Some counterexamples on higher order and p-Bergman kernels

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Abstract

We present four groups of counterexamples in the theory of higher order and p-Bergman kernels.

The first group is built on the domains $D_q := \{x \in \mathbb{C} : |x|^2 < |1 - qx^2|\}$ and their r-fold symmetric analogues, together with the higher order p-Bergman kernels on the associated Green sublevel sets. This construction gives three counterexamples: it answers in the negative a convexity question of Błocki–Zwonek; it shows that the monotonicity of the higher order p-Bergman kernels along the Green sublevel sets, which holds for $p \in [1, 2]$ by the results of Błocki–Zwonek and Bao–Guan–Sun, fails for every $p > 2$; and it shows that the L^p analogue of Guan's concavity theorem for minimal L^2 integrals on sublevel sets of plurisubharmonic functions fails for every $p > 2$.

The second group exhibits a four-connected planar domain on which the equality in the higher order Suita conjecture fails at every point; the same Baire-category construction works for every connectivity $n \geq 4$.

The third group is based on the unit disk with $H(z) = z$ and $0 < p < 1$: it shows that the natural identity between higher order p-Bergman kernels and the p-Bergman kernels with respect to ξ fails for $0 < p < 1$, and the same example also gives a counterexample to the L^p analogue of the identity $B(F, I, D) = C_{F,I}(D)$ of Bao–Guan for $0 < p < p_*$, where $p_* := 2(\sqrt{2} - 1)$.

The fourth group concerns the classical fiberwise p-Bergman kernel. A finite-dimensional weighted entire-function model, lifted to a pseudoconvex Hartogs family, shows that fiberwise log-plurisubharmonicity fails for every $p > 4$. In the simplest case $p = 6$, the Levi form of the logarithm of the fiberwise kernel along a fixed holomorphic section is exactly $-1/8$ at the origin. The construction also yields bounded pseudoconvex counterexamples by exhaustion.

The constructions and proofs of these counterexamples come from AI tools.

Contents

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*The counterexamples presented in this paper come from GPT-5.6-sol (xhigh) and DeepSeek-V4 flash and pro. The author has independently verified the statements and all computations. The author thanks Ting Lin for assistance with the use of GPT, and Xun Sun for independently rechecking the second group of counterexamples and for helpful discussions.

1 Introduction

Let $D \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and let $w \in D$. The Bergman kernel of D at w is defined by

$$K_D(w) := \sup\Big\{|f(w)|^2 : \ f \in \mathcal{O}(D), \ \|f\|_D := \Big(\int_D |f|^2 \, d\lambda\Big)^{1/2} \le 1\Big\},$$

where $d\lambda$ denotes the Lebesgue measure. Following [5], for a homogeneous polynomial H of degree k, $H(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$, we write $P_H(f) := \sum_{|\alpha|=k} a_\alpha D^\alpha f$, and we define the higher order Bergman kernel

$$K_D^H(z) := \sup\Big\{|P_H(f)(z)|^2 : \ f \in \mathcal{O}(D), \ f^{(j)}(z) = 0 \ (0 \le j < k), \ \|f\|_D \le 1\Big\}.$$

In dimension one, $H(z) = z^k$ and $K_D^{z^k}(z)$ will also be denoted by $K_D^{(k)}(z)$.

Let $G_D(\cdot, w)$ be the (pluricomplex) Green function of D with pole at w, and assume $w = 0$. For $a \le 0$ we denote

$$D_a := e^{-a}\{G_D(\cdot, 0) < a\}.$$

The main theorem of [5] asserts that $a \mapsto K_{D_a}^H(0)$ is non-decreasing, and it is noted that

$$K_{D_a}^H(0) = e^{2(n+k)a} K_{\{G_D(\cdot,0)<a\}}^H(0).$$

The counterexamples in this paper are grouped into four blocks according to the underlying construction.

1.1 Counterexamples from the Green sublevel families

1.1.1 The convexity question of Błocki–Zwonek

The main theorem of [5] also motivates the following question raised in [5, Remark 3].

Question 1.1 (Błocki–Zwonek, [5, Remark 3]). *Is the function*

$$(-\infty, 0] \ni a \longmapsto \log K_{D_a}^H(w)$$

convex, as it is in the case $H \equiv 1$?

The convexity in the case $H \equiv 1$ follows from Berndtsson's theorem on the plurisubharmonic variation of Bergman kernels (see the final remark in [3]). Our first result shows that Question 1.1 has a negative answer in general.

Theorem 1.2. *For $Q > 1$ sufficiently large set $c := 2Q$ and*

$$D_c := \{z \in \mathbb{C} : \ |z|^2 < |1 - cz^2|\}.$$

Then D_c is a domain in $\mathbb{C}$ (indeed $\mathbb{C}$ minus two disjoint Jordan regions, hence unbounded and doubly connected), $0 \in D_c$, and with $H(z) = z^6$ the function

$$(-\infty, 0] \ni a \longmapsto \log K_{(D_c)_a}^{z^6}(0)$$

is not convex. More precisely, for

$$a_- := -\log 2, \qquad a_0 := -\tfrac{1}{2}\log 2, \qquad a_+ := 0,$$

so that $a_0 = (a_- + a_+)/2$, we have

$$\log K_{(D_c)_{a_0}}^{z^6}(0) > \frac{1}{2}\Big(\log K_{(D_c)_{a_-}}^{z^6}(0) + \log K_{(D_c)_{a_+}}^{z^6}(0)\Big),$$

and the difference is

$$\frac{3}{64Q^2} + O(Q^{-4}) \qquad (Q \to \infty).$$

The proof of Theorem 1.2 is divided into three steps, each of independent interest:

1. an explicit formula for the Green function of D_c and the observation that the scaled domains $(D_c)_a$ are again of the same form (Subsection 2.1.1);
2. an exact reduction of the extremal problem defining $K^{z^6}_{D_q}(0)$ to the third Taylor coefficient functional on a weighted Bergman space of the unit disk (Subsection 2.1.2);
3. a rigorous asymptotic expansion of the associated infinite Gram matrix, valid up to order q^{-4}, which yields the precise size of the convexity defect (Subsection 2.1.3).

1.1.2 Monotonicity fails for $p > 2$

Recall that for $p > 0$, a homogeneous polynomial H of degree k, and a domain $D \subset \mathbb{C}^n$ containing the origin, the higher order p-Bergman kernel $K^{H,p}_D(0)$ is defined in Section 4, and that $D_a := e^{-a}\{G_D(\cdot, 0) < a\}$ for $a \le 0$. The main theorem of [5] asserts that, for $p = 2$,

$$(-\infty, 0] \ni a \longmapsto K^{H,p}_{D_a}(0) \quad \text{is non-decreasing.}$$

In [14] this was extended to all $p \in [1, 2]$: by [14, Theorem 1.7] one has $K^{H,p}_D(z) = \inf_{\xi \in S_H} K_{\xi,D,p}(z)$ for $p \ge 1$, while by [14, Theorem 1.8] the functions

$$(-\infty, 0] \ni a \longmapsto \log K_{\xi,\{G_D(\cdot,0)<a\},p}(0)$$

are convex and the functions

$$(-\infty, 0] \ni a \longmapsto e^{(2n+pk)a} K_{\xi,\{G_D(\cdot,0)<a\},p}(0)$$

are non-decreasing for $p \in (0, 2]$; combining the two results, [14, Corollary 1.10] yields the monotonicity of $a \mapsto K^{H,p}_{D_a}(0)$ for every $p \in [1, 2]$. Since the two ingredients have complementary ranges ($p \ge 1$ for the inf–min identity and $p \le 2$ for the convexity of the p-Bergman kernels with respect to ξ), it is natural to ask whether the monotonicity persists in the two remaining regimes $p \in (0, 1)$ and $p \in (2, +\infty)$.

This construction also answers this question in the negative for every $p > 2$. The simplest instance, treated in detail below, uses again the domain D_c and the polynomial $H(z) = z^6$ of Theorem 1.2 and covers $p > 3$; the full range $p > 2$ is then reached through the r-fold symmetric families of Theorem 1.4. The counterexamples were obtained by Codex with Deepseek v4 pro.

THEOREM 1.3. *Let $p > 3$ and $H(z) = z^6$. Then there is a constant $c > 1$ such that, with $D_{1/c} = (D_c)_{-\log c}$,*

$$K^{z^6,p}_{D_{1/c}}(0) > K^{z^6,p}_{D_c}(0).$$

Consequently the function

$$(-\infty, 0] \ni a \longmapsto K^{z^6,p}_{(D_c)_a}(0)$$

is <u>not</u> non-decreasing. More precisely, there are constants $0 < c_p < C_p < +\infty$, depending only on p, such that for every $q \ge 8$,

$$c_p\, q^{-(p-3)} \;\le\; K^{z^6,p}_{D_q}(0) \;\le\; C_p\, q^{-(p-3)},$$

while

$$\lim_{q\to 0^+} K^{z^6,p}_{D_q}(0) = \frac{(6!)^p(6p+2)}{2\pi} = K^{z^6,p}_{\mathbb{D}}(0) = K^{z^6,p}_{I_{D_c}(0)}(0),$$

where $\mathbb{D}$ *is the unit disk and* $I_{D_c}(0)$ *is the Azukawa indicatrix of* D_c *at* 0.

THEOREM 1.4. *For every* $p>2$ *there is an integer* $r>2/(p-2)$ *with the following property. For every integer* $m\ge 1$, *with* $H(z)=z^{rm}$ *and*

$$D^{(r)}_q := \{x\in\mathbb{C}:\ |x|^r < |1-qx^r|\},$$

there are constants $0<c_{p,r,m}<C_{p,r,m}<+\infty$, *depending only on* p,r,m, *such that for every* $q\ge 8$,

$$c_{p,r,m}\, q^{-(p-2-2/r)} \ \le\ K^{z^{rm},p}_{D^{(r)}_q}(0) \ \le\ C_{p,r,m}\, q^{-(p-2-2/r)},$$

while

$$\lim_{q\to 0^+} K^{z^{rm},p}_{D^{(r)}_q}(0) = \frac{(rm)!^p(rmp+2)}{2\pi}.$$

Consequently, for every sufficiently large $c>1$, *with* $D^{(r)}_{1/c} = (D^{(r)}_c)_{-\frac{2}{r}\log c}$, *one has*

$$K^{z^{rm},p}_{D^{(r)}_{1/c}}(0) > K^{z^{rm},p}_{D^{(r)}_c}(0),$$

and the function $a\mapsto K^{z^{rm},p}_{(D^{(r)}_c)_a}(0)$ *is not non-decreasing.*

Thus the monotonicity part of the Błocki–Zwonek and Bao–Guan–Sun results has no analogue for any $p>2$. For the family D_q of Theorem 1.3 the borderline exponent q^{3-p} flips sign exactly at $p=3$; the r-fold families of Theorem 1.4 push this threshold arbitrarily close to 2 by taking $r>2/(p-2)$ large. The limiting comparison with the Azukawa indicatrix, however, still holds with equality for all these families. The regime $p\in(0,1)$ remains open.

1.1.3 The L^p concavity of minimal integrals fails for $p>2$

Following [9], let $D\subset\mathbb{C}^n$ be a pseudoconvex domain containing the origin o, let I be an ideal of $\mathcal{O}_o$, and let $(F,o)\in\mathcal{O}_o$. The minimal L^2 integral of D is

$$C_{F,I}(D) := \inf\Big\{\int_D |\tilde F|^2 :\ (\tilde F - F, o)\in I,\ \tilde F\in\mathcal{O}(D)\Big\}.$$

Guan [9] established a concavity property of the minimal L^2 integrals on the sublevel sets of a plurisubharmonic function: if φ is a negative plurisubharmonic function on D with $\varphi(o)=-\infty$, and

$$G(t) := C_{F,\mathcal{I}(\varphi)_o}\big(\{\varphi<-t\}\cap D\big), \qquad t\ge 0,$$

where $\mathcal{I}(\varphi)_o$ denotes the multiplier ideal of φ at o (the ideal of germs $f\in\mathcal{O}_o$ with $|f|^2e^{-\varphi}$ locally integrable near o), then $G(0)<+\infty$ implies that $G(-\log r)$ is concave in $r\in(0,1]$. This concavity has been a basic tool in the characterization of the equality cases of the optimal L^2 extension theorem and in the higher dimensional generalization of the Suita conjecture.

The monotonicity results of [5] and [14] for the higher order p-Bergman kernels along the Green sublevel sets, recalled in the previous subsection, are the kernel-side counterparts of this structural property. In their spirit, it is natural to ask whether the L^p analogue of Guan's concavity holds: for $p > 0$ define the minimal L^p integral

$$C_{F,I,p}(D) := \inf\Big\{ \int_D |\tilde{F}|^p : \ (\tilde{F} - F, o) \in I, \ \tilde{F} \in \mathcal{O}(D) \Big\},$$

set

$$G_p(t) := C_{F,\mathcal{I}(\varphi)_o,p}\big(\{\varphi < -t\} \cap D\big), \qquad t \geq 0,$$

and ask whether $G_p(0) < +\infty$ implies that $G_p(-\log r)$ is concave in $r \in (0,1]$. For $p = 2$ this is exactly Guan's theorem.

Although we do not prove the full concavity for $1 \leq p < 2$, the combination of the logarithmic convexity of the p-Bergman kernels with respect to ξ and the L^p identity $B_p = C_{F,I,p}$ gives the sharp chord inequality

$$G_p(t) \geq e^{-t} G_p(0), \qquad t \geq 0, \quad 1 \leq p \leq 2;$$

see Corollary 4.6. For $p = 2$, this inequality is the chord bound implied by Guan's concavity theorem. Thus Corollary 4.6 extends that consequence of Guan's result to every $p \in [1,2]$. It also passes from the Green sublevel sets and homogeneous higher order functionals considered by Błocki–Zwonek, and subsequently by Bao–Guan–Sun for $p \in [1,2]$, to the sublevel sets of an arbitrary negative plurisubharmonic function and the quotient data modulo its multiplier ideal. In this sense it is a further extension, on the dual minimal-integral side, of the Bao–Guan–Sun generalization of the Błocki–Zwonek theorem.

This construction also answers this question in the negative for every $p > 2$: the r-fold families $D_q^{(r)}$ and the polynomial $H(z) = z^{rm}$ of Theorem 1.4 are transplanted into the minimal L^p integral setting through the multiplier ideal of the Green function. The counterexample was obtained by Codex with Deepseek v4 pro.

THEOREM 1.5. *Let $p > 2$, let $r \geq 1$ and $m \geq 1$ be integers, and set $k := rm$. Fix $c \in (0,1)$, put*

$$D := D_c^{(r)} = \big\{ x \in \mathbb{C} : \ |x|^r < |1 - cx^r| \big\},$$

which is a bounded pseudoconvex domain containing 0, and define

$$\varphi := 2(k+1)\, G_D(\cdot, 0), \qquad F(x) := \frac{x^k}{k!},$$

where $G_D(\cdot,0)$ is the Green function of D with pole at 0. Then $G_p(0) < +\infty$ and, as $\rho \to 0^+$,

$$G_p(-\log \rho) \sim \frac{2\pi}{(k!)^p (kp+2)}\, \rho^{\mu}, \qquad \mu := \frac{pk+2}{2(k+1)} = 1 + \frac{k(p-2)}{2(k+1)} > 1.$$

Consequently $G_p(-\log \rho)$ is not concave in $\rho \in (0,1]$, and the L^p analogue of Guan's concavity theorem fails for every $p > 2$.

The exponent μ makes the threshold $p = 2$ visible: for $p = 2$ one has $\mu = 1$, and the asymptotic is the linear growth compatible with Guan's concavity; for $0 < p < 2$ one has $\mu < 1$, so the same construction produces no contradiction, and the question remains open in that range. The proof is given in Subsection 2.3.

1.2 Higher order Suita's conjecture

Let Ω be a planar domain in $\mathbb{C}$ bounded by n analytic Jordan curves: $\Gamma_1, \ldots, \Gamma_n$, where we set Γ_n to be the boundary of the unbounded component of $\mathbb{C} \setminus \Omega$. According to the solvability of the Dirichlet problem on planar domains, for each $j \in \{1, \ldots, n-1\}$ there exists a harmonic function u_j on $\overline{\Omega}$ such that

$$u_j|_{\Gamma_j} \equiv 1, \qquad u_j|_{\Gamma_k} \equiv 0 \quad \forall k \in \{1, \ldots, n\} \setminus \{j\}.$$

We recall the higher order Suita conjecture in the form in which it is used below. The classical Suita conjecture asserts that for a planar domain Ω with Green function G_Ω and $w \in \Omega$,

$$K_\Omega(w) \geq \frac{c_\Omega(w)^2}{\pi},$$

where

$$c_\Omega(w) := \exp\Big(\lim_{z \to w}\big(G_\Omega(z,w) - \log|z-w|\big)\Big)$$

is the logarithmic capacity of $\mathbb{C} \setminus \Omega$ with respect to w; for the unit disk the two sides coincide. Its m-th order analogue, proved in [4], states that for every $m \geq 0$,

$$K_\Omega^{(m)}(w) \geq \frac{m!(m+1)!\, c_\Omega(w)^{2m+2}}{\pi},$$

with equality again for the unit disk. In higher dimensions, Błocki–Zwonek [5] proved the general inequality $K_D^H(z) \geq K_{I_D(z)}^H(0)$ for every homogeneous polynomial H, where $I_D(z)$ denotes the Azukawa indicatrix of D at z; in dimension one, for $H(z) = z^m$, it is exactly the inequality above.

