdot f} R/\mathfrak{a}_t \longrightarrow R/(\mathfrak{a}_t + (f)) \longrightarrow 0.$$

Its first map is well-defined by the displayed colon identity; the same identity gives injectivity. Its cokernel is the last term by definition. Indeed, the image consists of the classes of multiples of f in $R/\mathfrak{a}_t$, namely $(fR + \mathfrak{a}_t)/\mathfrak{a}_t$; quotienting by it gives $R/(\mathfrak{a}_t + (f))$. All three modules have finite length (the first is zero if $t - s \leq 0$). We may therefore use additivity of length in a short exact sequence. Consequently

$$(3.3) \quad L_f(t) := \ell_S(S/\mathfrak{a}_t S) = H(t) - H(t - s).$$

Step 2: obtain the hypersurface limit before using it. The ideals $\mathfrak{a}_m S$ are a graded primary family in the reduced analytic local ring S of dimension $n - 1$. By [theorem 3.1](#), there is a finite number B_f such that

$$(3.4) \quad \lim_{m \rightarrow \infty} \frac{(n-1)!L_f(m)}{m^{n-1}} = B_f = \lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m; S)}{m^{n-1}}.$$

Since L_f is increasing, its asymptotic formula again holds for real indices. Notice that L_f is increasing because it is a quotient length for decreasing ideals, not merely because it is the difference of two increasing functions.

Step 3: sum the exact differences. Evaluating (3.3) at $t = s, 2s, \dots, qs$ and using $H(0) = 0$, we obtain

$$\begin{aligned} H(qs) &= \sum_{j=1}^q L_f(js) \\ &= \frac{B_f s^{n-1}}{(n-1)!} \sum_{j=1}^q j^{n-1} + o(q^n) \\ &= \frac{B_f s^{n-1}}{n!} q^n + o(q^n). \end{aligned}$$

The summed error is $o(q^n)$: for every $\varepsilon > 0$, all sufficiently large individual errors are bounded by εj^{n-1} ; the finitely many earlier errors have bounded sum. On the other hand, (3.1) yields

$$H(qs) = \frac{\text{vol}(w)s^n}{n!} q^n + o(q^n).$$

Comparison gives $B_f = s \text{vol}(w)$, proving the irreducible case.

The power sum used here is a Riemann sum:

$$\frac{1}{q^n} \sum_{j=1}^q j^{n-1} = \frac{1}{q} \sum_{j=1}^q (j/q)^{n-1} \longrightarrow \int_0^1 x^{n-1} dx = \frac{1}{n}.$$

Dividing the two expressions for $H(qs)$ by q^n gives $B_f s^{n-1}/n! = \text{vol}(w)s^n/n!$. Since $s > 0$, we can cancel $s^{n-1}/n!$. The argument also covers $\text{vol}(w) = 0$.

Step 4: restore multiplicities of components. For general f , unique factorization in R gives $f = a \prod_{i=1}^r f_i^{b_i}$, with a a unit and f_i pairwise nonassociate irreducibles. The associativity formula for multiplicity [10, Chapter 11] gives

$$e(\mathfrak{a}_m; R/(f)) = \sum_{i=1}^r b_i e(\mathfrak{a}_m; R/(f_i)).$$

Indeed, $R_{(f_i)}$ is a discrete valuation ring and $\ell((R/(f))_{(f_i)}) = b_i$. At this localization the other factors are units, so we are taking the quotient of a discrete valuation ring by the b_i -th power of a uniformizer, which has length b_i . Divide by m^{n-1} and use the irreducible case together with $w(f) = \sum_i b_i w(f_i)$. $\square$

Remark 3.3. An asymptotic formula $H(t) = \text{vol}(w)t^n/n! + o(t^n)$ alone does not allow a fixed-width difference of order t^{n-1} to be computed. Here the separate existence theorem on $R/(f)$ supplies that missing limit before the telescoping argument is used. For reducible f , the factorization step avoids applying Cutkosky's reduced-completion theorem to a nonreduced hypersurface.

Example (A direct two-dimensional check). Let $w(x) = a > 0$, $w(y) = b > 0$ be a monomial valuation. A basis of $R/\mathfrak{a}_m(w)$ consists of $x^i y^j$ with $ai + bj < m$. The triangle bounded by this line has area $m^2/(2ab)$; the boundary contributes only $O(m)$ lattice points. Thus $\text{vol}(w) = 1/(ab)$. On $x = 0$, the ideal becomes $(y^{\lceil m/b \rceil})$, whose multiplicity is $\lceil m/b \rceil$. After division by m the limit is $1/b = w(x) \text{vol}(w)$, as predicted. The general proof replaces the monomial count by an exact sequence; it does not assume monomial coordinates for w .

4. THE MASS FORMULA AND EQUALITY FOR ZHOU WEIGHTS

Fix a tame maximal representative Φ of a local Zhou weight and put $w = w_\Phi$. We prove [theorem 1.1](#) in three steps: comparison with its valuation ideals, equality on holomorphic logarithms, and extension to arbitrary psh germs.

4.1. The valuation-ideal approximation. The ideal $\mathfrak{a}_m(w)$ consists of the holomorphic tests whose relative type is at least m . The psh germ $\Psi_m = m^{-1} \log |\mathfrak{a}_m(w)|$ combines all these tests into one analytic singularity. Tameness will make the discrepancy between its singularity and that of Φ a relative error of size C/m . No convergence of Monge–Ampère measures for arbitrary approximating sequences will be needed: we will compare their masses directly.

