Notes on Nagata's Conjecture

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摘要

这是一个有关于 Nagata 猜想的简短笔记, 主要是为了应付讨论班 (划掉) 介绍一下这个猜想的相关问题与进展, 尤其是最近的由 Stéphanie Nivoche [Niv21] 利用多重位势理论给出的一个 Nagata 猜想的等价命题.

目录

1 What is Nagata's Conjecture?

为了方便, 本文中我们不妨设一切事情均发生在基域 $\mathbb{C}$ 上. 并且我们用 $\|\cdot\|$ 来表示 $\mathbb{C}^n$ 中的标准范数 $\|z\|^2 = |z_1|^2 + \cdots + |z_n|^2$, 用 $|\cdot|$ 来表示范数 $|z| = \max\{|z_1|, \ldots, |z_n|\}$, 其中 $z = (z_1, \ldots, z_n)$.

1.1 Nagata's Conjecture

1.1.1 Hilbert's 14-th Problem and Nagata's Conjecture

一切问题的来源是 Hilbert 第十四问题:

Let k be a field and let $x_1, \ldots, x_n$ be algebraically independent elements over k. Let K be a subfield of $k(x_1, \ldots, x_n)$ containing k. Is $k[x_1, \ldots, x_n] \cap K$ finitely generated over k ?

Nagata 在 [Nag59] 中给出了上述 Hilbert 第十四问题的反例. 其中一个关键步骤是考虑满足下面条件的正整数 r:

Suppose $p_1, \ldots, p_r$ are r general points in $\mathbb{P}^2$. If a curve C of degree d that passes through every p_i with multiplicity at least m, then $d/m > \sqrt{r}$.

其中成一个性质 $\mathcal{P}$ 对 $\mathbb{P}^2$ 上的 r general points 成立是指如果存在 $(\mathbb{P}^2)^r$ 的一个 Zariski-open 子集 W 使 $\mathcal{P}$ 在 W 上成立.

Nagata 证明了如果 $r = s^2$ 为完全平方数, 其中正整数 $s \geq 4$, 则 r 满足上面的条件. 并且他在 [Nag59] 文中最后提出了著名的 Nagata's Conjecture:

Conjecture 1.1 (Nagata's Conjecture). *... It will be enough for r to be greater than 9.*

1.1.2 Necessity of $r > 9$

条件 $r > 9$ 是必要的. 对 $\mathbb{P}^2$ 上的 r general points $p_1, \ldots, p_r$, 及正整数 m, 记 $\delta(r, m)$ 为最小的正整数 d 是的存在次数为 d 的曲线在每个 p_i 点处的 vanishing order $\geq m$. 对 $r \leq 9$, Harbourne [Harb01] 得到了 $\delta(r, m) = \lceil c_r m \rceil$, 其中

r	1	2	3	4	5	6	7	8	9
c_r	1	1	3/2	2	2	12/5	21/8	48/17	3

可以注意一下 $r = 1, 4, 9$ 时候的情况.

1.1.3 Different Multiplicities

Nagata's Conjecture 实际上也等价于如下重数不相等的情形 ([Nag59]):

Conjecture 1.2. *Suppose $p_1, \dots, p_r$ are r general points in $\mathbb{P}^2$ and that $m_1, \dots, m_r$ are given positive integers. Then for $r > 9$ any curve C that passes through every p_i with multiplicity m_i must satisfy*

$$\deg C > \frac{1}{\sqrt{r}} \sum_{i=1}^{r} m_i.$$

1.1.4 Progress in Nagata's Conjecture

- Nagata [Nag59]: $r > 9$ 且为完全平方数时, 猜想成立;
- Geng Xu [Xu94]:
$$\deg C \geq \frac{\sqrt{r-1}}{r} \cdot \sum_{i=1}^{r} m_i;$$
- H. Tutaj-Gasinska [TGa03]:
$$\deg C \geq \frac{1}{\sqrt{r + \frac{1}{12}}} \cdot \sum_{i=1}^{r} m_i.$$

1.2 Variations of Nagata's Conjecture

1.2.1 Positivity of Divisors

Nagata's Conjecture 也常被描述为如下关于在 $\mathbb{P}^2$ 上对 r 个点做 blow-up 的猜想:

Let $\mu \colon P = \mathrm{Bl}_r(\mathbb{P}^2) \to \mathbb{P}^2$ be the blow-up of the projective plane along r very general points. Denote by H the pull-back of a line, and let $E \subseteq P$ be the exceptional divisor of μ (so that $E = \sum_{i=1}^{r} E_i$).