Guan–Sun–Yuan [8] proved a weighted version of the Suita conjecture for higher derivatives, and their result yields the following characterization of the equality locus: for a planar domain bounded by n analytic Jordan curves, the equality in the m-th order Suita conjecture holds at z if and only if

$$(m+1)u_j(z) \in \mathbb{Z}, \qquad j = 1, \ldots, n-1.$$

Thus, if all $u_j(z)$ are rational, then choosing $m+1$ to be a common denominator of the $u_j(z)$ gives equality at z; conversely any such equality forces $u_j(z) \in \mathbb{Q}$ for all j. Hence a domain in which, at every point, some triple $(u_1(z), u_2(z), u_3(z))$ fails to be rational has the property that equality in the higher order Suita conjecture never holds.

Recently, in the Annales de l'Institut Fourier, Xu–Zhou [11] constructed, for every $n \geq 3$ and every sufficiently large integer M, a family of smoothly bounded n-connected planar domains on which the equality in the m-th order Suita conjecture fails at every point for each $m = 0, 1, \ldots, M$. Our next result complements and strengthens this: it exhibits a four-connected domain on which the equality in the higher order Suita conjecture fails at every point and for every order $m \geq 0$.

THEOREM 1.6. *There exists a planar domain Ω in $\mathbb{C}$ bounded by 4 analytic Jordan curves, such that for the corresponding harmonic functions u_j, $j = 1, 2, 3$, there does not exist $z \in \Omega$ with*

$$u_j(z) \in \mathbb{Q}, \qquad j = 1, 2, 3.$$

In particular, at no point of Ω does the equality in the higher order Suita conjecture hold.

We prove Theorem 1.6 by a Baire-category (generic parameter) construction. The same construction works for every connectivity $n \geq 4$; we record this generalization in Subsection 3.1.

1.3 The ξ-reduction fails for $0 < p < 1$

For $p > 0$ and a homogeneous polynomial H of degree k, the higher order p-Bergman kernel $K_D^{H,p}(z)$ and, for a functional $\xi \in \ell_1^{(n)}$, the p-Bergman kernel with respect to ξ $K_{\xi,D,p}(z)$ are defined in Section 4. Bao–Guan–Sun [14] proved that for every $p \in [1, +\infty)$,

$$K_D^{H,p}(z) = \inf_{\xi \in S_H} K_{\xi,D,p}(z),$$

where S_H is the affine family of functionals whose degree-k part is prescribed by H; see Theorem 1.7 in [14].

It is natural to ask: whether the identity persists for $0 < p < 1$? This counterexample answers this question in the negative. Already in the simplest one-dimensional case the equality fails.

THEOREM 1.7. *Let $\mathbb{D} \subset \mathbb{C}$ be the unit disk, $p = 1/2$, $H(z) = z$ and $z = 0$. Then*

$$K_{\mathbb{D}}^{H,1/2}(0) = \frac{5}{4\pi},$$

while

$$\inf_{\xi \in S_H} K_{\xi,\mathbb{D},1/2}(0) > \frac{5}{4\pi}.$$

Consequently the identity

$$K_{\mathbb{D}}^{H,1/2}(0) = \inf_{\xi \in S_H} K_{\xi,\mathbb{D},1/2}(0)$$

fails, and the infimum is not attained at the common value.

As explained at the end of Section 4, the same example also answers in the negative the natural L^p analogue of the identity $B(F, I, D) = C_{F,I}(D)$ of Bao–Guan [12].

1.4 Fiberwise log-plurisubharmonicity fails for classical p-Bergman kernels

For a domain $D \subset \mathbb{C}^n$ and $p > 0$, the classical p-Bergman kernel is defined by

$$K_{D,p}(z) := \sup_{f \in \mathcal{O}(D) \cap L^p(D)} \frac{|f(z)|^p}{\int_D |f|^p \, d\lambda}.$$

For a pseudoconvex domain $\mathcal{X} \subset \mathbb{C}^n \times \mathbb{C}^m$ with fibers $\mathcal{X}_w$, the logarithm of the fiberwise kernel $K_{\mathcal{X}_w,p}(z)$ is plurisubharmonic for $0 < p \leq 2$; see [14]. The persistence of this property for $p > 2$, both in the classical case and for kernels with respect to a functional ξ, was left open there. The following fourth group of counterexamples gives a negative answer in the classical case for every $p > 4$. This construction was obtained by Codex with GPT-5.6-sol (xhigh).

THEOREM 1.8. *Let $p > 4$, choose*

$$\frac{5}{2} < \alpha \leq \frac{p+1}{2},$$

and put

$$S(z, w) := 1 + |z|^4 + |1 + wz^2|^2.$$

The Hartogs domain

$$\mathcal{X}_\alpha := \left\{ (z, \eta, w) \in \mathbb{C}^2 \times \mathbb{D} : |\eta|^2 S(z, w)^\alpha < 1 \right\}$$

is pseudoconvex. If

$$X_{\alpha,w} := \{(z,\eta) \in \mathbb{C}^2 : |\eta|^2 S(z,w)^\alpha < 1\},$$

then

$$\frac{\partial^2}{\partial w\,\partial \overline{w}} \log K_{X_{\alpha,w},p}(0,0)\bigg|_{w=0} = \frac{5-2\alpha}{8} < 0.$$

Consequently the logarithm of the classical fiberwise p-Bergman kernel is not plurisubharmonic on $\mathcal{X}_\alpha$. In particular, for $p = 6$ one may take $\alpha = 3$, and the displayed Levi form is $-1/8$. Moreover, bounded pseudoconvex counterexamples exist for every $p > 4$.

The full proof is given in Section 5. It reduces the extremal problem to a weighted A^p space of entire functions. The polynomial decay of the weight forces that space to consist only of affine functions; an evenness argument then shows that the constant function is extremal. The negative Levi form follows from two beta integrals. Bounded examples are obtained by truncating the z-variable and using exhaustion of the p-Bergman kernels.

2 Counterexamples from the Green sublevel families

2.1 Non-convexity of the logarithm of a higher order Bergman kernel

The proof of Theorem 1.2 is divided into three steps, which are treated in the next three subsubsections.

2.1.1 The Green function of D_c and the scaling law

Throughout this subsubsection we write $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$.

PROPOSITION 2.1. *Let $c > 0$ and*

$$F_c(z) := \frac{z^2}{1 - cz^2}.$$

Then the domain $D_c = \{|F_c| < 1\}$ satisfies $0 \in D_c$, and the mapping

$$F_c : D_c \longrightarrow \mathbb{D} \setminus \{-1/c\}$$

is proper of degree 2, vanishing to order 2 at 0. Consequently the (negative) pluricomplex Green function of D_c with pole at 0 is

$$G_{D_c}(z,0) = \frac{1}{2}\log|F_c(z)| = \log|z| - \frac{1}{2}\log|1 - cz^2|.$$

Proof. The identity $D_c = \{|F_c| < 1\}$ is immediate from the definitions. The rational function F_c is holomorphic on D_c: the poles $\pm c^{-1/2}$ are not contained in D_c, since $|F_c| \to +\infty$ near them. The equation $F_c(z) = \xi$ reads $z^2 = \xi(1 - cz^2)$, i.e. $z^2 = \xi/(1 + c\xi)$, which has two solutions for every $\xi \notin \{-1/c, \infty\}$, so F_c is proper of degree 2 and $F_c^{-1}(0) = \{0\}$, with multiplicity 2 at 0.

The last multiplicity statement is essential here: the pullback formula for a general proper map would contain a pole at every point of $F_c^{-1}(0)$. In the present situation

$$u(z) := \frac{1}{2} G_{\mathbb{D}\setminus\{-1/c\}}(F_c(z), 0)$$

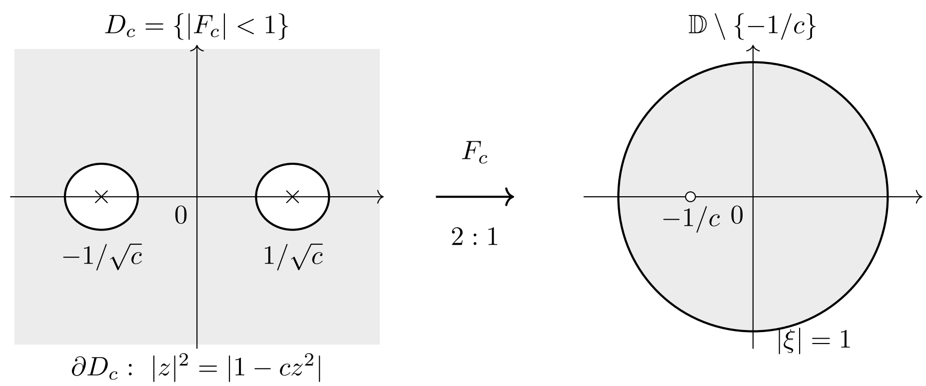

Figure 1: The domain $D_c = \{z \in \mathbb{C} : |z|^2 < |1-cz^2|\}$ in the z-plane (shaded; the two complementary components surround the poles $\pm 1/\sqrt{c}$ and the boundary is $|z|^2 = |1-cz^2|$) and its image $F_c(D_c) = \mathbb{D} \setminus \{-1/c\}$ in the ξ-plane. The map F_c is proper of degree 2 with $F_c(0) = 0$ (a double zero) and $F_c(\infty) = -1/c$.

is negative harmonic on $D_c \setminus \{0\}$, vanishes at the regular boundary, and satisfies $u(z) = \log|z| + O(1)$ at 0. By uniqueness it is $G_{D_c}(z, 0)$. Removing one point does not change a planar Green function, because a singleton is polar; see, for example, [16, Chapters 4–5]. Thus $G_{\mathbb{D}\setminus\{-1/c\}}(\xi, 0) = \log|\xi|$, and the displayed formula follows. ■

For $a \le 0$ we use the notation of [5]:

$$(D_c)_a := e^{-a}\big\{G_{D_c}(\cdot, 0) < a\big\}.$$

Proposition 2.2. *For $a \le 0$ and $q := ce^{2a}$ we have*

$$(D_c)_a = D_q := \big\{x \in \mathbb{C} : \ |x|^2 < |1 - qx^2|\big\}.$$

Proof. By Proposition 2.1, $G_{D_c}(z, 0) < a$ is equivalent to $|F_c(z)| < e^{2a}$. Writing $z = e^a x$, we have

$$F_c(e^a x) = e^{2a} \frac{x^2}{1 - ce^{2a}x^2} = e^{2a} F_q(x),$$

with $q = ce^{2a}$. Hence $|F_c(e^a x)| < e^{2a}$ is equivalent to $|F_q(x)| < 1$, i.e. $x \in D_q$. Since $(D_c)_a = \{x : e^a x \in \{G_{D_c}(\cdot, 0) < a\}\}$, the claim follows. ■

Since $\log q = \log c + 2a$ is affine in a, the convexity of $a \mapsto \log K^{z^6}_{(D_c)_a}(0)$ is equivalent to the convexity of $s \mapsto \log K^{z^6}_{D_{e^s}}(0)$ in $s = \log q$. We shall therefore study the family D_q, $q > 1$.

2.1.2 Reduction to a weighted Bergman kernel on the unit disk

For $q > 1$ we abbreviate $K_q := K^{z^6}_{D_q}(0)$. Since D_q is invariant under $x \mapsto -x$, the sixth derivative at 0 only sees the even part of a holomorphic function, and we may restrict to

$$f(x) = g(x^2), \qquad g \in \mathcal{O}(U_q),$$

where

$$U_q := \{y \in \mathbb{C} : \ |y| < |1 - qy|\} = \mathbb{C} \setminus \overline{B\Big(\frac{q}{q^2-1}, \frac{1}{q^2-1}\Big)}$$

is the image of D_q under the two-to-one map $y = x^2$.

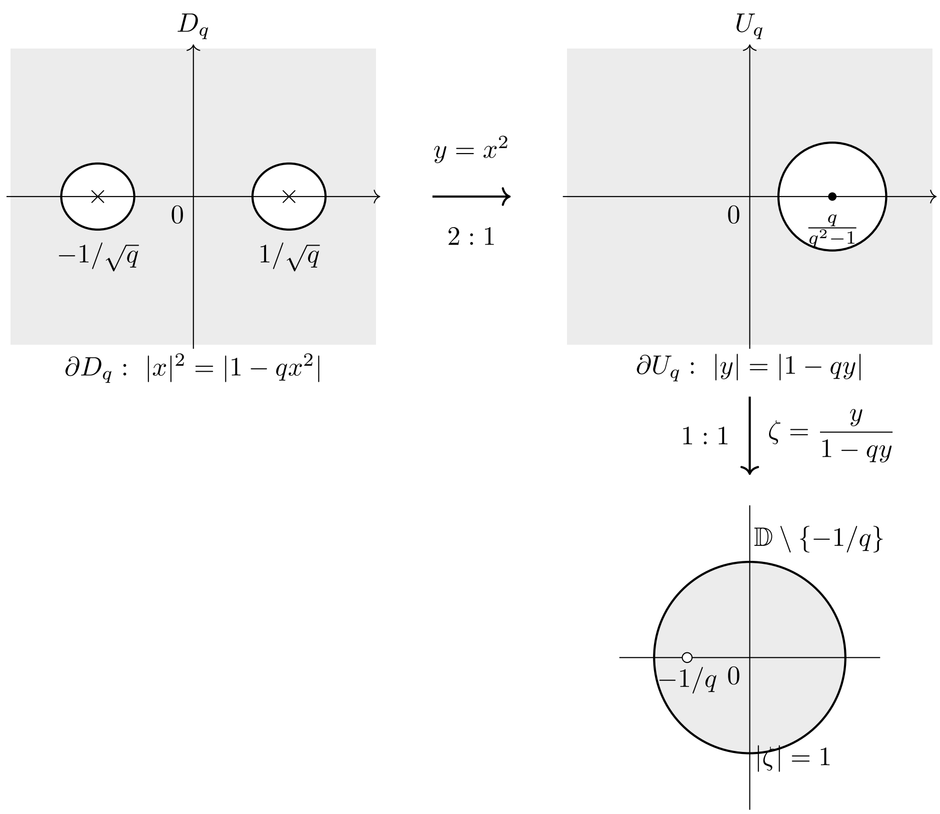

Figure 2: The domain $D_q = \{x \in \mathbb{C} : |x|^2 < |1 - qx^2|\}$ in the x-plane (shaded; the two complementary components surround the poles $\pm 1/\sqrt{q}$), its image $U_q = \{y \in \mathbb{C} : |y| < |1 - qy|\} = \mathbb{C} \setminus \overline{B\big(q/(q^2-1), 1/(q^2-1)\big)}$ in the y-plane under the two-to-one map $y = x^2$, and the unit disk $\mathbb{D} \setminus \{-1/q\}$ in the ζ-plane obtained via the Möbius map $\zeta = y/(1 - qy)$ $(1:1)$.

We record the elementary facts used below.