Lemma 4.1. *There is an integer $C > 0$ independent of m such that, with $\Psi_m = m^{-1} \log |\mathfrak{a}_m(w)|$,*

$$(4.1) \quad (1 + C/m)\Phi \leq \Psi_m \leq \Phi \quad \text{modulo bounded functions.}$$

Proof. Step 1: use tameness to get an inclusion of ideals. Choose an integer C at least the tameness constant, and choose a negative representative of Φ on a small neighborhood. If $f \in \mathcal{J}((m + C)\Phi)$, tameness says $\log |f| \leq m\Phi + O(1)$ (with an even stronger coefficient if the chosen C exceeds the original constant). Thus $w(f) \geq m$. Equation (2.1) implies $\mathcal{J}((m + C)\Phi) \subset \mathfrak{a}_m(w)$.

Step 2: obtain the lower bound for the weight. Demailly approximation and inclusion of ideals give

$$(m + C)\Phi \leq \log |\mathcal{J}((m + C)\Phi)| + O(1) \leq \log |\mathfrak{a}_m(w)| + O(1).$$

Here $I \subset J$ implies $\log |I| \leq \log |J| + O(1)$: express the finitely many generators of I as combinations of those of J , with holomorphic coefficients bounded on a smaller neighborhood. Division by m gives $(1 + C/m)\Phi \leq \Psi_m + O(1)$.

Step 3: obtain the upper bound from the definition of type. Each generator f of $\mathfrak{a}_m(w)$ satisfies $\sigma(\log |f|, \Phi) \geq m$. Attainment of relative type implies $\log |f| \leq m\Phi + O(1)$. Taking the maximum over finitely many generators and dividing by m proves (4.1). Bounded terms may depend on m ; comparison of residual masses is insensitive to these terms. Indeed, for r generators,

$\max_i \log |f_i| \leq \log |\mathbf{a}_m| \leq \max_i \log |f_i| + \frac{1}{2} \log r$. Each generator has its own bounded constant, but the maximum of finitely many such constants is still finite. Uniformity in m of those additive constants is not claimed or required. $\square$

Proposition 4.2. *One has $\tau_\Phi = \text{vol}(w) > 0$ and*

$$(4.2) \quad \nu_\Phi(\log |f|) = \text{vol}(w)w(f) \quad \text{for every } f \in R \setminus \{0\}.$$

Proof. Step 1: identify the ambient mass. By [proposition 2.2](#) and [lemma 4.1](#),

$$(4.3) \quad \tau_\Phi \leq \frac{e(\mathbf{a}_m(w); R)}{m^n} \leq (1 + C/m)^n \tau_\Phi.$$

The middle quantity is the residual mass of Ψ_m . Its limit is $\text{vol}(w)$ by (3.1), so the squeeze proves $\tau_\Phi = \text{vol}(w)$. Positivity follows from the positive residual mass of the maximal weight.

For the direction of these inequalities, $\Psi_m \leq \Phi + O(1)$ means Ψ_m is the more singular weight and has mass at least τ_Φ . The other half of the sandwich bounds that mass above by the mass of $(1 + C/m)\Phi$. The normalization m^{-n} comes from the n factors of $dd^c \Psi_m$.

Step 2: identify the mass tested by one hypersurface. For irreducible f , the hypersurface multiplicity formula gives

$$\nu_{\Psi_m}(\log |f|) = \frac{e(\mathbf{a}_m(w); R/(f))}{m^{n-1}}.$$

Comparison in the weight variable has degree $n - 1$, hence

$$(4.4) \quad \nu_\Phi(\log |f|) \leq \frac{e(\mathbf{a}_m(w); R/(f))}{m^{n-1}} \leq (1 + C/m)^{n-1} \nu_\Phi(\log |f|).$$

By [proposition 3.2](#), the middle term tends to $w(f) \text{vol}(w)$. This proves (4.2) for irreducible f . Both end terms have the same limit because $(1 + C/m)^{n-1} \rightarrow 1$; therefore the fixed number $\nu_\Phi(\log |f|)$ equals the computed middle limit. The possible singularity of $f = 0$ is covered by the analytic multiplicity formula already specified in [Section 2](#).

Step 3: handle all nonzero holomorphic germs. Factorization and additivity of both $dd^c \log |f|$ and $w(f)$ give the general case. A unit contributes zero to both sides. $\square$

4.2. From holomorphic tests to all psh germs. The next lemma isolates the use of the BFJ formulas. Its hypothesis is exactly what [proposition 4.2](#) has established.

The point of this step is that additivity on holomorphic logarithms alone would not justify approximation of a mixed mass in its first argument. Instead, we determine the representing measure. Once it is a point mass, integrating any test function is just evaluating it at that point. The simultaneous minimization needed for relative types is checked separately, using primary-ideal approximation.

Lemma 4.3 (Holomorphic testing). *Let Φ be tame and maximal, and suppose*

$$\nu_\Phi(\log |f|) = \tau_\Phi \sigma(\log |f|, \Phi) \quad \text{for every nonzero } f \in R.$$

Then there is a unique $v_0 \in \mathcal{V}$ such that

$$(4.5) \quad \text{MA}(g) = \frac{\tau_\Phi}{-g(v_0)} \delta_{v_0}, \quad \frac{\nu_\Phi(u)}{\tau_\Phi} = \sigma(u, \Phi) = \frac{-\hat{u}(v_0)}{-g(v_0)}, \quad g = \hat{\Phi},$$

for every psh germ u .