Conjecture 1.3. *The $\mathbb{R}-$divisor*

$$H - \frac{1}{\sqrt{r}} \cdot E$$

is nef on P provided that $r \geq 9$, and ample provided that $r > 9$.

1.2.2 Seshadri Constants

利用 Demailly 引入的 Seshadri constants, Nagata 猜想也可以推广到其他曲面上.

Let X be a smooth algebraic surface and L be an ample line bundle on X of degree d. Let $\mu \colon P = X' \to X$ be the blow-up of the X along r general points, with exceptional divisor

$E \subseteq X'$. The Seshadri constant

$$\epsilon(p_1, \ldots, p_r; X, L) \coloneqq \sup\left\{\epsilon \geq 0 \mid \mu^* L - \epsilon \cdot E \text{ is nef }\right\}.$$

CONJECTURE 1.4. *For sufficiently large r the Seshadri constant satifies*

$$\epsilon(p_1, \ldots, p_r; X, L) = \frac{d}{\sqrt{r}}.$$

如果 $\mu^* L - \epsilon \cdot E$ 是 nef 的, 首先我们需要有 $\left((\mu^* L - \epsilon \cdot E)^2\right) \geq 0$, 即

$$(L^2) - \epsilon^2 (E^2) \geq 0 \;\Rightarrow\; d^2 - \epsilon^2 \cdot r \geq 0 \;\Rightarrow\; \epsilon \geq \frac{d}{\sqrt{r}}.$$

1.2.3 Higher Dimensional Cases

Iarrobino 将 Nagata 猜想推广到了任意 $n \geq 2$ 维.

CONJECTURE 1.5 (Iarrobino, [Iar97]). *Any hypersurface in $\mathbb{P}^n$ passing through r generic points with multiplicity m has a degree $> r^{1/n} \cdot m$, except for an explicit finite list of (r, n).*

Evain [Eva02] 证明了当 r 为 s^n 形式的正整数时此猜想成立 (除了 $(4,2), (9,2), (8,3)$.)

1.2.4 Symplectic Packings

Nagata's Conjecture 在 symplectic packing 问题中有重要作用.

令 B^4 为 4 维球, 考虑所有可能的互不相交的 r 个半径相等的 4 维球在 B^4 中的辛嵌入. 记 $\hat{v}(B^4, r)$ 为这些嵌入所能填充的体积的上确界, 并记

$$v(B^4, r) = \frac{\hat{v}(B^4, r)}{\text{Volume}(B^4)}.$$

若 $v(B^4, r) = 1$, 则称对应填充为 "full symplectic packing".

McDuff-Polterovich [MP94] 证明了: Nagata's Conjecture would imply that the four-dimensional ball B^4 admits full symplectic packings of $r \geq 9$ balls.

特别地, $v(B^4, r^2) = 1$. 以及例如 G. Xu [Xu94] 的关于 Nagata 猜想的结果可对应推出 $v(B^4, r) \geq 1 - \frac{1}{r}$.

高维的 Nagata 猜想 (即 Iarrobino 猜想) 也可以对应给出高维的辛球填充相关的结果.

Nagata's Conjecture 还与在数论中有大量应用的 singular degree 相关问题紧密相关. 这个我们将在后面讨论.

1.3 Proof of Xu's Theorem

THEOREM 1.6 ([Xu94]). *Let $p_1, \ldots, p_k$ be k generic points in $\mathbb{P}^2$, and C be a reduced and irreducible curve of degree d with $\mathrm{mult}_{p_i}(C) = m_i$. Then*

(1)

$$d \geq \frac{\sqrt{k-1}}{k} \sum_{i=1}^{k} m_i.$$

(2)

$$d + \frac{1}{2\sqrt{k-1}} > \frac{1}{\sqrt{k}} \sum_{i=1}^{k} m_i \quad \text{if } k \geq 2.$$

我们首先证明如下引理.