Lemma 2.3. *Let $p \geq 1$, $\gamma \geq 2$, and let Φ be holomorphic on a punctured disk $0 < |z| < \delta$. If*

$$\int_{|z|<\delta} |\Phi(z)|^p |z|^{-\gamma} \, dA(z) < +\infty,$$

then Φ extends holomorphically across 0 and

$$\mathrm{ord}_0 \Phi > \frac{\gamma - 2}{p}.$$

Proof. Put

$$M_p(s) := \frac{1}{2\pi}\int_0^{2\pi} |\Phi(se^{i\theta})|^p\, d\theta.$$

The hypothesis says that $\int_0^\delta M_p(s)s^{1-\gamma}ds < +\infty$. Consequently there is a sequence $s_\nu \to 0$ such that $s_\nu^{2-\gamma}M_p(s_\nu) \to 0$. If $\Phi(z) = \sum_{j\in\mathbb{Z}} a_j z^j$ is its Laurent expansion, Cauchy's formula and Hölder's inequality give, for every $n \geq 1$,

$$|a_{-n}| \leq s_\nu^n M_p(s_\nu)^{1/p} = \big(s_\nu^{2-\gamma}M_p(s_\nu)\big)^{1/p} s_\nu^{n+(\gamma-2)/p} \longrightarrow 0.$$

Hence all negative Laurent coefficients vanish. If $m = \mathrm{ord}_0\Phi$, then the integrand is comparable to $|z|^{pm-\gamma}$ near 0, and its area integral is finite precisely when $pm + 2 - \gamma > 0$. ■

Lemma 2.4. *With the notation above, the following hold.*

(i) For $f(x) = g(x^2)$,

$$\|f\|^2_{L^2(D_q)} = \frac{1}{2}\int_{U_q} |g(y)|^2\, \frac{dA(y)}{|y|}.$$

(ii) The conditions $f(0) = f'(0) = \cdots = f^{(5)}(0) = 0$ and $f^{(6)}(0) = 1$ are equivalent to

$$g(y) = b_3 y^3 + O(y^4) \quad (y \to 0), \qquad b_3 = \frac{1}{6!}.$$

(iii) Under the Möbius transformation

$$\zeta = \frac{y}{1-qy}, \qquad y = \frac{\zeta}{1+q\zeta},$$

which maps U_q biholomorphically onto $\mathbb{D}\setminus\{-1/q\}$, the square integrability of $|g|^2/|y|$ forces $g(y(\zeta))$ to extend across and vanish at $\zeta = -1/q$. Hence there exists $h \in \mathcal{O}(\mathbb{D})$ with

$$g(y(\zeta)) = (1+q\zeta)h(\zeta),$$

and

$$\|f\|^2_{L^2(D_q)} = \frac{1}{2}\int_{\mathbb{D}} |h(\zeta)|^2\, \frac{dA(\zeta)}{|\zeta|\,|1+q\zeta|}.$$

(iv) In the variable ζ, the normalization $b_3 = 1/6!$ becomes

$$h(\zeta) = \frac{1}{6!}\zeta^3 + O(\zeta^4).$$

Proof. (i) The map $y = x^2$ is two-to-one from D_q onto U_q, and $dA(y) = 4|x|^2 dA(x) = 4|y|\,dA(x)$, whence $dA(x) = dA(y)/(4|y|)$.

(ii) Writing $g(y) = \sum_{m\geq 0} g_m y^m$, we have $f(x) = \sum_m g_m x^{2m}$; the first six derivatives vanish at 0 iff $g_0 = g_1 = g_2 = 0$, and $f^{(6)}(0) = 6!\, g_3$.

(iii) The Möbius map is a biholomorphism between U_q and $\mathbb{D}\setminus\{-1/q\}$ because $y = \zeta/(1+q\zeta)$ and

$$1 - qy = \frac{1}{1+q\zeta}, \qquad |y| < |1-qy| \iff |\zeta| < 1.$$

Near $\zeta_0 := -1/q$ one has $y \sim -1/(q^2(\zeta - \zeta_0)) \to \infty$. The function $\Phi(\zeta) := g(y(\zeta))$ is therefore initially holomorphic only on a punctured neighborhood of ζ_0. Before any factorization, the change of variables gives

$$\frac{1}{2}\int |g(y)|^2 \frac{dA(y)}{|y|} = \frac{1}{2}\int \frac{|\Phi(\zeta)|^2}{|\zeta|\,|1+q\zeta|^3}\, dA(\zeta).$$

Since $|\zeta|$ is bounded above and below near ζ_0, Lemma 2.3 with $(p, \gamma) = (2, 3)$ shows that Φ extends across ζ_0 and vanishes there. Thus $h = \Phi/(1 + q\zeta)$ is holomorphic on $\mathbb{D}$. The change of variables $y = \zeta/(1 + q\zeta)$ has Jacobian $dA(y) = |1 + q\zeta|^{-4} dA(\zeta)$, and $|y|^{-1} = |1 + q\zeta|/|\zeta|$, so

$$\frac{1}{2}\int_{U_q} |g|^2 \frac{dA(y)}{|y|} = \frac{1}{2}\int_{\mathbb{D}} |h|^2 \frac{dA(\zeta)}{|\zeta|\,|1+q\zeta|}.$$

(iv) From $y = \zeta + O(\zeta^2)$ we get $g(y(\zeta)) = g_3\zeta^3 + O(\zeta^4)$; dividing by $(1 + q\zeta)$ does not change the leading term. ■

We now introduce the infinite Gram matrix. For $i, j \geq 3$ set

$$M_{ij}(q) := \int_{\mathbb{D}} \frac{\zeta^i \bar{\zeta}^j}{|\zeta|\,|1+q\zeta|}\, dA(\zeta),$$

and let $K_3(q) := \big(M(q)^{-1}\big)_{33}$ be the $(3, 3)$-entry of the inverse of the infinite matrix $M(q)$ (which is positive definite on the space of holomorphic functions h with $\int |h|^2/(|\zeta||1 + q\zeta|)\, dA < \infty$).

Proposition 2.5. *For every $q > 1$,*

$$K_q = K^{z^6}_{D_q}(0) = 2(6!)^2\, K_3(q).$$

Proof. By definition,

$$K_q^{-1} = \min\Big\{\|f\|^2_{L^2(D_q)} : \ f \in \mathcal{O}(D_q),\ f^{(j)}(0) = 0\ (0 \leq j < 6),\ f^{(6)}(0) = 1\Big\}.$$

By Lemma 2.4 this equals

$$\min\Big\{\frac{1}{2}\int_{\mathbb{D}} |h|^2 \frac{dA}{|\zeta|\,|1+q\zeta|} : \ h \in \mathcal{O}(\mathbb{D}),\ h(\zeta) = \frac{1}{6!}\zeta^3 + O(\zeta^4)\Big\}.$$

Writing $h(\zeta) = \frac{1}{6!}\zeta^3 + \sum_{m\geq 4} c_m \zeta^m$, the minimum over the higher coefficients is

$$\frac{1}{2(6!)^2\big(M(q)^{-1}\big)_{33}},$$

and taking the reciprocal gives the claim. ■

2.1.3 Asymptotics of the Gram matrix

Set $\varepsilon := 1/q$ and $A(\varepsilon) := qM(q)$, so that

$$A_{ij}(\varepsilon) = \int_{\mathbb{D}} \frac{\zeta^i \bar{\zeta}^j}{|\zeta|\,|\zeta+\varepsilon|}\, dA(\zeta), \qquad i, j \geq 3.$$

We write $D = \operatorname{diag}(\pi/3, \pi/4, \pi/5, \dots)$, and for the off-diagonal terms we use the convention that $B_{i,i+1} = B_{i+1,i}$ for $i \geq 3$.

Proposition 2.6. *The matrix function $A(\varepsilon)$ admits the expansion*

$$A(\varepsilon) = D + \varepsilon B + \varepsilon^2 C + \varepsilon^3 E + O(\varepsilon^4)$$

in the sense of bounded quadratic forms on the Hilbert space $\{h = \zeta^3 v : \ v \in L^2_h(\mathbb{D})\}$ with norm $\int |h|^2 |\zeta|^{-2} dA$. The nonzero leading terms are

$$D_{ii} = \frac{\pi}{i}, \qquad B_{i,i+1} = B_{i+1,i} = -\frac{\pi}{2i}, \qquad C_{ii} = \frac{\pi}{4(i-1)}, \qquad i \geq 3.$$

Consequently, for $\varepsilon > 0$ sufficiently small,

$$\big(A(\varepsilon)^{-1}\big)_{33} = \frac{3}{\pi} - \frac{1}{8\pi}\varepsilon^2 + O(\varepsilon^4).$$

Proof. Write $h(\zeta) = \zeta^3 v(\zeta)$ and denote the Hilbert norm in the statement by $\|h\|_{\mathcal{H}}$. We first record the elementary estimates

$$\int_{\mathbb{D}} |v|^2 dA \leq 3 \int_{\mathbb{D}} |v|^2 |\zeta|^4 dA = 3\|h\|^2_{\mathcal{H}}, \qquad \sup_{|\zeta| \leq 1/2} |v(\zeta)|^2 \leq C\|h\|^2_{\mathcal{H}}. \tag{2.1}$$

Indeed, both follow by writing $v = \sum_{j \geq 0} v_j \zeta^j$ and using

$$\int_{\mathbb{D}} |v|^2 dA = \pi \sum_{j \geq 0} \frac{|v_j|^2}{j+1}, \qquad \|h\|^2_{\mathcal{H}} = \pi \sum_{j \geq 0} \frac{|v_j|^2}{j+3},$$

together with Cauchy–Schwarz for the supremum estimate.

On the outer region $|\zeta| \geq 2\varepsilon$, Taylor's formula gives, through third order,

$$\begin{aligned} \frac{1}{|\zeta+\varepsilon|} &= \frac{1}{|\zeta|}\Big(1 + \frac{2\varepsilon \operatorname{Re}\zeta}{|\zeta|^2} + \frac{\varepsilon^2}{|\zeta|^2}\Big)^{-1/2} \\ &= \frac{1}{|\zeta|} - \frac{\varepsilon \operatorname{Re}\zeta}{|\zeta|^3} + \varepsilon^2 \frac{3(\operatorname{Re}\zeta)^2 - |\zeta|^2}{2|\zeta|^5} + \varepsilon^3 R_3(\zeta) + O\Big(\frac{\varepsilon^4}{|\zeta|^5}\Big), \end{aligned}$$

where $|R_3(\zeta)| \leq C|\zeta|^{-4}$. After multiplication by the additional factor $|\zeta|^{-1}$ in $A(\varepsilon)$, the fourth-order remainder is bounded by $C\varepsilon^4|\zeta|^{-6}$. Hence (2.1) gives the uniform quadratic-form estimate

$$C\varepsilon^4 \int_{\mathbb{D}} |v|^2 \, dA \leq C'\varepsilon^4 \|h\|^2_{\mathcal{H}}.$$

It remains to justify that the inner disk does not alter these coefficients. On $|\zeta| < 2\varepsilon$, the substitution $\zeta = \varepsilon s$ and the second estimate in (2.1) give

$$\int_{|\zeta|<2\varepsilon} \frac{|h(\zeta)|^2}{|\zeta|\,|\zeta+\varepsilon|} \, dA(\zeta) \leq C\varepsilon^6 \|h\|^2_{\mathcal{H}} \int_{|s|<2} \frac{|s|^5}{|s+1|} \, dA(s).$$

Each of the Taylor terms of orders $0, 1, 2, 3$, integrated over the same disk, is likewise $O(\varepsilon^6)\|h\|^2_{\mathcal{H}}$. This proves the asserted expansion in bounded quadratic-form norm, with a bounded third-order form E whose entries are not needed below.

The angular Fourier integrals give

$$\int_0^{2\pi} e^{id\theta}\,d\theta = 2\pi\delta_{d0}, \qquad \int_0^{2\pi} \cos\theta\, e^{i\theta} d\theta = \pi, \qquad \int_0^{2\pi} \big(3\cos^2\theta - 1\big)d\theta = \pi,$$

which yield, after the radial integration,

$$D_{ii} = \int_0^1 \rho^{2i}\,\frac{2\pi}{\rho}\,d\rho = \frac{\pi}{i},$$

$$B_{i,i+1} = -\int_0^1 \rho^{2i+1}\,\frac{\pi}{\rho^2}\,d\rho = -\frac{\pi}{2i},$$

$$C_{ii} = \int_0^1 \rho^{2i}\,\frac{\pi}{2\rho^3}\,d\rho = \frac{\pi}{4(i-1)}.$$

The first-order off-diagonal entries of B with $|i-j| \neq 1$ vanish by angular orthogonality.

The form D is exactly $\|\cdot\|^2_{\mathcal{H}}$, so it is coercive on $\mathcal{H}$. The bounded-form expansion therefore implies, for small $|\varepsilon|$, a convergent Neumann expansion (see, for example, [10, Chapter IV])

$$A^{-1} = D^{-1} - \varepsilon D^{-1}BD^{-1} + \varepsilon^2\big(D^{-1}BD^{-1}BD^{-1} - D^{-1}CD^{-1}\big) + \varepsilon^3 T + O(\varepsilon^4)$$

for some bounded operator T. Since D is diagonal, the second-order contribution to the $(3,3)$-entry receives only the diagonal term C_{33} and the unique off-diagonal path $3 \to 4 \to 3$, so that

$$\begin{aligned}
\big(A^{-1}\big)_{33} &= \frac{3}{\pi} + \varepsilon^2\Big(\frac{B_{34}^2}{D_{33}^2 D_{44}} - \frac{C_{33}}{D_{33}^2}\Big) + O(\varepsilon^4)\\
&= \frac{3}{\pi} + \varepsilon^2\Big(\frac{(\pi/6)^2}{(\pi/3)^2(\pi/4)} - \frac{\pi/8}{(\pi/3)^2}\Big) + O(\varepsilon^4)\\
&= \frac{3}{\pi} - \frac{1}{8\pi}\varepsilon^2 + O(\varepsilon^4).
\end{aligned}$$

There is no third-order term: the transformation $\zeta \mapsto -\zeta$ unitarily identifies $A(\varepsilon)$ with $A(-\varepsilon)$, while the coefficient functional is unchanged, so the expansion of $\big(A^{-1}\big)_{33}$ contains only even powers. ■

Combining Proposition 2.5 with Proposition 2.6 gives the asymptotic law for the kernel.

Corollary 2.7. *As $q \to \infty$,*

$$K_3(q) = q\big(A(q^{-1})^{-1}\big)_{33} = \frac{3q}{\pi}\Big(1 - \frac{1}{24q^2} + O(q^{-4})\Big),$$

and therefore, with $C_0 := \log\big(2(6!)^2 \cdot 3/\pi\big)$,

$$\log K^{z^6}_{D_q}(0) = C_0 + \log q - \frac{1}{24q^2} + O(q^{-4}).$$

2.1.4 Proof of the main theorem

Proof of Theorem 1.2. Let $Q > 1$ and $c = 2Q$. By Proposition 2.2, the three values

$$a_- = -\log 2, \qquad a_0 = -\tfrac{1}{2}\log 2, \qquad a_+ = 0$$

correspond to

$$q_- = \frac{Q}{2}, \qquad q_0 = Q, \qquad q_+ = 2Q.$$

Since $a_0 = (a_- + a_+)/2$ and $\log q = \log c + 2a$ is affine in a, it suffices to prove

$$\log K^{z^6}_{D_Q}(0) - \frac{1}{2}\Big(\log K^{z^6}_{D_{Q/2}}(0) + \log K^{z^6}_{D_{2Q}}(0)\Big) > 0$$

for Q sufficiently large. By Corollary 2.7, the terms C_0 cancel, and so do the linear terms $\log q$:

$$\log Q - \frac{1}{2}\big(\log(Q/2) + \log(2Q)\big) = 0.$$

The remaining contribution is

$$-\frac{1}{24}\Big(\frac{1}{Q^2} - \frac{1}{2}\Big(\frac{4}{Q^2} + \frac{1}{4Q^2}\Big)\Big) + O(Q^{-4}) = \frac{3}{64Q^2} + O(Q^{-4}) > 0$$

for Q large enough. This proves the strict midpoint inequality, hence the failure of convexity. ■

Remark 2.8 (A different convexity question). *We recall that Błocki and Zwonek also conjectured in [4] the convexity of the* area *function $t \mapsto \log \lambda(\{G_{\Omega,w} < t\})$; counterexamples on annuli were found numerically in [2] and proved in full generality in [4]. That conjecture concerns the volume of sublevel sets and is unrelated to Question 1.1, which concerns the Bergman kernel.*

2.2 Monotonicity of higher order p-Bergman kernels fails for $p > 2$

In Sections 2.2.1 to 2.2.5 we treat the family D_q of Theorem 1.2, which yields the counterexample for $p > 3$; the full range $p > 2$ is then reached in Section 2.2.6 by the r-fold symmetric families.

Throughout Sections 2.2.1 to 2.2.5, $p > 3$ is fixed, $q > 1$, and we use freely the notation of Subsection 2.1. By Proposition 2.1 and Proposition 2.2, $(D_c)_a = D_q$ with $q = ce^{2a}$, so the monotonicity of $a \mapsto K^{z^6,p}_{(D_c)_a}(0)$ is equivalent to that of $q \mapsto K^{z^6,p}_{D_q}(0)$.