Proof. Step 1: turn the representing measure into a probability. Applying (2.4) to $u = \Phi$ gives $\int_{\mathcal{V}} (-g) d\text{MA}(g) = \tau_\Phi$. Thus

$$dP = \frac{-g}{\tau_\Phi} d\text{MA}(g)$$

is a Radon probability measure: its mass is 1, and its density is positive and continuous. It has the same support as $\text{MA}(g)$ since $-g$ is bounded above and bounded away from zero.

Step 2: equality of an average and its lower bound. For a nonzero holomorphic germ f , set $R_f(v) = v(f)/(-g(v))$ and $a_f = \sigma(\log |f|, \Phi)$. The hypotheses and theorem 2.3 give

$$\inf_{\mathcal{V}} R_f = a_f = \int_{\mathcal{V}} R_f dP.$$

The nonnegative function $R_f - a_f$ therefore has integral zero. The two equalities in that display have different sources. The infimum is the BFJ relative-type formula. For the integral, substitute the density defining P and cancel $-g$:

$$\int_{\mathcal{V}} R_f dP = \frac{1}{\tau_\Phi} \int_{\mathcal{V}} v(f) d\text{MA}(g) = \frac{\nu_\Phi(\log |f|)}{\tau_\Phi} = a_f.$$

In particular the integral is finite even if R_f is infinite at some semivaluations. It is continuous with values in $[0, +\infty]$, because evaluation of f is continuous in the valuation topology and $-g$ is continuous and bounded away from zero. If $R_f(v) > a_f$ at a point of $\text{supp } P$, some neighborhood of that point has positive P -mass and $R_f - a_f \geq \varepsilon > 0$ there, a contradiction. Consequently

$$(4.6) \quad \frac{v(f)}{-g(v)} = a_f \quad \text{for every } v \in \text{supp } P \text{ and every } f \neq 0.$$

This argument is made for each f on the support itself, so no intersection of uncountably many full-measure sets is involved.

More explicitly, the support consists of points every neighborhood of which has positive measure. If $R_f(v) > a_f$, continuity gives a neighborhood U and $\varepsilon > 0$ with $R_f - a_f > \varepsilon$ on U , even when $R_f(v) = +\infty$. Then $\int (R_f - a_f) dP \geq \varepsilon P(U) > 0$, the required contradiction.

Step 3: the holomorphic tests separate support points. The support is nonempty. If v_1, v_2 belong to it, then $v_1/(-g(v_1))$ and $v_2/(-g(v_2))$ agree on every holomorphic germ. Taking their minima on the coordinate generators

of $\mathfrak{m}$ and using $v_i(\mathfrak{m}) = 1$ shows $-g(v_1) = -g(v_2)$, and therefore $v_1 = v_2$. Thus $P = \delta_{v_0}$ for one point v_0 .

Here equality on convergent germs also gives equality on formal series. For a formal F and its degree- N truncation F_N , a normalized semivaluation satisfies $v(F - F_N) \geq N + 1$. If $v(F) < N + 1$, the valuation inequality gives $v(F_N) = v(F)$. If $v(F) = +\infty$, all truncations have values tending to infinity. Thus the values on the polynomial truncations determine $v(F)$. Finally, a Radon probability on a compact space with singleton support is the Dirac probability there: every compact subset of the complement is covered by finitely many zero-measure neighborhoods, and inner regularity gives zero measure to the complement.

Step 4: the same point minimizes every psh test. It remains to identify the relative type for an arbitrary u . By (4.6) and the definition of the infimum,

$$\frac{v_0(f)}{-g(v_0)} \leq \frac{v(f)}{-g(v)} \quad \text{for every } v \in \mathcal{V} \text{ and every } f \neq 0.$$

Taking the minimum over generators proves this inequality for any primary ideal, and positive scaling preserves it. Indeed, for $I = (f_1, \dots, f_r)$ both ideal values are the minima of the respective generator values; if $a_i \leq b_i$ for all i , then $\min_i a_i \leq \min_i b_i$. Multiplication by $c > 0$ preserves order. Choose $h_j = c_j \log |I_j| \downarrow \hat{u}$ as in [theorem 2.3](#). For each fixed v , pass to the limit in

$$\frac{-h_j(v_0)}{-g(v_0)} \leq \frac{-h_j(v)}{-g(v)}.$$

It follows that v_0 attains the infimum in (2.5) for u . This is a pointwise passage to the limit for each fixed v ; monotone limits in $[0, +\infty]$ preserve the inequality. No uniform convergence of h_j is asserted.