LEMMA 1.7 ([Xu94]). *Let $p_1, \ldots, p_k$ be k generic points in $\mathbb{P}^2$, and C be a reduced and irreducible curve of degree d with $\mathrm{mult}_{p_i}(C) = m_i$. Then*

$$d^2 \geq \sum_{i=1}^{k} m_i^2 - m_q$$

for any $q \in \{1, \ldots, k\}$ such that $m_q > 0$.

证明. 假设引理结论错误, 那么对于任意满足 $p_i(0) = p_i$ $(i \in \{1, \ldots, k\})$ 的 deformation $p_i(t)$, 我们可以得到曲线 C 的 deformation $C(t)$ 满足 $C(0) = C$, 且

$$\deg(C(t)) = d, \quad \mathrm{mult}_{p_i(t)}(C(t)) \geq m_i, \quad \text{and} \quad d^2 < \sum m_i^2 - m_q.$$

令 $[X, Y, Z]$ 为一个使得 $p_i(t) = [c_i(t), d_i(t), 1] = (c_i(t), d_i(t))$ 的 generic 齐次坐标, 且设 $F_t(X, Y, Z) = 0$ 为 $C(t)$ 在 $\mathbb{P}^2$ 中的定义方程. 由假设, 我们有, 在点 $p_i(0) = p_i$ 附近,

$$\begin{aligned} F_t(x, y, 1) &= F_t\left(\frac{X}{Z}, \frac{Y}{Z}, 1\right) \\ &= \sum_{r+s \geq m_i} a^i_{rs}(t)\big(x - c_i(t)\big)^r \big(y - d_i(t)\big)^s. \end{aligned}$$

从而

$$\frac{\mathrm{d}F_t}{\mathrm{d}t}\bigg|_{t=0}(x,y,1) = -\frac{\mathrm{d}c_i(t)}{\mathrm{d}t}\bigg|_{t=0}\cdot\frac{\partial F_0}{\partial X}\bigg|_{p_i} - \frac{\mathrm{d}d_i(t)}{\mathrm{d}t}\bigg|_{t=0}\cdot\frac{\partial F_0}{\partial Y}\bigg|_{p_i} + \sum_{r+s\geq m_i}\frac{\mathrm{d}a^i_{rs}(t)}{\mathrm{d}t}\bigg|_{t=0}\left(x-c_i(0)\right)^r\left(y-d_i(0)\right)^s.$$

如果我们取一个 p_i 的 deformation, 满足:

$$\frac{\mathrm{d}c_i(t)}{\mathrm{d}t}\bigg|_{t=0} = \frac{\mathrm{d}d_i(t)}{\mathrm{d}t}\bigg|_{t=0} = 0, \quad \forall\, i \neq q,$$

以及

$$\frac{\mathrm{d}c_i(t)}{\mathrm{d}t}\bigg|_{t=0} = c_q \neq 0, \quad \frac{\mathrm{d}d_i(t)}{\mathrm{d}t}\bigg|_{t=0} = d_q \neq 0,$$

并令 D 为由方程 $\frac{\mathrm{d}F_t}{\mathrm{d}t}\Big|_{t=0} = 0$ 所定义的曲线, 那么通过选取恰当的 c_q 与 d_q, 我们可以发现:

$$\deg(D) = d, \quad \mathrm{mult}_{p_i}(D) \geq m_i,\ \forall\, i\neq q, \quad \text{and} \quad \mathrm{mult}_{p_q}(D) = m_q - 1.$$

由于 C 是 reduced 且 irreducible, 由 Bezout 定理, 我们有

$$d^2 = C\cdot D \geq \sum_{i\neq q} m_i^2 + m_q(m_q-1) = \sum_{i=1}^k m_i^2 - m_q,$$

矛盾. ■

下面给出 Theorem 1.6 的证明.

Proof of Theorem 1.6. 不妨设 $0 < m_1 \leq m_2 \leq \cdots \leq m_l$, 且对 $i > l$, $m_i = 0$, 那么 $l \leq k$. 由于 $l = 1$ 的情形是 trivial 的, 以下假设 $l \geq 2$.