2.2.1 Reduction to a weighted A^p problem on the disk

For $f \in \mathcal{O}(D_q)$ write

$$f(x) = g_e(x^2) + x g_o(x^2), \qquad g_e, g_o \in \mathcal{O}(U_q),$$

with U_q as in Subsection 2.1.2. Since $q > 1$, the domain D_q is unbounded and contains a neighborhood of ∞. It is important not to assume that an arbitrary function holomorphic on D_q is holomorphic at ∞. Instead, suppose that $f \in L^p(D_q)$, as in the extremal problem. Because $p > 3 > 1$, its even and odd parts are separately in $L^p(D_q)$. With $\zeta_0 = -1/q$ and $\Phi_e(\zeta) := g_e(y(\zeta))$, the changes of variables $x^2 = y$ and $y = \zeta/(1 + q\zeta)$ show, near ζ_0, that

$$\int |\Phi_e(\zeta)|^p |\zeta - \zeta_0|^{-3}\, dA(\zeta) < +\infty.$$

For $\Phi_o(\zeta) := g_o(y(\zeta))$, the extra factor $|x|^p = |y|^{p/2}$ gives instead

$$\int |\Phi_o(\zeta)|^p |\zeta - \zeta_0|^{-(p/2+3)}\, dA(\zeta) < +\infty.$$

Lemma 2.3 gives

$$\operatorname{ord}_{\zeta_0} \Phi_e > \frac{1}{p}, \qquad \operatorname{ord}_{\zeta_0} \Phi_o > \frac{1}{2} + \frac{1}{p}.$$

Both orders are therefore at least one. Consequently, in the variable $\zeta = y/(1 - qy)$ of Lemma 2.4,

$$g_e(y(\zeta)) = (1 + q\zeta) h_e(\zeta), \qquad g_o(y(\zeta)) = (1 + q\zeta) h_o(\zeta),$$

for some $h_e, h_o \in \mathcal{O}(\mathbb{D})$, because $y = \infty$ corresponds to $\zeta = -1/q$.

Lemma 2.9. *With the notation above, the following hold.*

(i) The jet conditions $f^{(j)}(0) = 0$ for $0 \le j < 6$ and $f^{(6)}(0) = 1$ are equivalent to

$$h_e(\zeta) = \frac{\zeta^3}{6!} + O(\zeta^4), \qquad h_o(\zeta) = O(\zeta^3).$$

(ii) For $b := \sqrt{\zeta/(1 + q\zeta)}$, one has

$$\int_{D_q} |f|^p\, dA = \frac{1}{4} \int_{\mathbb{D}} \big(|h_e + bh_o|^p + |h_e - bh_o|^p\big) \frac{|1 + q\zeta|^{p-3}}{|\zeta|}\, dA(\zeta).$$

Proof. (i) follows exactly as in Lemma 2.4, since $y = \zeta + O(\zeta^2)$. For (ii), the map $x \mapsto y = x^2$ is two-to-one from D_q onto U_q with $dA(y) = 4|x|^2 dA(x) = 4|y|dA(x)$, and the two preimages of y contribute $|g_e(y) \pm w g_o(y)|^p$, $w^2 = y$. Since $w = \sqrt{\zeta/(1+q\zeta)}$ and $dA(y) = |1+q\zeta|^{-4} dA(\zeta)$, $|y|^{-1} = |1+q\zeta|/|\zeta|$, we obtain

$$\int_{D_q} |f|^p \, dA = \frac{1}{4} \int_{\mathbb{D}} |1+q\zeta|^p \big(|h_e + b h_o|^p + |h_e - b h_o|^p \big) |1+q\zeta|^{-3} |\zeta|^{-1} \, dA(\zeta),$$

with $b := \sqrt{\zeta/(1+q\zeta)}$, which is the claimed formula. ■

For $p \geq 1$ the function $\mathbb{C} \ni z \mapsto |z|^p$ is convex, so Jensen's inequality gives pointwise

$$\frac{1}{2}\big(|h_e + b h_o|^p + |h_e - b h_o|^p \big) \geq |h_e|^p,$$

and the odd part of f can only increase the L^p norm. Hence the infimum defining $K_{D_q}^{z^6,p}(0)$ may be taken among even functions, and by Lemma 2.9,

$$K_{D_q}^{z^6,p}(0) = 2(6!)^p M(q)^{-1},$$

where

$$M(q) := \inf \Big\{ \int_{\mathbb{D}} |h|^p \, \frac{|1+q\zeta|^{p-3}}{|\zeta|} \, dA(\zeta) : \ h \in \mathcal{O}(\mathbb{D}), \ h_3 = 1 \Big\},$$

and h_3 denotes the coefficient of ζ^3 in the Taylor expansion of h at 0. For $p = 2$ this is exactly the reduction of Lemma 2.4 and Proposition 2.5: the weight $|1+q\zeta|^{p-3}|\zeta|^{-1}$ becomes $|1+q\zeta|^{-1}|\zeta|^{-1}$.

2.2.2 Lower bound

Take $h_0(\zeta) = \zeta^3/6!$, which corresponds to the even admissible function

$$f_q(x) = \frac{x^6(1-qx^2)^{-4}}{6!},$$

because $1 + q\zeta = (1-qy)^{-1}$ and $g_e(y) = y^3(1-qy)^{-4}/6!$. By Lemma 2.9(ii),

$$\int_{D_q} |f_q|^p \, dA = \frac{1}{2(6!)^p} \int_{\mathbb{D}} |\zeta|^{3p-1} |1+q\zeta|^{p-3} \, dA(\zeta) \leq \frac{\pi(1+q)^{p-3}}{(6!)^p(3p+1)},$$

since $p > 3$ gives $|1+q\zeta|^{p-3} \leq (1+q)^{p-3}$ and $\int_{\mathbb{D}} |\zeta|^{3p-1} dA = 2\pi/(3p+1)$. Therefore

$$K_{D_q}^{z^6,p}(0) \geq \frac{(6!)^p(3p+1)}{\pi(1+q)^{p-3}} \geq c_p \, q^{-(p-3)}, \qquad c_p := \frac{(6!)^p(3p+1)}{\pi \, 2^{p-3}},$$

for $q \geq 1$.

2.2.3 Upper bound by duality

By homogeneity,

$$M(q)^{-1} = \sup\Big\{|h_3|^p : \int_{\mathbb{D}} |h|^p w_q \, dA \le 1\Big\} = \|\ell\|^p,$$

where $w_q(\zeta) := |1 + q\zeta|^{p-3}|\zeta|^{-1}$ and $\ell(h) := h_3$ on the weighted space $A^p(\mathbb{D}, w_q)$. Fix ζ_0 with $|\zeta_0| = 1/2$. Since $|h|^p$ is subharmonic for $p \ge 1$,

$$|h(\zeta_0)|^p \le \frac{16}{\pi} \int_{B(\zeta_0,1/4)} |h|^p \, dA.$$

On $B(\zeta_0, 1/4) \subset \mathbb{D}$ we have $|\zeta| \le 3/4$, and for $q \ge 8$,

$$|1 + q\zeta| \ge q|\zeta| - 1 \ge \frac{q}{8}, \qquad w_q(\zeta) \ge \frac{4}{3}\Big(\frac{q}{8}\Big)^{p-3}.$$

Hence

$$|h(\zeta_0)|^p \le \frac{12}{\pi}\Big(\frac{q}{8}\Big)^{-(p-3)} \int_{\mathbb{D}} |h|^p w_q \, dA.$$

Since $h_3 = (2\pi i)^{-1} \int_{|\zeta|=1/2} h(\zeta)\zeta^{-4} d\zeta$, we have $|h_3| \le 8 \max_{|\zeta|=1/2} |h(\zeta)|$. Thus

$$\|\ell\| \le 8\Big(\frac{12}{\pi}\Big)^{1/p}\Big(\frac{q}{8}\Big)^{-(p-3)/p},$$

and consequently

$$K^{z^6,p}_{D_q}(0) \le C_p \, q^{-(p-3)}, \qquad C_p := \frac{24(6!)^p 8^{2p-3}}{\pi}.$$

2.2.4 The limit as $q \to 0$

For $0 < q < 1$ we compare D_q with disks. If $|x|^2 < 1/(1 + q)$, then $(1 + q)|x|^2 < 1$, whence $|x|^2 < 1 - q|x|^2 \le |1 - qx^2|$; conversely, if $x \in D_q$, then $|x|^2 < |1 - qx^2| \le 1 + q|x|^2$, whence $|x|^2 < 1/(1 - q)$. Hence

$$D\Big(0, (1 + q)^{-1/2}\Big) \subset D_q \subset D\Big(0, (1 - q)^{-1/2}\Big).$$

Since the kernel is non-increasing under domain inclusion and $K^{z^6,p}_{D(0,R)}(0) = R^{-(6p+2)} K^{z^6,p}_{\mathbb{D}}(0)$ by rescaling, with

$$K^{z^6,p}_{\mathbb{D}}(0) = \frac{(6!)^p(6p + 2)}{2\pi},$$

we get

$$(1 - q)^{3p+1}\frac{(6!)^p(6p + 2)}{2\pi} \le K^{z^6,p}_{D_q}(0) \le (1 + q)^{3p+1}\frac{(6!)^p(6p + 2)}{2\pi},$$

and $K^{z^6,p}_{D_q}(0) \to (6!)^p(6p + 2)/(2\pi)$ as $q \to 0^+$.

2.2.5 Proof of Theorem 1.3

By Sections 2.2.2 and 2.2.3,

$$c_p q^{-(p-3)} \le K^{z^6,p}_{D_q}(0) \le C_p q^{-(p-3)}, \qquad q \ge 8,$$

so $K^{z^6,p}_{D_q}(0) \to 0$ as $q \to \infty$, while Subsection 2.2.4 gives $K^{z^6,p}_{D_q}(0) \to (6!)^p(6p+2)/(2\pi) > 0$ as $q \to 0^+$. Choose $c > 1$ so large that $C_p c^{-(p-3)} < (1-c^{-1})^{3p+1}(6!)^p(6p+2)/(2\pi)$. Then

$$K^{z^6,p}_{D_c}(0) \le C_p c^{-(p-3)} < (1-c^{-1})^{3p+1}\frac{(6!)^p(6p+2)}{2\pi} \le K^{z^6,p}_{D_{1/c}}(0).$$

By Proposition 2.2, $D_{1/c} = D_{ce^{2a}}$ with $a = -\log c$, i.e. $D_{1/c} = (D_c)_{-\log c}$, while $D_c = (D_c)_0$. Thus $-\log c < 0$ but $K^{z^6,p}_{(D_c)_{-\log c}}(0) > K^{z^6,p}_{(D_c)_0}(0)$, and $a \mapsto K^{z^6,p}_{(D_c)_a}(0)$ is not non-decreasing. This proves the theorem.

Finally, the Green function formula of Proposition 2.1 gives $G_{D_c}(z,0) = \log|z| + O(1)$ near 0, so the Azukawa indicatrix of D_c at 0 is the unit disk $\mathbb{D}$, and the limit

$$\lim_{q\to 0^+} K^{z^6,p}_{D_q}(0) = \frac{(6!)^p(6p+2)}{2\pi} = K^{z^6,p}_{I_{D_c}(0)}(0)$$

holds with equality.

2.2.6 Extension to every $p > 2$ by r-fold symmetry

Fix integers $r \ge 2$ and $m \ge 1$, set $H(z) = z^{rm}$, and consider

$$D^{(r)}_q := \{x \in \mathbb{C} : \ |x|^r < |1 - qx^r|\}, \qquad q > 1.$$

The map

$$F_q(x) := \frac{x^r}{1-qx^r}$$

maps $D^{(r)}_q$ properly, r-to-one, onto $\mathbb{D} \setminus \{-1/q\}$ and vanishes to order r at 0, so the Green function of $D^{(r)}_q$ with pole at 0 is

$$G_{D^{(r)}_q}(x,0) = \frac{1}{r}\log|F_q(x)| = \log|x| - \frac{1}{r}\log|1-qx^r|,$$

and, exactly as in Proposition 2.2,

$$(D^{(r)}_c)_a = D^{(r)}_q, \qquad q = ce^{ra}.$$

From now on assume $p > 2 + 2/r$, which is exactly the regime used in the counterexample below. Let $f \in \mathcal{O}(D^{(r)}_q) \cap L^p(D^{(r)}_q)$ satisfy $f^{(j)}(0) = 0$ for $0 \le j < rm$ and $f^{(rm)}(0) = 1$, and let $\omega := e^{2\pi i/r}$. Averaging f over the r-th roots of unity,

$$f_0(x) := \frac{1}{r}\sum_{j=0}^{r-1} f(\omega^j x),$$

we get $f_0(\omega x) = f_0(x)$, hence $f_0(x) = g(x^r)$ with $g \in \mathcal{O}(U_q)$. Convexity gives $\|f_0\|_{L^p} \le \|f\|_{L^p}$, and since $\omega^{jrm} = 1$, the average f_0 satisfies the same jet conditions as f. Thus the extremal problem for $K^{z^{rm},p}_{D_q^{(r)}}(0)$ may be restricted to r-fold symmetric functions. Put $\Phi(\zeta) := g(y(\zeta))$ and $\zeta_0 = -1/q$. The first integral in (2.2), derived just below, shows near ζ_0 that

$$\int |\Phi(\zeta)|^p |\zeta - \zeta_0|^{-(2+2/r)}\, dA(\zeta) < +\infty.$$

Since $p > 2 + 2/r$, Lemma 2.3 implies $\operatorname{ord}_{\zeta_0} \Phi > 2/(rp)$, hence $\operatorname{ord}_{\zeta_0} \Phi \ge 1$. Therefore $g(y(\zeta)) = (1 + q\zeta)h(\zeta)$ with $h \in \mathcal{O}(\mathbb{D})$ and $h(\zeta) = \zeta^m/(rm)! + O(\zeta^{m+1})$.

Since $dA(y) = r^2|x|^{2r-2}dA(x) = r^2|y|^{2-2/r}dA(x)$ and every y has exactly r preimages, each contributing $|g(y)|^p$, we obtain

$$\int_{D_q^{(r)}} |f|^p\, dA = \frac{1}{r}\int_{U_q} |g(y)|^p\, |y|^{2/r-2}\, dA(y) = \frac{1}{r}\int_{\mathbb{D}} |h(\zeta)|^p\, |\zeta|^{2/r-2}\, |1 + q\zeta|^{p-2-2/r}\, dA(\zeta), \tag{2.2}$$

using $dA(y) = |1 + q\zeta|^{-4}dA(\zeta)$ and $|y|^{2/r-2} = |\zeta|^{2/r-2}|1 + q\zeta|^{2-2/r}$. Therefore

$$K^{z^{rm},p}_{D_q^{(r)}}(0) = r(rm)!^p\, M(q)^{-1}, \qquad M(q) := \inf\Big\{\int_{\mathbb{D}} |h|^p w_q\, dA :\ h \in \mathcal{O}(\mathbb{D}),\ h_m = 1\Big\},$$

where $w_q(\zeta) := |\zeta|^{2/r-2}|1 + q\zeta|^{p-2-2/r}$ and h_m is the coefficient of ζ^m. This is the natural r-fold analogue of the reduction of Section 2.2.1; for $r = 2$, $m = 3$ it recovers Theorem 2.9 with $H(z) = z^6$.

<u>Lower bound.</u> Set $\beta := p - 2 - 2/r > 0$. Take $h_0(\zeta) = \zeta^m$, which corresponds to the symmetric admissible function

$$f_q(x) = \frac{x^{rm}(1 - qx^r)^{-(m+1)}}{(rm)!}.$$

Since $|1 + q\zeta| \le 1 + q$ on $\mathbb{D}$ and $\int_{\mathbb{D}} |\zeta|^{pm+2/r-2}dA = 2\pi/(pm + 2/r)$,

$$M(q) \le \int_{\mathbb{D}} |\zeta|^{pm+2/r-2}|1 + q\zeta|^{\beta}dA \le \frac{2\pi(1 + q)^\beta}{pm + 2/r},$$

hence

$$K^{z^{rm},p}_{D_q^{(r)}}(0) \ge \frac{r(rm)!^p(pm + 2/r)}{2\pi(1 + q)^\beta} \ge c_{p,r,m}\, q^{-\beta}$$

for $q \ge 1$.