Step 5: return to the original measure. Integration against $P = \delta_{v_0}$ gives the generalized Lelong number in (4.5). The measure formula follows from the definition of P . This limit argument does not exchange an infimum and a limit. $\square$

Proof of theorem 1.1. By [proposition 4.2](#), the hypothesis of [lemma 4.3](#) holds. Hence $\nu_\Phi(u) = \tau_\Phi \sigma(u, \Phi)$ for all u . On holomorphic germs, that lemma also gives

$$w(f) = \frac{v_0(f)}{-g(v_0)}.$$

Taking minima on generators of $\mathfrak{m}$ yields $\lambda = 1/(-g(v_0))$, and therefore $v_0 = w/\lambda$. Since $\tau_\Phi = \text{vol}(w)$, all formulas in (1.1) and (1.2) follow. In particular the point v_0 just constructed is the normalization of the actual real valuation w_Φ ; it is not a semivaluation with nonzero kernel on R . $\square$

Corollary 4.4 (Normalization and quantitative estimates). *For $v = w_\Phi/\lambda$ as above,*

$$\tau_\Phi = \lambda^{-n} \text{vol}(v), \quad \text{MA}(\hat{\Phi}) = \lambda^{1-n} \text{vol}(v) \delta_v.$$

The constant C of [lemma 4.1](#) also satisfies, for every $m \geq 1$,

$$\text{vol}(w) \leq \frac{e(\mathfrak{a}_m(w); R)}{m^n} \leq (1 + C/m)^n \text{vol}(w),$$

and, for every $0 \neq f \in \mathfrak{m}$,

$$w(f) \text{vol}(w) \leq \frac{e(\mathfrak{a}_m(w); R/(f))}{m^{n-1}} \leq (1 + C/m)^{n-1} w(f) \text{vol}(w).$$

Proof. Since $\mathfrak{a}_t(bv) = \mathfrak{a}_{t/b}(v)$, the real-index volume limit gives $\text{vol}(bv) = b^{-n} \text{vol}(v)$. The estimates are (4.3) and (4.4), with the masses now identified. For reducible f , sum over the irreducible factors. $\square$

5. CONTROLLED DENSITY OF ALGEBRAIC ZHOU VALUATIONS

At this stage flatness is proved. To obtain separation, we must show that the valuations associated to Zhou weights test enough directions. Density on holomorphic functions is not yet sufficient: the psh transform need not be continuous in the weak valuation topology. The solution is to approximate a psh germ by a multiplier ideal and to bound the error *uniformly for the valuations being chosen*.

5.1. Algebraic Zhou valuations and analytic correspondence.

Integrability thresholds. The number $c_o^{\mathfrak{q}}(u)$ below is the largest exponent for which $|\mathfrak{q}|^2 e^{-2tu}$ remains integrable. A larger t usually makes the singularity worse, whereas the holomorphic factor $|\mathfrak{q}|^2$ helps integrability. For example, in one variable $u = a \log |z|$ and $\mathfrak{q} = (z^p)$ give

$$\int_0^\varepsilon r^{2p-2ta+1} dr < \infty \iff ta < p + 1.$$

Thus $c_o^{(z^p)}(a \log |z|) = (p + 1)/a$ for $a > 0$. The log discrepancy in higher dimensions plays the role of the 1 in this numerator, after change of variables on a modification.

The algebraic setting and the discrepancy. The ring $R = \mathcal{O}_{\mathbb{C}^n, o}$ is a regular excellent Noetherian local domain of characteristic zero. Thus $X = \text{Spec } R$ is among the schemes considered in [3]. All centered valuations and all ideals below are taken on this same analytic local ring; there is no change from convergent germs to rational functions on affine space.

For a graded sequence $I_\bullet$, write

$$w(I_\bullet) = \lim_m \frac{w(I_m)}{m} = \inf_m \frac{w(I_m)}{m}, \quad \text{lct}^{\mathfrak{q}}(I_\bullet) = \sup_m m c_o^{\mathfrak{q}}(\log |I_m|),$$

where $c_o^{\mathfrak{q}}(u) = \sup\{t \geq 0 : |\mathfrak{q}|^2 e^{-2tu} \text{ is locally integrable}\}$. For ideals on R , this analytic threshold agrees with the algebraic jumping number; see [3, Proposition 11.1]. The valuative formula gives, in particular,

$$(5.1) \quad \text{lct}^{\mathfrak{q}}(I_\bullet) \leq \frac{A(v) + v(\mathfrak{q})}{v(I_\bullet)} \quad \text{when } v(I_\bullet) > 0, A(v) < \infty.$$

Here A is the usual log discrepancy, as in [11, Section 5]. On a log-smooth model, A of a quasi-monomial valuation is the weighted sum of the log discrepancies of the defining divisors. On an arbitrary valuation it is the supremum of the discrepancies of its quasi-monomial retractions. In particular,

$$A(bv) = bA(v), \quad A(r_\pi(v)) \leq A(v).$$

A centered nontrivial valuation of finite discrepancy has $A(v) > 0$.

More concretely, if E is a prime divisor on a smooth model and the Jacobian of π vanishes to order k_E along E , then $A(\text{ord}_E) = k_E + 1$ and $A(b \text{ord}_E) = b(k_E + 1)$. For the blowup of $o \in \mathbb{C}^n$, the chart $(t, s_2, \dots, s_n) \mapsto (t, ts_2, \dots, ts_n)$ has Jacobian t^{n-1} ; hence its exceptional order has discrepancy n . In coordinates with no modification, a monomial valuation of weights α_i has $A = \sum_i \alpha_i$. These examples explain the convention for A ; some sources use a discrepancy without the added 1, which is not our convention.

Definition 5.1 ([3, Definition 3.2]). An algebraic Zhou valuation related to a nonzero ideal $\mathfrak{q}$ is a maximal element, for pointwise order on R , among valuations w satisfying $\text{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet(w)) \leq 1$. We write $\mathcal{Z}$ for the cone of positive multiples of algebraic Zhou valuations centered at o .