(1) 由 Lemma 1.7, 对所有 $q \leq l$ 有 $d^2 \geq \sum m_i^2 - m_q$, 因此

$$ld^2 \geq l\sum m_i^2 - \sum m_i \geq \left(\sum m_i\right)^2 - \sum m_i \geq \left(1-\frac{1}{l}\right)\left(\sum m_i\right)^2.$$

即

$$d \geq \frac{\sqrt{l-1}}{l}\sum m_i \geq \frac{\sqrt{k-1}}{k}\sum m_i.$$

(2) 若 $2 \le l \le k-2$, 则 $(k-l)l \ge k$. 我们有

$$(k-l)\left(\sum m_i^2\right) \ge (k-l)lm_1^2 \ge km_1,$$

即

$$k\left(\sum m_i^2 - m_1\right) \ge l\sum m_i^2 \ge \left(\sum m_i\right)^2.$$

由 Lemma 1.7, 有 $d^2 \ge \sum m_i^2 - m_1$, 从而 $d \ge \left(\sum m_i\right)/\sqrt{k}$, 在此种情况下.

若 $l = k-1$, 我们仍然有不等式 $(k-l)\sum m_i^2 \ge km_1$ 成立, 除非 $m_1 = \cdots = m_l = 1$. 当 $m_1 = \cdots = m_l = 1$ 时, 由 Lemma 1.7, $d^2 \ge k-2$, 故: 若 $k \ge 3$,

$$d + \frac{1}{2(k-1)} \ge \sqrt{k-2} + \frac{1}{2\sqrt{k-1}} > \frac{k-1}{\sqrt{k}} = \frac{\sum m_i}{\sqrt{k}};$$

若 $k = 2$, 则是 trivial 的.

若 $l = k$, 由 Lemma 1.7, $d^2 \ge \sum m_i^2 - m_1$, 我们有 $d \ge \sqrt{k-1}m_1$. 从而

$$\frac{\left(\sum m_i\right)^2}{k} \le \sum_{i=1}^{k} m_i^2 \le d^2 + m_1 < \left(d + \frac{m_1}{2d}\right)^2 \le \left(d + \frac{1}{2\sqrt{k-1}}\right)^2,$$

即

$$d + \frac{1}{2\sqrt{k-1}} > \frac{\sum m_i}{\sqrt{k}}.$$

■

2 Singular Degree

2.1 Waldschmidt's Singular Degree

令 S 为 $\mathbb{C}^n$ 中一有限点集. Waldschmidt 引入了所谓的 singular degree. 对任意整数 $l > 0$, 记 $\Omega(S,l)$ 为在 S 中每点处的 vanishing order $\geq l$ 的多项式 $P \in \mathbb{C}[z_1,\dots,z_n]$ 的最小次数:

$$\Omega(S,l) \coloneqq \min\left\{ \deg P \mid P \in \mathbb{C}[z],\ \mathrm{ord}(P,p) \geq l,\ \forall\, p \in S \right\}.$$

例如 $n = 1$ 时, 显然有 $\Omega(S) = \Omega(S,l) = \mathrm{Card}\, S =: |S|$.

易知

$$\Omega(S, l_1 + l_2) \leq \Omega(S, l_1) + \Omega(S, l_2).$$

由此可以定义

$$\Omega(S)\colon \inf_{l>0} \frac{\Omega(S,l)}{l} = \lim_{l\to+\infty} \frac{\Omega(S,l)}{l}.$$

称 $\Omega(S,1)$ 为 S 的 degree (包含 S 的代数超曲面的最小次数), $\Omega(S)$ 为 S 的 singular degree. 利用 Hörmander-Bombieri-Skoda 定理 ($\mathbb{C}^n$ 中从一个点的 L^2 延拓定理), Waldschmidt [Wald87(79)] 证明了:

THEOREM 2.1 ([Wald87(79)]). *For positive integers l_1 and l_2,*

$$\frac{\Omega(S,l_1)}{l_1 + n - 1} \leq \Omega(S) \leq \frac{\Omega(S,l_2)}{l_2}.$$

我们将在后面回顾一下这个定理的证明.