<u>Upper bound.</u> Let $\ell(h) := h_m$ on $A^p(\mathbb{D}, w_q)$. By homogeneity $M(q)^{-1} = \|\ell\|^p$. Fix $|\zeta_0| = 1/2$. Subharmonicity gives $|h(\zeta_0)|^p \le \frac{16}{\pi}\int_{B(\zeta_0,1/4)} |h|^p dA$. On $B(\zeta_0, 1/4)$ we have $|\zeta| \le 3/4$; since $2/r - 2 \le 0$, $|\zeta|^{2/r-2} \ge (3/4)^{2/r-2}$, and for $q \ge 8$, $|1 + q\zeta| \ge q/8$. Hence

$$w_q(\zeta) \ge \Big(\frac{3}{4}\Big)^{2/r-2}\Big(\frac{q}{8}\Big)^{\beta},$$

and

$$|h(\zeta_0)|^p \le \frac{16}{\pi}\Big(\frac{3}{4}\Big)^{2-2/r}\Big(\frac{q}{8}\Big)^{-\beta}\int_{\mathbb{D}} |h|^p w_q\, dA.$$

Since $h_m = (2\pi i)^{-1}\int_{|\zeta|=1/2} h(\zeta)\zeta^{-m-1}d\zeta$, one has $|h_m| \le 2^m \max_{|\zeta|=1/2}|h(\zeta)|$, and consequently

$$K^{z^{rm},p}_{D_q^{(r)}}(0) \le C_{p,r,m}\, q^{-\beta}, \qquad q \ge 8.$$

The limit as $q \to 0$. For $0 < q < 1$, exactly as in Section 2.2.4,

$$D\Big(0,(1+q)^{-1/r}\Big) \subset D_q^{(r)} \subset D\Big(0,(1-q)^{-1/r}\Big),$$

and rescaling gives $K^{z^{rm},p}_{D(0,R)}(0) = R^{-(2+prm)}K^{z^{rm},p}_{\mathbb{D}}(0)$ with $K^{z^{rm},p}_{\mathbb{D}}(0) = (rm)!^p(rmp+2)/(2\pi)$. Therefore

$$\lim_{q\to 0^+} K^{z^{rm},p}_{D_q^{(r)}}(0) = \frac{(rm)!^p(rmp+2)}{2\pi},$$

and the Green function formula above gives $I_{D_c^{(r)}}(0) = \mathbb{D}$, so the comparison with the Azukawa indicatrix holds with equality.

Proof of Theorem 1.4. For $p > 2$ choose an integer $r > 2/(p-2)$, so that $\beta = p - 2 - 2/r > 0$, and take any $m \ge 1$. The two bounds above give

$$c_{p,r,m}\, q^{-\beta} \le K^{z^{rm},p}_{D_q^{(r)}}(0) \le C_{p,r,m}\, q^{-\beta}, \qquad q \ge 8,$$

so the kernel tends to 0 as $q \to \infty$, while its limit at $q \to 0^+$ is the positive constant $(rm)!^p(rmp+2)/(2\pi)$. Choose $c > 1$ so large that $C_{p,r,m}c^{-\beta} < (1-c^{-1})^{(2+prm)/r}(rm)!^p(rmp+2)/(2\pi)$; then

$$K^{z^{rm},p}_{D_c^{(r)}}(0) \le C_{p,r,m}c^{-\beta} < (1-c^{-1})^{(2+prm)/r}\frac{(rm)!^p(rmp+2)}{2\pi} \le K^{z^{rm},p}_{D_{1/c}^{(r)}}(0).$$

Since $q = ce^{ra}$, the identity $q = 1/c$ corresponds to $a = -\frac{2}{r}\log c$, i.e. $D_{1/c}^{(r)} = (D_c^{(r)})_{-\frac{2}{r}\log c}$, while $D_c^{(r)} = (D_c^{(r)})_0$. Thus the kernel at the smaller value $a = -\frac{2}{r}\log c < 0$ exceeds its value at $a = 0$, and $a \mapsto K^{z^{rm},p}_{(D_c^{(r)})_a}(0)$ is not non-decreasing. ■

2.3 The L^p concavity of minimal integrals fails for $p > 2$

Throughout this subsection we fix $p > 2$ and integers $r \geq 1$, $m \geq 1$, and set $k := rm$. We fix $c \in (0,1)$ and write

$$D := D_c^{(r)} = \{x \in \mathbb{C} : \ |x|^r < |1 - cx^r|\}, \qquad F_c(x) := \frac{x^r}{1 - cx^r}.$$

Since $c < 1$, the domain D is bounded: for $x \in D$,

$$|x|^r < |1 - cx^r| \leq 1 + c|x|^r,$$

so $|x| < (1-c)^{-1/r}$.

2.3.1 The multiplier ideal of the Green function

Exactly as in Theorem 2.1 (whose proof applies verbatim to the map F_c), the map $F_c : D \to \mathbb{D} \setminus \{-1/c\}$ is proper of degree r, it vanishes to order r at 0, and consequently

$$G_D(x,0) = \frac{1}{r} \log |F_c(x)| = \log |x| - \frac{1}{r} \log |1 - cx^r|$$

is the Green function of D with pole at 0. Set

$$\varphi := 2(k+1)\, G_D(\cdot, 0), \qquad F(x) := \frac{x^k}{k!}.$$

Then φ is a negative plurisubharmonic function on D with $\varphi(0) = -\infty$, and near 0,

$$e^{-\varphi} = |x|^{-2(k+1)}\, |1 - cx^r|^{2(k+1)/r} = |x|^{-2(k+1)} \big(1 + O(|x|^r)\big).$$

Lemma 2.10. *One has $\mathcal{I}(\varphi)_0 = \mathfrak{m}_0^{k+1}$.*

Proof. For $f \in \mathcal{O}_0$ with $\operatorname{ord}_0 f = j$, one has $|f|^2 e^{-\varphi} \asymp |x|^{2j-2(k+1)}$ near 0, and in polar coordinates

$$\int_{|x|<\delta} |f|^2 e^{-\varphi}\, dA \asymp \int_0^\delta s^{2j-2k-1}\, ds,$$

which is finite exactly when $j \geq k+1$. ∎

2.3.2 Reduction to a higher order p-Bergman kernel

Let $\Omega \subset \mathbb{C}$ be a domain containing 0. Since $(\tilde{F} - F, 0) \in \mathfrak{m}_0^{k+1}$ is equivalent to

$$\tilde{F}^{(j)}(0) = 0 \ \ (0 \leq j < k), \qquad \tilde{F}^{(k)}(0) = 1,$$

Lemma 2.10 gives

$$C_{F,\mathcal{I}(\varphi)_0,p}(\Omega) = \inf\Big\{ \int_\Omega |\tilde{F}|^p : \ \tilde{F} \in \mathcal{O}(\Omega),\ \tilde{F}^{(j)}(0) = 0\ (0 \leq j < k),\ \tilde{F}^{(k)}(0) = 1 \Big\} = \frac{1}{K_\Omega^{z^k,p}(0)},$$

where $K_\Omega^{z^k,p}(0)$ is the higher order p-Bergman kernel of Section 4. Indeed, every admissible $\tilde{F}$ satisfies $\|\tilde{F}\|^p_{L^p(\Omega)} \geq 1/K_\Omega^{z^k,p}(0)$ by the definition of the kernel, and rescaling functions with $f^{(j)}(0) = 0$ $(0 \leq j < k)$ and $f^{(k)}(0) \neq 0$ by $1/f^{(k)}(0)$ produces admissible functions with norms arbitrarily close to $1/K_\Omega^{z^k,p}(0)$.

2.3.3 Sublevel sets and the scaling law

For $t \geq 0$ set

$$a := -\frac{t}{2(k+1)}, \qquad q := ce^{ra}.$$

As in Theorem 2.2 (applied to the r-fold map F_c, exactly as in Subsection 2.2.6),

$$\big\{G_D(\cdot,0) < a\big\} = e^a D_q^{(r)}, \qquad D_q^{(r)} := \big\{x \in \mathbb{C} : \ |x|^r < |1 - qx^r|\big\} :$$

indeed $G_D(z,0) < a$ is equivalent to $|F_c(z)| < e^{ra}$, and writing $z = e^a x$ gives $F_c(e^a x) = e^{ra} F_q(x)$.

Lemma 2.11. *For every $t \geq 0$,*

$$G_p(t) = \frac{e^{(pk+2)a}}{K^{z^k,p}_{D_q^{(r)}}(0)}, \qquad a = -\frac{t}{2(k+1)}, \quad q = ce^{ra}.$$

Proof. Since $G_D(\cdot,0) < 0$ on D,

$$\{\varphi < -t\} \cap D = \{G_D(\cdot,0) < a\} = e^a D_q^{(r)}.$$

For $\tilde{F} \in \mathcal{O}(e^a D_q^{(r)})$ set $g(x) := \tilde{F}(e^a x) \in \mathcal{O}(D_q^{(r)})$. Then $g^{(j)}(0) = e^{aj} \tilde{F}^{(j)}(0)$, so the admissibility of $\tilde{F}$ is equivalent to $g^{(j)}(0) = 0$ $(0 \leq j < k)$ and $g^{(k)}(0) = e^{ak}$, while

$$\int_{e^a D_q^{(r)}} |\tilde{F}|^p \, dA = e^{2a} \int_{D_q^{(r)}} |g|^p \, dA.$$

By the homogeneity of the previous subsection,

$$G_p(t) = e^{2a} \cdot \frac{e^{pak}}{K^{z^k,p}_{D_q^{(r)}}(0)} = \frac{e^{(pk+2)a}}{K^{z^k,p}_{D_q^{(r)}}(0)}.$$

■

2.3.4 The limit as $t \to +\infty$ and the failure of concavity

As computed in Subsection 2.2.6 — the comparison

$$D\big(0,(1+q)^{-1/r}\big) \subset D_q^{(r)} \subset D\big(0,(1-q)^{-1/r}\big), \qquad 0 < q < 1,$$

together with the rescaling $K^{z^k,p}_{D(0,R)}(0) = R^{-(pk+2)} K^{z^k,p}_{\mathbb{D}}(0)$ and the disk value $K^{z^k,p}_{\mathbb{D}}(0) = (k!)^p(kp+2)/(2\pi)$ — one has, for every $p \geq 1$,

$$\lim_{q \to 0^+} K^{z^k,p}_{D_q^{(r)}}(0) = \frac{(k!)^p(kp+2)}{2\pi} =: L.$$

Proof of Theorem 1.5. First, $G_p(0) < +\infty$: since $(F - F, 0) \in \mathfrak{m}_0^{k+1}$ and $F \in L^p(D)$ (D is bounded), we have $G_p(0) \leq \|F\|^p_{L^p(D)} < +\infty$. Moreover $G_p(0) > 0$: every admissible $\tilde{F}$ has

$\tilde{F}^{(k)}(0) = 1$, hence positive L^p norm on a neighborhood of 0; equivalently $G_p(0) = 1/K_D^{z^k,p}(0) \in (0, +\infty)$.

As $t \to +\infty$ we have $a \to -\infty$ and $q = ce^{ra} \to 0$, so Lemma 2.11 and the limit above give

$$G_p(t) \sim \frac{1}{L}\, e^{(pk+2)a} = \frac{1}{L}\, \exp\Big(-\frac{pk+2}{2(k+1)}\, t\Big).$$

Let $g(\rho) := G_p(-\log \rho)$ for $\rho \in (0, 1]$. Then

$$g(\rho) \sim \frac{1}{L}\, \rho^{\mu}, \qquad \mu := \frac{pk+2}{2(k+1)} = 1 + \frac{k(p-2)}{2(k+1)} > 1,$$

as $\rho \to 0^+$.

The function g is non-decreasing, because G_p is non-increasing in t. Moreover $\lim_{t\to+\infty} G_p(t) = 0$, hence $g(0+) = 0$: indeed,

$$G_p(t) \leq \int_{\{\varphi<-t\}} |F|^p \, dA \to 0$$

by dominated convergence as the sublevel sets shrink to $\{0\}$. Also $g(1) = G_p(0) > 0$.

If g were concave on $(0, 1]$, then for $0 < \varepsilon < \rho < 1$,

$$g(\rho) \;\geq\; \frac{\rho - \varepsilon}{1 - \varepsilon}\, g(1) + \frac{1 - \rho}{1 - \varepsilon}\, g(\varepsilon),$$

and letting $\varepsilon \to 0^+$ gives $g(\rho) \geq \rho\, g(1)$ for every $\rho \in (0, 1]$. But $\mu > 1$ and $g(\rho) \sim \rho^{\mu}/L$ imply $g(\rho) < \rho\, g(1)$ for all sufficiently small ρ — a contradiction. Hence $g = G_p(-\log \rho)$ is not concave in $\rho \in (0, 1]$, which proves the theorem. ■

3 A four-connected domain with no point of equality in the higher order Suita conjecture

Fix three pairwise distinct points $c_1, c_2, c_3 \in \mathbb{D}$ and choose $\rho_0 > 0$ so small that the closed disks $\overline{\mathbb{D}(c_j, 2\rho_0)}$ are pairwise disjoint and contained in $\mathbb{D}$. Let

$$B := \left(\tfrac{1}{2}\rho_0, \rho_0\right) \times \left(\tfrac{1}{2}\rho_0, \rho_0\right) \times \left(\tfrac{1}{2}\rho_0, \rho_0\right) \subset \mathbb{R}^3,$$

where $(\frac{1}{2}\rho_0, \rho_0)$ denotes the open interval in $\mathbb{R}$. For $\mathbf{r} = (r_1, r_2, r_3) \in B$ put

$$D_j(\mathbf{r}) := \overline{\mathbb{D}(c_j, r_j)}, \qquad \Gamma_j(\mathbf{r}) := \partial D_j(\mathbf{r}), \quad j = 1, 2, 3, \qquad \Gamma_4 := \partial\mathbb{D},$$

and

$$\Omega(\mathbf{r}) := \mathbb{D} \setminus \Big(D_1(\mathbf{r}) \cup D_2(\mathbf{r}) \cup D_3(\mathbf{r})\Big).$$

Then $\Omega(\mathbf{r})$ is bounded by the four pairwise disjoint analytic Jordan curves $\Gamma_1(\mathbf{r}), \Gamma_2(\mathbf{r}), \Gamma_3(\mathbf{r}), \Gamma_4$ (see Figure 3).

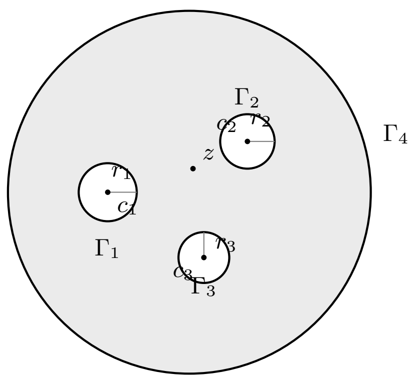

Figure 3: The four-connected domain $\Omega(\mathbf{r}) = \mathbb{D}\setminus\big(D_1(\mathbf{r})\cup D_2(\mathbf{r})\cup D_3(\mathbf{r})\big)$ with boundary components $\Gamma_1, \Gamma_2, \Gamma_3$ and Γ_4.

For $j = 1, 2, 3$ let $u_j(\cdot\,; \mathbf{r})$ be the harmonic function on $\Omega(\mathbf{r})$ with boundary values 1 on $\Gamma_j(\mathbf{r})$ and 0 on the other three curves, and set $u_4 := 1 - u_1 - u_2 - u_3$. Basic facts on harmonic measure used below can be found in [7].

Lemma 3.1. *For every $z \in \Omega(\mathbf{r})$ and $j, k \in \{1, 2, 3\}$,*

$$\frac{\partial u_j}{\partial r_j}(z; \mathbf{r}) > 0, \qquad \frac{\partial u_j}{\partial r_k}(z; \mathbf{r}) < 0 \quad (j \neq k), \qquad \frac{\partial u_4}{\partial r_k}(z; \mathbf{r}) < 0. \tag{3.3}$$

Proof. The functions u_j are real-analytic in $(z, \mathbf{r})$; this follows from the standard analytic dependence of solutions of the Dirichlet problem on analytic boundary perturbations; see [15, 17]. The first inequality in (3.3) is the usual monotonicity of harmonic measure under enlargement of the corresponding boundary component (Figure 4); it follows from the maximum principle, since on the new, smaller domain the old solution is < 1 on the new boundary and $= 0$ on all other boundary components.

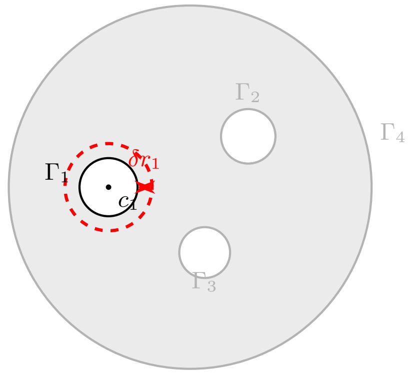

Figure 4: Enlargement of the inner disk bounded by $\Gamma_1(\mathbf{r})$ by $\delta r_1 > 0$: the dashed circle is the new boundary, and the red arrow indicates the radial displacement corresponding to the increase of r_1 in the variational formula (3.4).