We use the following precise results from [3].

Proposition 5.2 (Bao–Guan–Zhou). (i) *If $\tilde{w}$ is an algebraic Zhou valuation related to $\mathfrak{q}_0$, then*

$$(5.2) \quad A(\tilde{w}) = 1 - \tilde{w}(\mathfrak{q}_0) < \infty.$$

- (ii) *If v is centered and $C = \text{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet(v))$ is finite and positive, there is an algebraic Zhou valuation $\tilde{w}$ related to $\mathfrak{q}$ such that $C\tilde{w} \geq v$.*
- (iii) *A valuation on R centered at o is an algebraic Zhou valuation related to $\mathfrak{q}_0$ if and only if it equals w_Φ for an analytic Zhou weight Φ related to $|\mathfrak{q}_0|^2$.*

Part (i) is [3, Theorem 4.1]; part (ii) follows by applying [3, Corollary 3.4] to v/C , as in the proof of its Lemma 9.1; part (iii) is its Theorem 11.2. For (ii), the scaling identity $\mathfrak{a}_t(v/C) = \mathfrak{a}_{Ct}(v)$ makes the threshold of v/C equal to one. Also $\tilde{w} \geq v/C$ implies $\tilde{w}(\mathfrak{m}) > 0$, so the analytic correspondence applies.

The scaling is worth making explicit: $\text{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet(bv)) = b \text{lct}^{\mathfrak{q}}(\mathfrak{a}_\bullet(v))$. It follows from reindexing the filtration, or from the valuative formula since $\mu(\mathfrak{a}_\bullet(bv)) = b^{-1} \mu(\mathfrak{a}_\bullet(v))$ for a test valuation μ . Thus division by C , not multiplication by C , gives threshold one before the Zhou majorant is chosen.

Consequently every $w \in \mathcal{Z}$ has finite log discrepancy and can be written $w = bw_\Phi$ with $b > 0$ and Φ a local Zhou weight. By [theorem 1.1](#),

$$(5.3) \quad w[u] = b\sigma(u, \Phi).$$

The scalar b is kept explicit: the normalization fixed by a datum $\mathfrak{q}_0$ is not the normalization $w(\mathfrak{m}) = 1$.

5.2. A bounded-discrepancy refinement of density. For finitely many ideals I_i we will encode all required values in one product $\mathfrak{q} = \mathfrak{m}I_1 \cdots I_r$. A majorant $w \geq v$ makes each difference $w(I_i) - v(I_i)$ nonnegative. If their sum is small, each is small. The same construction will also bound $A(w)$; this is what later controls the psh approximation error.

Proposition 5.3. *Let v be a centered valuation with $A = A(v) < \infty$. Given nonzero ideals $I_1, \dots, I_r$ and an integer $k \geq 1$, there exists $w \in \mathcal{Z}$ such that*

$$(5.4) \quad w \geq v, \quad 0 < A(w) \leq A, \quad 0 \leq w(I_i) - v(I_i) \leq A/k \quad (1 \leq i \leq r).$$

The same choice can be required to satisfy $0 \leq w(\mathfrak{m}) - v(\mathfrak{m}) \leq A/k$.

Proof. Step 1: choose one auxiliary ideal and its threshold. Put $\mathfrak{q} = \mathfrak{m}I_1 \cdots I_r$ and $C_k = \text{lct}^{\mathfrak{q}^k}(\mathfrak{a}_\bullet(v))$. First, $v(\mathfrak{a}_\bullet(v)) = 1$: $v(\mathfrak{a}_m(v)) \geq m$, while, for any fixed germ f of value $s = v(f) > 0$, the power $f^{\lceil m/s \rceil}$ belongs to $\mathfrak{a}_m(v)$ and has value less than $m + s$. Thus (5.1) gives

$$(5.5) \quad C_k \leq A + kv(\mathfrak{q}) < \infty.$$

Moreover, $\mathfrak{m}^{\lceil m/v(\mathfrak{m}) \rceil} \subset \mathfrak{a}_m(v)$, so the ordinary threshold of $\mathfrak{a}_m(v)$ is at least $n/\lceil m/v(\mathfrak{m}) \rceil$. A holomorphic multiplier can only increase the threshold. Taking the supremum of m times these lower bounds gives $C_k \geq nv(\mathfrak{m}) > 0$.

For this lower bound, $\log |\mathfrak{m}^N| = N \log |z| + O(1)$ has threshold n/N , by the radial integral $\int_0^\varepsilon r^{2n-1-2tN} dr$. The inclusion of ideals gives a less singular weight for $\mathfrak{a}_m(v)$, hence a threshold no smaller than n/N . The factor $|\mathfrak{q}|^{2k}$ is bounded above and cannot destroy integrability. Finally $m/\lceil m/v(\mathfrak{m}) \rceil \rightarrow v(\mathfrak{m})$. These observations verify positivity rather than assume it.