Esnault-Viehweg [EV83] 利用代数几何工具证明, Azhari [Azh90] 利用 Hörmander L^2 估计重新证明:

$$\frac{\Omega(S,l)+1}{l+n-1} \leq \Omega(S).$$

Demailly [Dem82] 猜想:

CONJECTURE 2.2 (Chudnovsky-Demailly Conjecture). *对任意 $l \geq 1$:*

$$\frac{\Omega(S,l)+n-1}{l+n-1} \leq \Omega(S).$$

其中 $l = 1$ 情形是 Chudnovsky [Chud79] 的一个猜想, 且 $n = 2$ 情形是已知的 (即上面不等式, 当然最初是 Chudnovsky [Chud79] 利用相交理论证明, 并且 Demailly [Dem82] 利用分析方法独立证明).

这些结果及猜想与超越数论有紧密联系, 例如可以参见 Demailly 大书 [CADG].

Waldschmidt [Wald87(79)] 还证明了一个 $\Omega(S,l)$ 的上界估计:

THEOREM 2.3 ([Wald87(79)]). *We have*

$$\Omega(S,l) \le (l+n-1)|S|^{1/n} - (n-1).$$

证明. 这个结论其实是比较 trivial 的一个估计. 注意到, 以多项式 P 的系数为未知数的齐次线性方程组

$$\mathrm{D}^\alpha P(s) = 0,\ (s \in S,\ \alpha \in \mathbb{Z}^n_{\ge 0},\ |\alpha| \le l-1)$$

的方程个数至多为 $\binom{l+n-1}{n} \cdot |S|$. 因此若 P 的次数 d 满足:

$$\binom{d+n}{n} > \binom{l+n-1}{n} \cdot |S|,$$

则该方程组必有非平凡解. 注意到

$$\binom{d+n}{n} \Big/ \binom{l+n-1}{n} \ge \left(\frac{d+n}{l+n-1}\right)^n,$$

因此如果 $d > (l+n-1)|S|^{1/n} - n$, 换言之, $d = \lfloor (l+n-1)|S|^{1/n} - (n-1) \rfloor$, 我们就一定可以找到 d 次多项式 $P \in \mathbb{C}[z]$ 满足 $\mathrm{ord}(P,s) \ge l$, $\forall s \in S$. 即

$$\Omega(S) \le d \le (l+n-1)|S|^{1/n} - (n-1).$$

■

由此可得 $\Omega(S) \le |S|^{1/n}$.

利用不变量 $\Omega(S,l)$, 可以将 Nagata's Conjecture 表述为:

<u>In $\mathbb{C}^2$, if $r > 9$, then $\Omega(S,l) > l\sqrt{r}$, $\forall l \ge 1$ holds for a set S of r points in general position.</u>

如果对某个 $|S| = r$, 我们有 $\Omega(S) = |S|^{1/n}$, 那么根据 $\Omega(S)$ 的定义, 有 $\Omega(S,l) \ge l \cdot |S|^{1/n}$, 并且若 $|S|$ 不是 n 次幂, 就再有 $\Omega(S,l) > l \cdot |S|^{1/n}$.

2.2 Consequence of Hörmander-Bombieri-Skoda Theorem

Hörmander-Bombieri-Skoda Theorem was used to give estimates of $\Omega(S,l)$ ([Wald87(79)]).

LEMMA 2.4 (Hörmander-Bombieri-Skoda). *Let V be a plurisubharmonic function on a pseudoconvex domain $D \subset \mathbb{C}^n$, not identifying to 0. Let $\varepsilon > 0$. Then there exists $F \in \mathcal{O}(D)$ not* *identifying to* 0*, such that*

$$\int_D |F(z)|^2 e^{-V(z)} \left(1+\|z\|^2\right)^{-n-\varepsilon} \mathrm{d}\lambda(z) < +\infty,$$

where $\|z\|^2 := |z_1|^2 + \cdots + |z_n|^2$ *for* $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Proof of Theorem 2.1. By the definition, let P be a polynomial with degree $\Omega(S, l_2)$ such that $\mathrm{ord}_s(P) \geq l_2$ for any $s \in S$. Set $\mu > \dfrac{2l_1 + 2n - 2}{l_2}$. The function $V = \mu \log |P|$ is plurisubharmonic on $\mathbb{C}^n$. Then by Lemma 2.4, there exists an entire function F, not identifying 0, satisfying

$$\int_{\mathbb{C}^n} |F(z)|^2 |P(z)|^{-\mu} (1+\|z\|^2)^{-n-\varepsilon} \mathrm{d}\lambda(z) < +\infty.$$