For the strict derivative signs we use Hadamard's variational formula [17]. Let $G_{\mathbf{r}}(w, z)$ be the Green function of $\Omega(\mathbf{r})$ and let n denote the outward unit normal. Up to a fixed positive multiplicative constant,

$$\frac{\partial u_j}{\partial r_k}(z; \mathbf{r}) = \int_{\Gamma_k(\mathbf{r})} \frac{\partial u_j}{\partial n}(w) \, \frac{\partial G_{\mathbf{r}}}{\partial n}(w, z) \, ds(w). \tag{3.4}$$

For an inner boundary $\Gamma_k(\mathbf{r})$ one has $\partial G_{\mathbf{r}}/\partial n > 0$ on $\Gamma_k(\mathbf{r})$. By Hopf's lemma, $\partial u_j/\partial n$ on $\Gamma_k(\mathbf{r})$ is strictly positive if $j = k$ and strictly negative if $j \neq k$; similarly $\partial u_4/\partial n < 0$ on $\Gamma_k(\mathbf{r})$. Hence (3.3) follows from (3.4). ∎

For $z \in \Omega(\mathbf{r})$ put

$$A(z; \mathbf{r}) := \left(\tfrac{\partial u_j}{\partial r_k}(z; \mathbf{r}) \right)_{1 \leq j,k \leq 3}.$$

Lemma 3.2. *For every $\mathbf{r} \in B$ and every $z \in \Omega(\mathbf{r})$ the matrix $A(z; \mathbf{r})$ is invertible.*

Proof. By (3.3), $A_{jk} < 0$ for $j \neq k$, and

$$(A^T \mathbf{1})_j = \sum_{i=1}^{3} \frac{\partial u_i}{\partial r_j} = \frac{\partial}{\partial r_j}\big(u_1 + u_2 + u_3\big) = -\frac{\partial u_4}{\partial r_j} > 0.$$

Thus A^T has positive diagonal, nonpositive off-diagonal entries and positive row sums. If $A^T \mathbf{x} = 0$ for some $\mathbf{x} \neq 0$, choose an index i at which $x_i = \max_j x_j > 0$; if all $x_j \leq 0$, choose instead an index i at which $x_i = \min_j x_j < 0$. In the first case, since $A^T_{ik} \leq 0$ for $k \neq i$,

$$0 = (A^T \mathbf{x})_i = A^T_{ii} x_i + \sum_{k \neq i} A^T_{ik} x_k \geq A^T_{ii} x_i + \sum_{k \neq i} A^T_{ik} x_i = (A^T \mathbf{1})_i x_i > 0,$$

a contradiction. The second case is analogous with all inequalities reversed. Hence A^T, and therefore A, is nonsingular. ∎

Proof of Theorem 1.6. Let

$$X := \{(z, \mathbf{r}) \in \mathbb{C} \times B : z \in \Omega(\mathbf{r})\},$$

an open subset of $\mathbb{R}^2 \times \mathbb{R}^3 \cong \mathbb{R}^5$, and define

$$F : X \to \mathbb{R}^3, \qquad F(z, \mathbf{r}) := \big(u_1(z; \mathbf{r}), u_2(z; \mathbf{r}), u_3(z; \mathbf{r})\big).$$

By Lemma 3.1, F is real-analytic. Its derivative contains the invertible 3×3 block $D_{\mathbf{r}}F = A(z; \mathbf{r})$; hence F is a submersion at every point of X.

For $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{Q}^3$ set $S_{\mathbf{q}} := F^{-1}(\mathbf{q})$. By the implicit function theorem, $S_{\mathbf{q}}$ is a real-analytic submanifold of X of dimension $\dim_{\mathbb{R}} X - 3 = 2$ (possibly empty). Let $\pi_{\mathbf{q}} : S_{\mathbf{q}} \to B$ be the projection onto the parameter box. Each connected component of $S_{\mathbf{q}}$ is a two-dimensional C^1 manifold. Since $\dim_{\mathbb{R}} B = 3$, Sard's theorem shows that its image under $\pi_{\mathbf{q}}$ has empty interior (indeed Lebesgue measure zero). A second countable manifold has at most countably many connected components, so $\pi_{\mathbf{q}}(S_{\mathbf{q}})$ is a countable union of null sets, hence a null set. Consequently

$$E := \bigcup_{\mathbf{q} \in \mathbb{Q}^3} \pi_{\mathbf{q}}(S_{\mathbf{q}})$$

is a countable union of null sets, and $B \setminus E$ is nonempty (in fact of full measure).

Choose $\mathbf{r} \in B \setminus E$. We claim that $\Omega := \Omega(\mathbf{r})$ has the required property. Indeed, if for some $z \in \Omega$ all three numbers $u_j(z; \mathbf{r})$ were rational, then with $\mathbf{q} := \big(u_1(z; \mathbf{r}), u_2(z; \mathbf{r}), u_3(z; \mathbf{r})\big) \in \mathbb{Q}^3$ we would have $(z, \mathbf{r}) \in S_{\mathbf{q}}$ and hence $\mathbf{r} \in E$, a contradiction. This completes the proof. ■

3.1 Generalization to arbitrary connectivity

The construction of the preceding part extends verbatim to every $n \geq 4$.

THEOREM 3.3. *For every integer $n \geq 4$ there exists a planar domain $\Omega \subset \mathbb{C}$ bounded by n analytic Jordan curves such that, for the corresponding harmonic functions $u_1, \dots, u_{n-1}$, there does not exist $z \in \Omega$ with*

$$u_j(z) \in \mathbb{Q}, \qquad j = 1, \dots, n-1.$$

In particular, at no point of Ω does the equality in the higher order Suita conjecture hold.

Proof. Fix $n \geq 4$ and set $m := n - 1$. Choose m pairwise distinct points $c_1, \dots, c_m \in \mathbb{D}$ and $\rho_0 > 0$ so small that the closed disks $\overline{\mathbb{D}(c_j, 2\rho_0)}$ are pairwise disjoint and contained in $\mathbb{D}$. Let $B := (\rho_0/2, \rho_0)^m$. For $\mathbf{r} = (r_1, \dots, r_m) \in B$ define

$$D_j(\mathbf{r}) := \overline{\mathbb{D}(c_j, r_j)}, \qquad \Gamma_j(\mathbf{r}) := \partial D_j(\mathbf{r}) \quad (1 \leq j \leq m), \qquad \Gamma_{m+1} := \partial \mathbb{D},$$

and

$$\Omega(\mathbf{r}) := \mathbb{D} \setminus \bigcup_{j=1}^{m} D_j(\mathbf{r}).$$

Let $u_j(\cdot\,; \mathbf{r})$ be the harmonic function on $\Omega(\mathbf{r})$ with boundary values 1 on $\Gamma_j(\mathbf{r})$ and 0 on the other curves, $1 \leq j \leq m$, and put $u_{m+1} := 1 - \sum_{j=1}^{m} u_j$.

The variational argument in Lemma 3.1 applies unchanged with m in place of 3: for all $j, k \in \{1, \dots, m\}$,

$$\frac{\partial u_j}{\partial r_j}(z;\mathbf{r}) > 0, \qquad \frac{\partial u_j}{\partial r_k}(z;\mathbf{r}) < 0 \quad (j \neq k), \qquad \frac{\partial u_{m+1}}{\partial r_k}(z;\mathbf{r}) < 0.$$

Indeed, the off-diagonal and u_{m+1} estimates follow from Hadamard's formula (3.4) and Hopf's lemma exactly as in the proof of Lemma 3.1.

Let $A(z;\mathbf{r}) := (\partial u_j/\partial r_k)_{1\le j,k\le m}$. As in Lemma 3.2,

$$(A^T\mathbf{1})_j = \sum_{i=1}^{m} \frac{\partial u_i}{\partial r_j} = \frac{\partial}{\partial r_j}\Big(\sum_{i=1}^{m} u_i\Big) = -\frac{\partial u_{m+1}}{\partial r_j} > 0.$$

Thus A^T has positive diagonal, nonpositive off-diagonal entries and positive row sums; by the same maximum-entry argument as in Lemma 3.2, A^T, and hence A, is nonsingular.

Define

$$X := \big\{(z,\mathbf{r}) \in \mathbb{C} \times B : z \in \Omega(\mathbf{r})\big\}, \qquad F : X \to \mathbb{R}^m, \quad F(z,\mathbf{r}) := \big(u_1(z;\mathbf{r}), \dots, u_m(z;\mathbf{r})\big).$$

Then F is real-analytic and a submersion, since $D_{\mathbf{r}}F = A$ is invertible. For every $\mathbf{q} \in \mathbb{Q}^m$, the set $S_{\mathbf{q}} := F^{-1}(\mathbf{q})$ is a real-analytic submanifold of X of dimension

$$\dim_{\mathbb{R}} X - m = (2+m) - m = 2.$$

Let $\pi_{\mathbf{q}} : S_{\mathbf{q}} \to B$ be the projection onto the parameter box. Since $\dim_{\mathbb{R}} B = m = n-1 \ge 3$, Sard's theorem shows that $\pi_{\mathbf{q}}(S_{\mathbf{q}})$ is a null set: every connected component is a smooth image of a two-dimensional manifold, and there are at most countably many components. Therefore

$$E := \bigcup_{\mathbf{q}\in\mathbb{Q}^m} \pi_{\mathbf{q}}(S_{\mathbf{q}})$$

is a countable union of null sets, and $B \setminus E$ is nonempty.

Choose $\mathbf{r} \in B \setminus E$. If for some $z \in \Omega(\mathbf{r})$ all $u_j(z;\mathbf{r})$ were rational, then $(z,\mathbf{r}) \in S_{\mathbf{q}}$ for $\mathbf{q} := F(z,\mathbf{r}) \in \mathbb{Q}^m$, so $\mathbf{r} \in E$, a contradiction. Hence $\Omega := \Omega(\mathbf{r})$ satisfies the conclusion of the theorem. ∎

4 The ξ-reduction fails for $0 < p < 1$

4.1 The counterexample on the unit disk

Let $D \subset \mathbb{C}^n$ be a domain, $p > 0$, and let

$$H(z) = \sum_{|\alpha|=k} a_\alpha z^\alpha$$

be a homogeneous polynomial of degree k. Following [14], denote by

$$\ell_1^{(n)} := \Big\{ \xi = (\xi_\alpha)_{\alpha \in \mathbb{N}^n} : \ \xi_\alpha \in \mathbb{C}, \ \sum_{\alpha \in \mathbb{N}^n} |\xi_\alpha| \rho^{|\alpha|} < +\infty \text{ for every } \rho > 0 \Big\}$$

the space of functionals on holomorphic germs. For $\xi \in \ell_1^{(n)}$ and $f \in \mathcal{O}(D)$, the pairing is defined by

$$(\xi \cdot f)(z) := \sum_{\alpha \in \mathbb{N}^n} \xi_\alpha \frac{D^\alpha f(z)}{\alpha!}.$$

For the homogeneous polynomial H of degree k above, let S_H denote the affine family of all $\xi \in \ell_1^{(n)}$ such that

$$\xi_\alpha = \begin{cases} a_\alpha \alpha!, & |\alpha| = k, \\ 0, & |\alpha| > k, \end{cases}$$

the entries with $|\alpha| < k$ being free. With this notation, the higher order p-Bergman kernel and the p-Bergman kernel with respect to ξ are

$$K_D^{H,p}(z) := \sup \Big\{ |P_H(f)(z)|^p : \ f \in A^p(D), \ f^{(j)}(z) = 0 \ (0 \le j < k), \ \|f\|_{L^p(D)} \le 1 \Big\},$$

$$K_{\xi,D,p}(z) := \sup \Big\{ |(\xi \cdot f)(z)|^p : \ f \in A^p(D), \ \|f\|_{L^p(D)} \le 1 \Big\}.$$

For a positive homogeneous functional, equivalently,

$$K_{\xi,D,p}(z) = \frac{1}{\inf \big\{ \|f\|_{L^p(D)}^p : \ f \in A^p(D), \ (\xi \cdot f)(z) = 1 \big\}}.$$

We answer the question whether the identity in Theorem 1.7 of [14] extends to $0 < p < 1$ in the negative. Already the one-dimensional model $D = \mathbb{D}$, $H(z) = z$, $p = 1/2$ gives a counterexample. For $H(z) = z$ the set S_H consists of all sequences

$$\xi = (\xi_0, 1, 0, 0, \ldots), \qquad \xi_0 \in \mathbb{C}.$$

4.1.1 Proof of Theorem 1.7

Proof of Theorem 1.7. We first compute $K_{\mathbb{D}}^{H,1/2}(0)$. Let $f \in \mathcal{O}(\mathbb{D})$ satisfy $f(0) = 0$ and $f'(0) = 1$, and write $f(z) = zg(z)$ with $g(0) = 1$. Since $|g|^{1/2}$ is subharmonic, for every $r \in (0,1)$,

$$\frac{1}{2\pi} \int_0^{2\pi} |g(re^{i\theta})|^{1/2} \, d\theta \ge |g(0)|^{1/2} = 1.$$

Hence

$$\int_{\mathbb{D}} |f|^{1/2}\, dA = \int_0^1 r^{1/2} r \Big(\int_0^{2\pi} |g(re^{i\theta})|^{1/2}\, d\theta \Big) dr \geq 2\pi \int_0^1 r^{3/2}\, dr = \frac{4\pi}{5},$$

with equality for $g \equiv 1$, i.e. $f(z) = z$. Therefore

$$K_{\mathbb{D}}^{H,1/2}(0) = \frac{1}{4\pi/5} = \frac{5}{4\pi}.$$

Next, fix $\alpha = 3/2$ and set

$$M(\alpha) := \frac{1}{\pi} \int_{\mathbb{D}} |1 + \alpha z|\, dA(z).$$

We need the elementary estimate

$$\frac{M(\alpha)}{\sqrt{2\alpha}} < \frac{4}{5}. \tag{4.5}$$

For $0 \leq x \leq 1$ the fourth Taylor polynomial of $\sqrt{1-x}$ gives

$$\sqrt{1-x} \leq 1 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128}, \tag{4.6}$$

because all remaining Taylor coefficients of $\sqrt{1-x}$ are negative. Write $t = \alpha r$ and

$$\kappa := \frac{4t}{(1+t)^2}.$$

Then

$$|1 + te^{i\theta}| = (1+t)\sqrt{1 - \kappa \sin^2 \frac{\theta}{2}}.$$

Using (4.6) and the averages

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^2 \frac{\theta}{2}\, d\theta = \frac{1}{2}, \qquad \frac{1}{2\pi} \int_0^{2\pi} \sin^4 \frac{\theta}{2}\, d\theta = \frac{3}{8},$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin^6 \frac{\theta}{2}\, d\theta = \frac{5}{16}, \qquad \frac{1}{2\pi} \int_0^{2\pi} \sin^8 \frac{\theta}{2}\, d\theta = \frac{35}{128},$$

we obtain

$$\frac{1}{2\pi} \int_0^{2\pi} |1 + te^{i\theta}|\, d\theta \leq (1+t)\Big(1 - \frac{\kappa}{4} - \frac{3\kappa^2}{64} - \frac{5\kappa^3}{256} - \frac{175\kappa^4}{16384}\Big).$$

Consequently

$$M(\alpha) \leq \int_0^1 2r(1+\alpha r)\Big(1 - \frac{\kappa}{4} - \frac{3\kappa^2}{64} - \frac{5\kappa^3}{256} - \frac{175\kappa^4}{16384}\Big)\, dr.$$

For $\alpha = 3/2$ the integral on the right is

$$\frac{13771}{10000}.$$

Since

$$\Big(\frac{13771}{10000}\Big)^2 = \frac{189640441}{100000000} < \frac{48}{25} = \Big(\frac{4}{5}\sqrt{3}\Big)^2,$$

we have

$$\frac{13771}{10000} < \frac{4}{5}\sqrt{3} = \frac{4}{5}\sqrt{2\alpha},$$

which proves (4.5).