Step 2: choose a Zhou majorant and retain its discrepancy. By [proposition 5.2\(ii\)](#), there is $\tilde{w}_k$ related to $\mathfrak{q}^k$ with $w_k := C_k \tilde{w}_k \geq v$. Homogeneity and (5.2) give the exact relation

$$(5.6) \quad A(w_k) = C_k - kw_k(\mathfrak{q}).$$

Indeed, $C_k A(\tilde{w}_k) = C_k(1 - \tilde{w}_k(\mathfrak{q}^k)) = C_k - kw_k(\mathfrak{q})$. Since $\mathfrak{q}$ is nonzero and nonunit, all these values are finite, and $w_k \geq v$ implies $w_k(\mathfrak{q}) \geq v(\mathfrak{q})$. Consequently

$$0 < A(w_k) \leq A + kv(\mathfrak{q}) - kw_k(\mathfrak{q}) = A - k(w_k(\mathfrak{q}) - v(\mathfrak{q})).$$

The single inequality simultaneously bounds the discrepancy and the excess on the product. Equivalently, we obtain

$$0 < A(w_k) \leq A, \quad 0 \leq w_k(\mathfrak{q}) - v(\mathfrak{q}) \leq A/k.$$

Step 3: separate the finitely many ideal errors. Since

$$w_k(\mathfrak{q}) - v(\mathfrak{q}) = w_k(\mathfrak{m}) - v(\mathfrak{m}) + \sum_{i=1}^r (w_k(I_i) - v(I_i)),$$

and every summand is nonnegative, each is at most A/k . Take $w = w_k$. $\square$

Remark 5.4. The construction is that of [3, Lemma 9.1 and Theorem 9.2], with the additional bound supplied by (5.6) retained. The weak density theorem itself is already established there; an earlier analytic version appears in [2, Theorems 1.13–1.14]. Our use of density requires the bound on the ordinary log discrepancy A , not merely closeness on finitely many ideals.

5.3. Uniform multiplier-ideal approximation. For a fixed integer m , the ideal $\mathcal{J}(mu)$ packages the integrability of a possibly very irregular function into finitely many holomorphic generators. The next estimate says that its valuation, divided by m , approximates $v[u]$ with error at most $A(v)/m$. Thus a bound on $A(v)$ is exactly the uniformity needed while v varies. Coherence and generic nonvanishing of multiplier ideals imply that $\mathcal{J}(mu)$ is a nonzero ideal, although it may equal R and need not be primary; see [7].

Lemma 5.5. *Let $u \in \text{PSH}_o$ and let v be centered with $A(v) < \infty$. For every integer $m \geq 1$,*

$$(5.7) \quad 0 \leq v[u] - \frac{v(\mathcal{J}(mu))}{m} \leq \frac{A(v)}{m}.$$

In particular, this holds for every $v \in \mathcal{Z}$.

Proof. Step 1: the estimate on quasi-monomial valuations. For quasi-monomial v , this is the valuative Demailly estimate [4, Equation (5.3), Section 5.2], namely

$$v(\mathcal{J}(mu)) \leq mv[u] \leq v(\mathcal{J}(mu)) + A(v).$$

The left inequality follows from $u \leq m^{-1} \log |\mathcal{J}(mu)| + O(1)$. The role of $A(v)$ in the right inequality can be seen directly for $v = \text{ord}_E$. Write $E = (t = 0)$ at a general smooth point on a model, let $a = \text{ord}_E(f)$, and let k_E be the Jacobian order. The divisorial part of $u \circ \pi$ is $\gamma \log |t|$, where $\gamma = v[u]$; after subtracting it, the remaining local psh potential is bounded above. For $f \in \mathcal{J}(mu)$, change of variables therefore forces integrability of a positive constant times

$$|t|^{2a+2k_E-2m\gamma} dV.$$

The one-variable radial integral in t requires $a + k_E - m\gamma > -1$. In particular $mv[u] \leq v(f) + A(v)$. Choosing a generator of $\mathcal{J}(mu)$ of minimal v -value gives the asserted weak inequality. Scaling treats positive multiples of ord_E . The inequality extends to a fixed quasi-monomial simplex by rational approximation, since ideal evaluations, A , and the Kiselman numbers representing $v[u]$ are continuous on that simplex; see [4, Proposition 5.6].

Here rational monomial weights, up to positive scaling, give divisorial valuations after toroidal modification. Approximating positive real weights by rational ones stays in the same local simplex. The cited continuity justifies that passage; no global continuity of $v[u]$ on the entire valuative space is used.

Step 2: take the supremum over retractions. For a general finite-discrepancy v , the left inequality still follows from Demailly approximation and monotonicity of valuative transforms. For the right inequality, let $r_\pi(v)$ range over its quasi-monomial retractions. For unnormalized valuations the retractions are defined by scaling the normalized ones. They satisfy

$$r_\pi(v) \leq v \text{ on } R, \quad A(r_\pi(v)) \leq A(v), \quad v[u] = \sup_\pi r_\pi(v)[u].$$

The first two properties are part of the retraction and discrepancy formalism of [11, Sections 4–5]; the last is (2.6) with the sign reversed. Applying the quasi-monomial estimate gives

$$mr_\pi(v)[u] \leq r_\pi(v)(\mathcal{J}(mu)) + A(r_\pi(v)) \leq v(\mathcal{J}(mu)) + A(v).$$

Taking the supremum proves (5.7). Its right side is finite because $\mathcal{J}(mu)$ is a nonzero ideal and v is a real valuation. Thus $v[u]$ is finite as well. The right-hand bound is independent of π , which is why the supremum can be taken. We do not need the retracted multiplier-ideal values to converge individually. Dividing by m now yields both bounds in (5.7). $\square$