Now fix any $c \in \mathbb{C}$. Choose a unit complex number $e^{i\theta}$ such that c and $2\alpha e^{i\theta}$ have the same argument (take $e^{i\theta} = 1$ if $c = 0$), so that $c + 2\alpha e^{i\theta}$ has modulus $|c| + 2\alpha$. Choose a unit complex number $e^{i\psi}$ with $e^{i\psi}(c + 2\alpha e^{i\theta}) = |c| + 2\alpha$, and define

$$f_{c,\theta}(z) := e^{i\psi}\frac{(1 + \alpha e^{i\theta} z)^2}{|c| + 2\alpha}.$$

Then $f_{c,\theta}$ is a polynomial, and

$$f_{c,\theta}(0) = \frac{e^{i\psi}}{|c| + 2\alpha}, \qquad f'_{c,\theta}(0) = \frac{2\alpha e^{i\theta} e^{i\psi}}{|c| + 2\alpha},$$

so that for $\xi_c = (c, 1, 0, 0, \ldots) \in S_H$,

$$(\xi_c \cdot f_{c,\theta})(0) = c f_{c,\theta}(0) + f'_{c,\theta}(0) = \frac{e^{i\psi}(c + 2\alpha e^{i\theta})}{|c| + 2\alpha} = 1.$$

Moreover, by the rotation invariance of dA,

$$\int_{\mathbb{D}} |f_{c,\theta}|^{1/2}\, dA = \frac{1}{\sqrt{|c| + 2\alpha}} \int_{\mathbb{D}} |1 + \alpha e^{i\theta} z|\, dA = \frac{\pi M(\alpha)}{\sqrt{|c| + 2\alpha}} \le \frac{\pi M(\alpha)}{\sqrt{2\alpha}} < \frac{4\pi}{5},$$

since $|c| + 2\alpha \ge 2\alpha$. By the equivalent form of $K_{\xi,D,p}$, the same estimate gives the uniform lower bound

$$K_{\xi_c,\mathbb{D},1/2}(0) \ge \frac{1}{\int_{\mathbb{D}} |f_{c,\theta}|^{1/2}\, dA} \ge \frac{\sqrt{2\alpha}}{\pi M(\alpha)} > \frac{5}{4\pi} = K_{\mathbb{D}}^{H,1/2}(0).$$

Since the middle lower bound is independent of c, we may take the infimum and obtain

$$\inf_{\xi \in S_H} K_{\xi,\mathbb{D},1/2}(0) \ge \frac{\sqrt{2\alpha}}{\pi M(\alpha)} > \frac{5}{4\pi} = K_{\mathbb{D}}^{H,1/2}(0),$$

which proves the theorem. ■

4.2 The threshold $p_* = 2(\sqrt{2} - 1)$ for the counterexample

The counterexample in Theorem 1.7 is not an isolated phenomenon: the second variation at the extremal $f(z) = z$ shows that the identity

$$K_{\mathbb{D}}^{H,p}(0) = \inf_{\xi \in S_H} K_{\xi,\mathbb{D},p}(0) \tag{4.7}$$

fails for every $0 < p < p_*$, where

$$p_* := 2(\sqrt{2} - 1).$$

For $p > p_*$ the same computation shows that z is a strict local minimizer in the class $f'(0) = 1$, and at $p = p_*$ the second variation degenerates along a single explicit direction; numerically, z remains the global minimizer for every $p \ge p_*$ in this example.

Since $H(z)=z$, every $\xi\in S_H$ has the form $\xi_c=(c,1,0,0,\ldots)$, $c\in\mathbb{C}$, and

$$K_{\xi_c,\mathbb{D},p}(0)=\frac{1}{m_{\xi_c}(p)},\qquad m_{\xi_c}(p):=\inf\Big\{\int_{\mathbb{D}}|f|^p\,dA:\ f\in\mathcal{O}(\mathbb{D}),\ cf(0)+f'(0)=1\Big\}.$$

The function $f(z)=z$ is admissible for every ξ_c, and $\int_{\mathbb{D}}|z|^p\,dA=2\pi/(p+2)$; hence $m_{\xi_c}(p)\le 2\pi/(p+2)$ for all c.

THEOREM 4.1. *Let $p_*:=2(\sqrt{2}-1)$. For every $0<p<p_*$, with $D=\mathbb{D}$, $H(z)=z$ and $z=0$,*

$$\inf_{\xi\in S_H}K_{\xi,\mathbb{D},p}(0)>K_{\mathbb{D}}^{H,p}(0);$$

that is, the identity (4.7) fails for every $p\in(0,p_*)$. In particular, Theorem 4.1 contains the case $p=1/2$ of Theorem 1.7.

Proof. By the same argument as in the proof of Theorem 1.7, with p in place of $1/2$,

$$K_{\mathbb{D}}^{H,p}(0)=\frac{p+2}{2\pi}.$$

It suffices to show that $m_{\xi_0}(p)<2\pi/(p+2)$. For $\lambda>0$ and $\varepsilon\in\mathbb{R}$ consider

$$f_\varepsilon(z):=z+\varepsilon(1+\lambda z^2),$$

so that $f_\varepsilon'(0)=1$ and $f_0=z$. Write $J_p(\varepsilon):=\int_{\mathbb{D}}|f_\varepsilon|^p\,dA$. The first derivative at 0 vanishes by symmetry, and a direct expansion (given after the proof) yields

$$J_p''(0)=\pi\Big(p+\frac{p^2\lambda^2}{p+4}+\frac{2p(p-2)\lambda}{p+2}\Big). \tag{4.8}$$

The quadratic polynomial in λ on the right-hand side has minimum

$$p-\frac{(2-p)^2(p+4)}{(p+2)^2}=\frac{4(p^2+4p-4)}{(p+2)^2}=\frac{4(p-p_*)(p+2+2\sqrt{2})}{(p+2)^2},$$

attained at

$$\lambda_p:=\frac{(2-p)(p+4)}{p(p+2)}.$$

Since $p+2+2\sqrt{2}>0$ for $p>0$, the minimum is negative precisely for $p<p_*$. Fixing such a p and taking $\lambda=\lambda_p$, we have $J_p''(0)<0$; as $J_p'(0)=0$ and $J_p(0)=2\pi/(p+2)$, for all sufficiently small $\varepsilon>0$,

$$\int_{\mathbb{D}}|f_\varepsilon|^p\,dA<\frac{2\pi}{p+2}.$$

Thus $m_{\xi_0}(p)<2\pi/(p+2)$, and Proposition 4.2 yields $\sup_{c\in\mathbb{C}}m_{\xi_c}(p)<2\pi/(p+2)$. Consequently

$$\inf_{\xi\in S_H}K_{\xi,\mathbb{D},p}(0)=\frac{1}{\sup_{c\in\mathbb{C}}m_{\xi_c}(p)}>\frac{p+2}{2\pi}=K_{\mathbb{D}}^{H,p}(0),$$

∎

Proposition 4.2. *The identity (4.7) holds if and only if $m_{\xi_0}(p) = 2\pi/(p+2)$, i.e. if and only if z minimizes $\int_{\mathbb{D}} |f|^p \, dA$ among all holomorphic f with $f'(0) = 1$.*

Proof. Since
$$\inf_{\xi \in S_H} K_{\xi,\mathbb{D},p}(0) = \inf_{c \in \mathbb{C}} \frac{1}{m_{\xi_c}(p)} = \frac{1}{\sup_{c\in\mathbb{C}} m_{\xi_c}(p)},$$
it suffices to show that $\sup_{c\in\mathbb{C}} m_{\xi_c}(p)$ equals $2\pi/(p+2)$ if and only if it is attained at $c = 0$.

We first note that $c \mapsto m_{\xi_c}(p)$ is continuous. Write $J_p(f) := \int_{\mathbb{D}} |f|^p \, dA$ and
$$N(c) := \sup_{f \neq 0} \frac{|cf(0) + f'(0)|}{J_p(f)^{1/p}},$$
so that $m_{\xi_c}(p) = N(c)^{-p}$. Since $|f(0)| \le \pi^{-1/p} J_p(f)^{1/p}$ by subharmonicity,
$$|N(c) - N(d)| \le |c - d| \, \pi^{-1/p},$$
so N, and hence $c \mapsto m_{\xi_c}(p)$, is continuous. Taking $f \equiv 1$ shows $N(c) \ge |c| \, \pi^{-1/p}$, whence $m_{\xi_c}(p) \le \pi \, |c|^{-p} \to 0$ as $|c| \to \infty$.

It remains to check that $m_{\xi_c}(p) < 2\pi/(p+2)$ for every $c \neq 0$. Set $\eta = \bar{c}/|c|$ and $f_t(z) = z + t\eta(1 - cz)$. Then $cf_t(0) + f_t'(0) = 1$ for every t, and the local factorization argument used for the second variation below justifies differentiation at $t = 0$ for every $p > 0$. Hence
$$\frac{d}{dt}\Big|_{t=0} \int_{\mathbb{D}} |f_t|^p \, dA = p \, \Re \int_{\mathbb{D}} |z|^{p-2} \bar{z} \, \eta(1 - cz) \, dA = -p \, \Re(\eta c) \int_{\mathbb{D}} |z|^p \, dA = -p|c| \, \frac{2\pi}{p+2} < 0,$$
because $\int_{\mathbb{D}} |z|^{p-2} \bar{z} \, dA = 0$ by rotation invariance. Thus $J_p(f_t) < 2\pi/(p+2)$ for all sufficiently small $t > 0$, which gives $m_{\xi_c}(p) < 2\pi/(p+2)$.

Consequently, if $m_{\xi_0}(p) = 2\pi/(p+2)$ then the supremum is attained at $c = 0$; if instead $m_{\xi_0}(p) < 2\pi/(p+2)$, then $m_{\xi_c}(p) < 2\pi/(p+2)$ for every $c \in \mathbb{C}$, and continuity together with the decay at infinity yields $\sup_{c\in\mathbb{C}} m_{\xi_c}(p) < 2\pi/(p+2)$. ∎

For completeness we record and justify the second variation used above. Let $h(z) = \sum_{m\neq 1} h_m z^m$ be holomorphic on a neighborhood of $\overline{\mathbb{D}}$ with $h'(0) = 0$. Although the pointwise second derivative is singular at $z = 0$ when $p < 2$, differentiation under the integral is valid for every $p > 0$. Indeed, choose a cutoff supported in a small disk about 0. By the holomorphic implicit function theorem, for small real ε the function $z + \varepsilon h(z)$ has there a unique simple zero $a(\varepsilon)$ and factors as
$$z + \varepsilon h(z) = (z - a(\varepsilon)) u_\varepsilon(z),$$
where u_ε is nonvanishing and depends smoothly on ε. After the translation $w = z - a(\varepsilon)$, two derivatives fall only on smooth factors multiplying $|w|^p$, which is locally integrable. Away from 0 the usual dominated-convergence argument applies. Thus the integral is twice differentiable at 0, and we may expand $|z + \varepsilon h|^p = |z|^p |1 + \varepsilon h/z|^p$ for $z \neq 0$. Using the elementary averages $\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta} \, d\theta = \delta_{k,0}$, we obtain
$$\frac{d^2}{d\varepsilon^2}\Big|_{\varepsilon=0} \int_{\mathbb{D}} |z + \varepsilon h|^p \, dA = \pi \left(p^2 \sum_{m\neq 1} \frac{|h_m|^2}{p + 2m} + \frac{2p(p-2)}{p+2} \Re(h_0 h_2) \right), \tag{4.9}$$

which for $h(z) = 1 + \lambda z^2$ reduces to (4.8). The only potentially negative term is the cross term $\Re(h_0 h_2)$, which arises from the pairing of the modes $h_0 z^{-1}$ and $h_2 z$ in $(h/z)^2$. The threshold p_* is exactly the value at which the two-mode quadratic form $(x, y) \mapsto px^2 + p^2y^2/(p+4) + 2p(p-2)xy/(p+2)$ becomes positive semidefinite.

REMARK 4.3. *It remains open whether* (4.7) *holds for every $p \geq p_*$. For $p \geq 1$ this follows from [14, Theorem 1.7], and the numerical evidence above suggests that z remains the global minimizer for every $p \geq p_*$ in this example; a rigorous proof of* (4.7) *in the remaining range $p_* \leq p < 1$ is not known.*

4.3 The L^p analogue of $B = C$

Bao–Guan [12] proved that for $p = 2$,

$$B(F, I, D) := \sup_{\xi \in \ell_I} \frac{|(\xi \cdot F)(0)|^2}{K_{\xi, D, 2}(0)} = C_{F,I}(D) := \inf\Big\{\int_D |\tilde{F}|^2 : \ (\tilde{F} - F, 0) \in I, \ \tilde{F} \in \mathcal{O}(D)\Big\},$$

where

$$\ell_I := \big\{\xi \in \ell_1^{(n)} \setminus \{0\} : \ (\xi \cdot f)(0) = 0 \ \ \forall\, (f, 0) \in I\big\}.$$

It is natural to ask whether the L^p analogue

$$B_p(F, I, D) := \sup_{\xi \in \ell_I} \frac{|(\xi \cdot F)(0)|^p}{K_{\xi, D, p}(0)}, \qquad C_{F,I,p}(D) := \inf\Big\{\int_D |\tilde{F}|^p : \ (\tilde{F} - F, 0) \in I, \ \tilde{F} \in \mathcal{O}(D)\Big\}$$

holds for all $p \in (0, +\infty)$. The counterexample above answers this in the negative for every $0 < p < p_*$, while for $p \geq 1$ the identity holds for every ideal I.

THEOREM 4.4. *Let $p_* := 2(\sqrt{2} - 1)$. For every $0 < p < p_*$, with $D = \mathbb{D}$, $I = \mathfrak{m}_0^2$ and $F(z) = z$,*

$$B_p(z, \mathfrak{m}_0^2, \mathbb{D}) < C_{z, \mathfrak{m}_0^2, p}(\mathbb{D}) = \frac{2\pi}{p+2}.$$

Proof. Since $(\widetilde{F} - z, 0) \in \mathfrak{m}_0^2$ means $\widetilde{F}(0) = 0$ and $\widetilde{F}'(0) = 1$, the computation of $K_{\mathbb{D}}^{z,p}(0)$ above gives

$$C_{z, \mathfrak{m}_0^2, p}(\mathbb{D}) = \frac{1}{K_{\mathbb{D}}^{z,p}(0)} = \frac{2\pi}{p+2}.$$

On the other hand $\ell_{\mathfrak{m}_0^2} = \big\{(\xi_0, \xi_1, 0, 0, \ldots) : (\xi_0, \xi_1) \neq (0, 0)\big\}$, and for $\xi_c := (c, 1, 0, 0, \ldots)$ the homogeneity identity $K_{\xi_c, \mathbb{D}, p}(0) = 1/m_{\xi_c}(p)$ yields

$$B_p(z, \mathfrak{m}_0^2, \mathbb{D}) = \sup_{c \in \mathbb{C}} m_{\xi_c}(p) < \frac{2\pi}{p+2}$$

by Theorem 4.1 and Proposition 4.2. ■

THEOREM 4.5. *Let $p \geq 1$, let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing o, let I be an ideal of $\mathcal{O}_o$, and let $(F, o) \in \mathcal{O}_o$. Then*

$$B_p(F, I, D) = C_{F,I,p}(D).$$

Here both sides are allowed to be $+\infty$, with the conventions $a/0 = +\infty$ for $a > 0$, $0/0 = 0$, and $\sup \varnothing = 0$ *in the definition of B_p.*

Proof. The inequality $B_p \le C_{F,I,p}$ is elementary: for every admissible $\widetilde{F}$ and every $\xi \in \ell_I$ one has $(\xi \cdot F)(o) = (\xi \cdot \widetilde{F})(o)$, hence $|(\xi \cdot F)(o)|^p \le K_{\xi,D,p}(o) \int_D |\widetilde{F}|^p$.

Assume first that D is bounded. For the reverse inequality fix $k \ge 1$ and write $V_{I+\mathfrak{m}^k} := \{f \in A^p(D) : (f,o) \in I + \mathfrak{m}^k\}$. Since $\mathfrak{m}^k \subseteq I + \mathfrak{m}^k$, membership in $I + \mathfrak{m}^k$ is detected by the $(k-1)$-jet, so the quotient $Q_k := A^p(D)/V_{I+\mathfrak{m}^k}$ is finite-dimensional. As $p \ge 1$, $A^p(D)$ is a normed space, and $C_{F,I+\mathfrak{m}^k,p}(D)^{1/p} = \|[F]\|_{Q_k}$ (the class of F being represented by the polynomial $j_{k-1}F$). By the Hahn–Banach theorem,

$$C_{F,I+\mathfrak{m}^k,p}(D)^{1/p} = \|[F]\|_{Q_k} = \sup\Big\{|\varphi(F)| : \ \varphi \in V_{I+\mathfrak{m}^k}^{\perp}, \ \|\varphi\| \le 1\Big\}.$$

Every $\varphi \in V_{I+\mathfrak{m}^k}^{\perp}$ depends only on the $(k-1)$-jet, i.e. $\varphi(f) = (\xi \cdot f)(o)$ for some $\xi \in \ell_{I+\mathfrak{m}^k}$, and $\|\varphi\| = \|\xi\|_*$ with $\|\xi\|_* := K_{\xi,D,p}(o)^{1/p}$. Therefore

$$B_p(F, I + \mathfrak{m}^k, D) = C_{F,I+\mathfrak{m}^k,p}(D).$$

Since $I \subseteq I + \mathfrak{m}^k$, one has $\ell_{I+\mathfrak{m}^k} \subseteq \ell_I$, hence

$$B_p(F, I, D) \ \ge \ B_p(F, I + \mathfrak{m}^k, D) = C_{F,I+\mathfrak{m}^k,p}(D), \qquad k \ge 1.$$

It remains to note that $\lim_{k\to\infty} C_{F,I+\mathfrak{m}^k,p}(D) = C_{F,I,p}(D)$. Indeed $C_k := C_{F,I+\mathfrak{m}^k,p}(D)$ is increasing in k with $C_k \le C_{F,I,p}(D)$; if $C_\infty := \lim_k C_k < +\infty$, choose $G_k \in A^p(D)$ with $(G_k - F, o) \in I + \mathfrak{m}^k$ and $\int_D |G_k|^p \le C_k + 1/k$. By Montel's theorem (through the submean-value property of $|f|^p$) a subsequence converges locally uniformly to some $G \in \mathcal{O}(D)$; Fatou's lemma gives $\int_D |G|^p \le C_\infty$; and the jet conditions pass to the limit, so $(G - F, o) \in I + \mathfrak{m}^k$ for every k, whence $(G - F, o) \in I$ by the Krull intersection theorem [1, Chapter 10], $\bigcap_{k\ge1}(I + \mathfrak{m}^k) = I$. Therefore $C_{F,I,p}(D) \le C_\infty$, and combined with $C_\infty \le C_{F,I,p}(D)$ this gives the limit. Consequently

$$B_p(F, I, D) \ \ge \ \lim_{k\to\infty} C_{F,I+\mathfrak{m}^k,p}(D) = C_{F,I,p}(D) \ \ge \ B_p(F, I, D),$$

which proves the theorem when D is bounded.