Lemma 5.6 (Order and evaluation). *If w, v are centered, of finite log discrepancy, and $w \geq v$ on R , then $w[u] \geq v[u]$ for every $u \in \text{PSH}_o$.*

Proof. For every integer m , one has $w(\mathcal{J}(mu)) \geq v(\mathcal{J}(mu))$. This follows by taking the minimum of the pointwise inequalities on any finite list of generators of the ideal. By lemma 5.5, dividing by m and taking the limit gives $w[u] \geq v[u]$. Explicitly, the two sequences $w(\mathcal{J}(mu))/m$ and $v(\mathcal{J}(mu))/m$ converge to $w[u]$ and $v[u]$, since their errors are bounded by $A(w)/m$ and $A(v)/m$. Their termwise order survives the limit. $\square$

Theorem 5.7 (Finite-psh approximation by Zhou valuations). *Let $v(\mathbf{m}) = 1$ and $A = A(v) < \infty$, and fix finitely many psh germs $u_1, \dots, u_r$. There are $w_j \in \mathcal{Z}$, $j \geq 1$, with*

$$(5.8) \quad w_j \geq v, \quad A(w_j) \leq A, \quad 1 \leq w_j(\mathbf{m}) \leq 1 + A/j,$$

$$(5.9) \quad 0 \leq w_j[u_i] - v[u_i] \leq \frac{A}{j} + \frac{A}{j^2} \quad (1 \leq i \leq r).$$

In particular $v_j = w_j/w_j(\mathbf{m})$ is a normalized Zhou ray and $v_j[u_i] \rightarrow v[u_i]$ simultaneously for all i .

Proof. Step 1: choose ideals at the same scale as the approximation. For each j , apply proposition 5.3 with $k = j$ and $I_i = \mathcal{J}(ju_i)$. These ideals are nonzero, although they need not be primary and may equal R ; multiplication by $\mathbf{m}$ in that proof ensures a nonunit auxiliary ideal in every case. We obtain (5.8) and

$$0 \leq w_j(\mathcal{J}(ju_i)) - v(\mathcal{J}(ju_i)) \leq A/j.$$

Step 2: add the two different approximation errors. Applying lemma 5.5 to both valuations gives

$$\begin{aligned} w_j[u_i] &\leq \frac{w_j(\mathcal{J}(ju_i))}{j} + \frac{A(w_j)}{j} \\ &\leq \frac{v(\mathcal{J}(ju_i))}{j} + \frac{A}{j^2} + \frac{A}{j} \\ &\leq v[u_i] + \frac{A}{j} + \frac{A}{j^2}. \end{aligned}$$

The lower bound is lemma 5.6. The error A/j comes from replacing u_i by the multiplier-ideal weight at the valuation w_j . The error A/j^2 comes from approximating the ideal value to precision A/j and then dividing it by j . Both estimates hold for the *same* w_j and every i ; choosing separate sequences for separate test functions is not what the construction does.

Step 3: check the normalization instead of suppressing it. For normalization, set $d_j = w_j(\mathbf{m}) - 1$ and $e_{i,j} = w_j[u_i] - v[u_i]$. Since $v[u_i] < \infty$,

$$|v_j[u_i] - v[u_i]| = \left| \frac{e_{i,j} - d_j v[u_i]}{1 + d_j} \right| \leq \frac{A}{j} \left(1 + \frac{1}{j} + v[u_i] \right) \rightarrow 0.$$

□

This supplies exactly the finite set of psh tests needed for separation. It does not claim convergence of an arbitrary weakly convergent sequence of Zhou valuations. The uniform discrepancy bound and (5.7) supply the required control.

6. VALUATIVE SEPARATION AND THE BFJ CONVERSE

6.1. From equality of masses to equality of transforms. The proof now uses no new approximation theorem. The test family shrinks from all tame weights to Zhou weights, and the controlled density result restores all quasi-monomial directions. We keep the quantifiers explicit: the approximating sequence may depend on the target valuation and on the two psh germs being compared.

Proof of theorem 1.2. Step 1: recover every Zhou valuation test. Assume first that $\nu_\Phi(u_1) = \nu_\Phi(u_2)$ for every local Zhou weight. By theorem 1.1, division by the fixed positive mass τ_Φ gives $\sigma(u_1, \Phi) = \sigma(u_2, \Phi)$. Equation (5.3) therefore implies

$$(6.1) \quad w[u_1] = w[u_2] \quad \text{for every } w \in \mathcal{Z}.$$

For an unnormalized $w = bw_\Phi$, both evaluations are multiplied by the same $b > 0$. This is why equality for the original Zhou weights gives equality on the full cone $\mathcal{Z}$, not just on representatives with one particular scale.

Step 2: recover one arbitrary quasi-monomial direction. Fix $v \in \mathcal{V}_{\text{qm}}$. Its log discrepancy is finite. Apply [theorem 5.7](#) to the pair (u_1, u_2) to obtain the same sequence w_j for both germs. Using [\(6.1\)](#) and [\(5.9\)](#), we find

$$v[u_1] = \lim_j w_j[u_1] = \lim_j w_j[u_2] = v[u_2].$$

Thus $\hat{u}_1 = \hat{u}_2$ on $\mathcal{V}_{\text{qm}}$. One can avoid even writing two limits: with $\epsilon_j = A(v)/j + A(v)/j^2$, both $v[u_i]$ lie in the interval $[w_j[u_i] - \epsilon_j, w_j[u_i]]$, and the interval is the same for $i = 1, 2$. Hence $|v[u_1] - v[u_2]| \leq \epsilon_j \rightarrow 0$.

Step 3: recover all remaining semivaluations. For arbitrary $\mu \in \mathcal{V}$, the retraction formula [\(2.6\)](#) gives

$$\hat{u}_1(\mu) = \inf_{\pi} \hat{u}_1(r_{\pi}(\mu)) = \inf_{\pi} \hat{u}_2(r_{\pi}(\mu)) = \hat{u}_2(\mu).$$

Equality holds term by term inside these infima because every $r_{\pi}(\mu)$ is quasi-monomial. Infima of identical collections agree, including when their value is $-\infty$. This proves (ii).

Step 4: prove the reverse implication. Conversely, if the valuative transforms agree, their integrals against $\text{MA}(\hat{\Phi})$ agree by [\(2.4\)](#). This gives (i).

Step 5: identify the implication asked for by BFJ. Finally, every Zhou weight has a tame maximal representative, which is an allowed test in condition (4) of Theorem A of [\[4\]](#). The implication just proved yields equality of the valuative transforms. This is equivalent to equality of Lelong numbers of the pullbacks at every point of every modification above o , namely condition (1); see [\[4, Sections 5.1 and 5.6\]](#). $\square$

6.2. The BFJ question stated without shorthand. For two psh germs u_1, u_2 , Theorem A of [\[4\]](#) compares the following four kinds of information:

- (1) the ordinary Lelong numbers of $u_i \circ \pi$ at all points over o on all proper modifications;
- (2) the ideals $\mathcal{J}_+(tu_i)$ for all real $t > 0$;
- (3) the relative types against every tame maximal weight;
- (4) the generalized Lelong numbers against every tame weight.

BFJ proved the equivalence of the first three and their implication to the fourth, leaving the converse from the fourth open. In the present argument, equality of the data in (4) first gives equality for its subfamily of Zhou weights. Flatness changes those equalities of masses into equalities of relative types; controlled density then gives equality of the transforms. The latter is (1). Thus the new implication is specifically $(4) \Rightarrow (1)$, not a claim that each individual numerical invariant determines a germ.

There is also a concrete way to see why a valuative transform records the ordinary Lelong numbers in (1). Given a point p on a smooth model Y , blow up p . The generic Lelong number of the pulled-back psh function along the new exceptional divisor equals its ordinary Lelong number at p . This is the same familiar relation that identifies the order at a point of a

holomorphic function with its exceptional order after blowing up that point. The divisor on the composite modification gives a divisorial valuation, with its scale retained. Equality of transforms gives equality of that divisorial measurement, and hence equality at p . The precise equivalence, including the valuative-space conventions, is the one used by BFJ in Section 5.6.

6.3. Two consequences and the scope of the proof.

Corollary 6.1 (Order detection). *For $u_1, u_2 \in \text{PSH}_o$, the following are equivalent:*

$$\nu_\Phi(u_1) \geq \nu_\Phi(u_2) \quad \text{for every local Zhou weight } \Phi; \quad \widehat{u}_1 \leq \widehat{u}_2 \quad \text{on } \mathcal{V}.$$

Proof. Divide each test inequality by $\tau_\Phi > 0$ and use [theorem 1.1](#). This gives $w[u_1] \geq w[u_2]$ on $\mathcal{Z}$. The simultaneous approximation in [theorem 5.7](#) preserves this inequality at each $v \in \mathcal{V}_{\text{qm}}$; applying (2.6) extends the inequality of transforms to $\mathcal{V}$. The converse follows by integrating $-\widehat{u}_1 \geq -\widehat{u}_2$ against the positive measure $\text{MA}(\widehat{\Phi})$. $\square$

Corollary 6.2 (Multiplier ideals). *Under either condition of [theorem 1.2](#), $\mathcal{J}(tu_1) = \mathcal{J}(tu_2)$ for every $t > 0$.*

Proof. Theorem A of [4] gives equality of $\mathcal{J}_+(tu_i)$. The strong openness theorem of Guan–Zhou [9] identifies $\mathcal{J}_+(tu_i)$ with $\mathcal{J}(tu_i)$. $\square$

Remark 6.3 (Scope of the argument). The proof uses neither fundamental weights for all quasi-monomial valuations nor a conjectural identification of the Zhou cone with that class. Quasi-monomial valuations enter only as finite-discrepancy targets for the controlled density construction and as the determining points in (2.6). The mass formula is proved directly for Zhou weights. The one-dimensional case is elementary: a maximal weight is $a \log |z| + O(1)$, its mass is a , and its normalized generalized Lelong number and relative type are both $\nu(u, o)/a$.

DECLARATION ON THE USE OF ARTIFICIAL INTELLIGENCE

The author proposed the research questions and the organization around Zhou weights, their generalized Lelong numbers, and the density of Zhou valuations. The hypersurface-volume proof, its application to Zhou weights, and the controlled-density argument presented here were developed and written with GPT-6 Astra. Background formulations in the preceding draft were prepared with GPT-5.6-sol (high). The results of other authors used in these arguments are identified by citations. The AI-generated proofs require the author’s mathematical verification before submission.

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Email address: bsjie@amss.ac.cn