For a general pseudoconvex domain, choose a bounded pseudoconvex exhaustion

$$D_1 \Subset D_2 \Subset \cdots \Subset D, \qquad o \in D_1, \qquad \bigcup_{j\ge1} D_j = D.$$

The same Montel–Fatou argument used above gives

$$C_{F,I,p}(D_j) \nearrow C_{F,I,p}(D). \tag{4.10}$$

Indeed, the left-hand side is increasing and bounded above by the right-hand side; if its limit is finite, a diagonal limit of almost minimizers on D_j is an admissible function on D with no larger L^p integral.

For a fixed $\xi \in \ell_I$, apply the same argument to

$$m_{\xi,p}(D_j) := \inf\Big\{\int_{D_j} |f|^p : f \in A^p(D_j), \ (\xi \cdot f)(o) = 1\Big\} = \frac{1}{K_{\xi,D_j,p}(o)}.$$

It follows that

$$m_{\xi,p}(D_j) \nearrow m_{\xi,p}(D), \qquad K_{\xi,D_j,p}(o) \searrow K_{\xi,D,p}(o), \tag{4.11}$$

where the extended values are permitted. Hence, using the monotonicity in j to interchange the two suprema,

$$\begin{aligned} B_p(F,I,D) &= \sup_{\xi\in\ell_I} \lim_{j\to\infty} \frac{|(\xi\cdot F)(o)|^p}{K_{\xi,D_j,p}(o)} \\ &= \lim_{j\to\infty} \sup_{\xi\in\ell_I} \frac{|(\xi\cdot F)(o)|^p}{K_{\xi,D_j,p}(o)} = \lim_{j\to\infty} B_p(F,I,D_j) \\ &= \lim_{j\to\infty} C_{F,I,p}(D_j) = C_{F,I,p}(D), \end{aligned}$$

where the bounded case was used on every D_j. This completes the proof. ■

Corollary 4.6. *Let $1 \le p \le 2$, let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing o, and let φ be a negative plurisubharmonic function on D with $\varphi(o) = -\infty$. Set $I := \mathcal{I}(\varphi)_o$, let $(F,o) \in \mathcal{O}_o$, and define*

$$D_t := \{\varphi < -t\} \cap D, \qquad G_p(t) := C_{F,I,p}(D_t), \qquad t \ge 0.$$

If $G_p(0) < +\infty$, then

$$G_p(t) \ge e^{-t} G_p(0), \qquad t \ge 0. \tag{4.12}$$

Equivalently, $G_p(-\log r) \ge r G_p(0)$ for every $r \in (0,1]$.

Proof. If $I = \mathcal{O}_o$, then $G_p \equiv 0$ and there is nothing to prove. Assume that I is proper, and first suppose that D is bounded. For $\xi \in \ell_I$ put

$$K_\xi(t) := K_{\xi,D_t,p}(o), \qquad u_\xi(t) := \log K_\xi(t).$$

The logarithmic convexity theorem for the p-Bergman kernel with respect to ξ shows that u_ξ is convex on $[0,+\infty)$; see [14, Proposition 6.5]. The multiplier-ideal sublevel estimate gives

$$\sup_{t\ge 0} e^{-t} K_\xi(t) < +\infty, \qquad \xi \in \ell_{\mathcal{I}(\varphi)_o}. \tag{4.13}$$

For $p = 2$ this is the monotonicity statement [13, Proposition 1.3]; for $0 < p \le 2$ the same cutoff argument uses the optimal L^p extension theorem, as in [14, Lemma 6.3 and Proposition 6.4]. Thus $u_\xi(t) - t$ is a convex function bounded above on $[0,+\infty)$, and hence it is non-increasing. Consequently,

$$K_{\xi,D_t,p}(o) \le e^t K_{\xi,D,p}(o), \qquad t \ge 0, \quad \xi \in \ell_I. \tag{4.14}$$

Each connected component of D_t is pseudoconvex. Since the components not containing o do not affect either extremal problem, Theorem 4.5 applies to D_t and gives

$$\begin{aligned} G_p(t) = B_p(F,I,D_t) &= \sup_{\xi\in\ell_I} \frac{|(\xi\cdot F)(o)|^p}{K_{\xi,D_t,p}(o)} \\ &\ge e^{-t} \sup_{\xi\in\ell_I} \frac{|(\xi\cdot F)(o)|^p}{K_{\xi,D,p}(o)} = e^{-t} B_p(F,I,D) = e^{-t} G_p(0), \end{aligned}$$

which is (4.12).

If D is unbounded, take a bounded pseudoconvex exhaustion (D_j) as in the proof of Theorem 4.5, and set

$$G_{p,j}(t) := C_{F,I,p}(D_t \cap D_j).$$

The bounded case gives $G_{p,j}(t) \geq e^{-t} G_{p,j}(0)$. For each fixed $t \geq 0$, the exhaustion argument (4.10), applied to the component of D_t containing o, yields

$$G_{p,j}(t) \nearrow G_p(t).$$

Letting $j \to \infty$ proves (4.12) on D. ■

Remark 4.7. *For $p = 2$, Guan's concavity theorem says that $r \mapsto G_2(-\log r)$ is concave on $(0, 1]$ and has limit 0 at $r = 0$; hence it implies $G_2(-\log r) \geq rG_2(0)$. Corollary 4.6 extends precisely this lower bound to every $p \in [1, 2]$. On the kernel side, Błocki–Zwonek proved the corresponding Green-sublevel monotonicity for $p = 2$, and Bao–Guan–Sun extended it to $p \in [1, 2]$. The corollary allows an arbitrary negative plurisubharmonic function and arbitrary quotient data modulo $\mathcal{I}(\varphi)_o$; through Theorem 4.5, it is therefore a further minimal-integral extension of those kernel results. Both the theorem and the corollary hold without a boundedness assumption on D, as the exhaustion arguments above show.*

Remark 4.8. *For $p = 2$, Theorem 4.5 is the result of [12]; the argument above is the same chain with the L^2 orthogonal projection replaced by the Hahn–Banach duality. In the example of Theorem 4.4, the identity $B_p(z, \mathfrak{m}_0^2, \mathbb{D}) = C_{z,\mathfrak{m}_0^2,p}(\mathbb{D})$ for $p \geq 1$ is equivalent to the ξ-reduction $K_{\mathbb{D}}^{z,p}(0) = \inf_{\xi \in S_z} K_{\xi,\mathbb{D},p}(0)$ of [14, Theorem 1.7]. Thus the failure established here occurs in the subadditive range $0 < p < p_*$; the status of this example for $p_* \leq p < 1$ remains open, as stated in Remark 4.3.*

5 Failure of fiberwise log-plurisubharmonicity for classical p-Bergman kernels

In this section we prove Theorem 1.8. The construction is a Hartogs lift of a finite-dimensional weighted extremal problem on the complex plane. Throughout the section, let $p > 4$ and fix

$$\frac{5}{2} < \alpha \leq \frac{p+1}{2}. \tag{5.15}$$

Set

$$S(z, w) := 1 + |z|^4 + |1 + wz^2|^2 = |1|^2 + |z^2|^2 + |1 + wz^2|^2. \tag{5.16}$$

5.1 The Hartogs lift and the extremal function

Since the three entries in the last expression in (5.16) are holomorphic in (z, w),

$$\Phi_\alpha(z, w) := \alpha \log S(z, w)$$

is plurisubharmonic on $\mathbb{C} \times \mathbb{D}$. The complete Hartogs domain

$$\mathcal{X}_\alpha = \left\{(z, \eta, w) \in \mathbb{C}^2 \times \mathbb{D} : |\eta|^2 < e^{-\Phi_\alpha(z,w)}\right\} \tag{5.17}$$

is therefore pseudoconvex. Its fiber over w is

$$X_{\alpha,w} = \left\{(z, \eta) \in \mathbb{C}^2 : |\eta|^2 < S(z, w)^{-\alpha}\right\}.$$

Define

$$I_\alpha(w) := \int_{\mathbb{C}} S(z,w)^{-\alpha}\, d\lambda(z). \tag{5.18}$$

The integral is finite since $\alpha > 1/2$.

Lemma 5.1. *For every $w \in \mathbb{D}$, if f is entire and*

$$\int_{\mathbb{C}} |f(z)|^p S(z,w)^{-\alpha}\, d\lambda(z) < +\infty,$$

then f is affine.

Proof. Uniformly for $w \in \mathbb{D}$ one has

$$1 + |z|^4 \le S(z,w) \le 3(1 + |z|^4).$$

Write $f(z) = \sum_{k\ge 0} a_k z^k$. Since $p > 1$, the coefficient estimate on the circle of radius r gives

$$|a_k|^p r^{kp} \le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{\mathrm{i}\theta})|^p\, d\theta.$$

If $a_k \ne 0$, integration in r and the lower bound $S^{-\alpha} \ge 3^{-\alpha}(1+|z|^4)^{-\alpha}$ show that necessarily

$$kp - 4\alpha < -2, \qquad \text{or equivalently} \qquad k < \frac{4\alpha - 2}{p}.$$

By (5.15), $(4\alpha - 2)/p \le 2$. Thus $a_k = 0$ for every $k \ge 2$. ■

Proposition 5.2. *For every $w \in \mathbb{D}$,*

$$K_{X_{\alpha,w},p}(0,0) = \frac{1}{\pi I_\alpha(w)}. \tag{5.19}$$

In particular, the constant function is the unique normalized extremal function at $(0,0)$.

Proof. Let $F \in A^p(X_{\alpha,w})$ satisfy $F(0,0) = 1$, and put $f(z) := F(z,0)$. Applying the submean inequality to $|F(z,\cdot)|^p$ on the η-disk of radius $S(z,w)^{-\alpha/2}$ gives

$$\int_{X_{\alpha,w}} |F|^p\, d\lambda \ge \pi \int_{\mathbb{C}} |f(z)|^p S(z,w)^{-\alpha}\, d\lambda(z).$$

The right-hand side is finite, so Lemma 5.1 gives $f(z) = 1 + bz$ for some $b \in \mathbb{C}$.

The density $S(z,w)^{-\alpha}$ is even in z. Since $p > 1$, convexity gives

$$\frac{|1+bz|^p + |1-bz|^p}{2} \ge 1.$$

Pairing z and $-z$ in the weighted integral, we obtain

$$\int_{\mathbb{C}} |f(z)|^p S(z,w)^{-\alpha}\, d\lambda(z) \ge I_\alpha(w).$$

Consequently every admissible F has p-th power norm at least $\pi I_\alpha(w)$. Equality is attained by $F \equiv 1$, which proves (5.19); strict convexity also gives uniqueness. ■

5.2 The negative Levi form

We now compute the variation of (5.19). At $w = 0$,

$$S(z,0) = 2 + |z|^4, \qquad S_w(z,0) = z^2, \qquad S_{w\overline{w}}(z,0) = |z|^4.$$

Differentiation under the integral sign in (5.18) gives

$$(I_\alpha)_w(0) = -\alpha \int_{\mathbb{C}} z^2 (2 + |z|^4)^{-\alpha-1}\, d\lambda(z) = 0 \tag{5.20}$$

by the angular integration, and

$$\begin{aligned}(I_\alpha)_{w\overline{w}}(0) = \alpha(\alpha+1) &\int_{\mathbb{C}} |z|^4 (2 + |z|^4)^{-\alpha-2}\, d\lambda(z) \\ &- \alpha \int_{\mathbb{C}} |z|^4 (2 + |z|^4)^{-\alpha-1}\, d\lambda(z).\end{aligned} \tag{5.21}$$

For $j \geq 0$ and $\beta > (j+1)/2$, the substitution $u = r^4/2$ yields

$$\int_{\mathbb{C}} |z|^{2j} (2 + |z|^4)^{-\beta}\, d\lambda(z) = \frac{\pi}{2}\, 2^{(j+1)/2-\beta} B\left(\frac{j+1}{2}, \beta - \frac{j+1}{2}\right). \tag{5.22}$$

Applying (5.22) to the two terms in (5.21) gives

$$\frac{\int_{\mathbb{C}} |z|^4 (2 + |z|^4)^{-\alpha-2}\, d\lambda(z)}{I_\alpha(0)} = \frac{\alpha - \frac{1}{2}}{4\alpha(\alpha+1)}, \tag{5.23}$$

$$\frac{\int_{\mathbb{C}} |z|^4 (2 + |z|^4)^{-\alpha-1}\, d\lambda(z)}{I_\alpha(0)} = \frac{1}{2\alpha}. \tag{5.24}$$

It follows that

$$\frac{\partial^2}{\partial w\, \partial \overline{w}} \log I_\alpha(w)\bigg|_{w=0} = \frac{2\alpha - 5}{8}. \tag{5.25}$$

Combining (5.19), (5.20), and (5.25), we conclude that

$$\frac{\partial^2}{\partial w\, \partial \overline{w}} \log K_{X_{\alpha,w},p}(0,0)\bigg|_{w=0} = \frac{5 - 2\alpha}{8} < 0. \tag{5.26}$$

This proves the explicit part of Theorem 1.8. Notice that for $p = 6$ and $\alpha = 3$, the right-hand side of (5.26) is exactly $-1/8$.

5.3 Bounded pseudoconvex counterexamples

Although the explicit family (5.17) is unbounded in the z-direction, the failure persists on a sufficiently large bounded truncation. For $R > 0$ put

$$\mathcal{X}_\alpha^{(R)} := \mathcal{X}_\alpha \cap \{|z| < R\}, \qquad X_{\alpha,w}^{(R)} := X_{\alpha,w} \cap \{|z| < R\}.$$

Then $\mathcal{X}_\alpha^{(R)}$ is bounded and pseudoconvex.

Corollary 5.3. *For every $p > 4$, there exists $R > 0$ such that*

$$(z, \eta, w) \longmapsto \log K_{X_{\alpha,w}^{(R)},p}(z, \eta)$$

is not plurisubharmonic on $\mathcal{X}_\alpha^{(R)}$.

Proof. Write

$$u_R(w) := \log K_{X_{\alpha,w}^{(R)},p}(0,0), \qquad u_\infty(w) := \log K_{X_{\alpha,w},p}(0,0).$$

The standard exhaustion property of the p-Bergman kernel gives, for every fixed w,

$$u_R(w) \searrow u_\infty(w) \qquad (R \to +\infty).$$

Indeed, monotonicity gives the existence of the limit; normalized extremal functions on the increasing domains form a normal family, and Fatou's lemma identifies the limit with the extremal problem on $X_{\alpha,w}$.

Both u_R and u_∞ are radial functions of w. In fact, if $w' = e^{\mathrm{i}\theta} w$, the unitary map

$$(z, \eta) \longmapsto (e^{-\mathrm{i}\theta/2} z, \eta)$$

identifies the corresponding fibers and fixes $(0, 0)$. By (5.26), there is a sufficiently small $r > 0$ such that $u_\infty(r) < u_\infty(0)$. Pointwise exhaustion at these two parameter values then gives $u_R(r) < u_R(0)$ for all sufficiently large R.

If the fiberwise logarithmic kernel on $\mathcal{X}_\alpha^{(R)}$ were plurisubharmonic, its restriction to the section $(0, 0, w)$ would be subharmonic. Radiality would then imply the submean inequality $u_R(0) \le u_R(r)$, a contradiction. ∎

Thus the classical fiberwise log-plurisubharmonicity question has a negative answer for every $p > 4$, even when the total space is required to be bounded and pseudoconvex. The range $2 < p \le 4$, as well as the corresponding sublevel-set convexity question, is not settled by this construction.

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