Let $\zeta \in \mathbb{C}^n$, and $r > 0$. Since $|F|^2$ is subharmonic, we have

$$|F(\zeta)|^2 \leq \frac{1}{\lambda\big(\mathbb{B}(o,1)\big) r^{2n}} \int_{\mathbb{B}(\zeta,r)} |F(z)|^2 \mathrm{d}\lambda(z).$$

Thus, there exists some constant $C_1 > 0$, independent of ζ and r, such that

$$|F(\zeta)|^2 \leq C_1 \frac{1}{r^{2n}} \sup_{z \in \mathbb{B}(\zeta,r)} \Big(|P(z)|^{\mu} (1+\|z\|^2)^{n+\varepsilon} \Big).$$

Choose $r = \|\zeta - \sigma\|$, where $\sigma \in S$. Then for ζ near σ, it holds that

$$|F(\zeta)|^2 \leq C_2 \|\zeta - \sigma\|^{\mu l_2 - 2n},$$

which implies $\mathrm{ord}_\sigma(F) \geq \dfrac{\mu}{2} l_2 - n > l_1 - 1$ for any $\sigma \in S$.

Choose $|\zeta| = R$ and $r = R/2$. Then we also get

$$\|F\|_R^2 \leq C_3 R^{\mu \cdot \Omega(S,l_2) + 2\varepsilon},$$

where C_3 does not depend on R. It follows that F is a polynomial of degree $\leq \dfrac{\mu}{2} \cdot \Omega(S, l_2) + \varepsilon$.

Let ε and $\mu - \dfrac{2(l_1 + n - 1)}{l_2}$ be sufficiently small, thus we get

$$\Omega(S, l_1) \leq \frac{l_1 + n - 1}{l_2} \cdot \Omega(S, l_2).$$

■

3 Convergence of Multipoled Pluricomplex Green Functions

3.1 Multipoled Pluricomplex Green Functions

令 S 为 $\mathbb{C}^n$ 中的一个有限点集, $|S|$ 为其点的个数. 令 R 为一个充分大的正实数并使得 $S \subset B(O,R)$. 令 $g_R(S,\cdot)$ 为 $B(O,R)$ 上的在 S 中的每一点处有一个对数极点的多复 Green 函数:

$$g_R(S,z) = \sup\Big\{u(z)\colon u \in \mathrm{PSH}^-(B(O,R)),\ u(z) \le \log\|z-p\| + O(1),\ \forall p \in S\Big\}.$$

$g_R(S,\cdot)$ 也是下面 Dirichlet 问题的解:

$$\begin{cases} u \in \mathrm{PSH}^-(B(O,R)),\ u \in C(\overline{D} \setminus S), \\ (\mathrm{dd}^c u)^n = 0 \text{ on } D \setminus S, \\ u(z) = \log\|z-p\| + O(1) \text{ as } z \to p,\ \forall\, p \in S, \\ u(z) \to 0 \text{ as } z \in \partial D. \end{cases}$$

对任意 $z \in B(O,R)$, 有 $g_R(S,z) = g_1(S/R, z/R)$. 从而可以定义一个 $B(O,1)$ 上的负值多次调和函数 g_∞ 如下:

$$g_\infty(z) = \Big(\limsup_{\mathbb{C}^* \ni t \to 0} g_1(tS,z)\Big)^*,$$

其中 $*$ 代表上半连续正则化. 显然,

$$|S| \cdot \log\|z\| \le g_\infty(z) \le g_{B(O,1)}(O,z) = \log\|z\|.$$

再令

$$\widetilde{g_1}(tS,z) \coloneqq \sup\{g_1(tS,w)\colon \|w\| = \|z\|\},$$

并定义

$$\widetilde{g_\infty}(z) \coloneqq \sup\{g_\infty(w)\colon \|w\| = \|z\|\}.$$

注意到 $\widetilde{g_\infty}(z)$ 关于 $\log\|z\|$ 是凸且递增的, 因此这里的定义里不需要上半连续正则化了.

Rashkovskii-Thomas [RT14, Theorem 1.1] 告诉我们:

The family $\big(g_1(tS,\cdot)\big)_{t\in\mathbb{C}^*}$ converges locally uniformly outside the origin in $B(O,1)$ to g_∞.

3.2 Convergence of Pluricomplex Green Functions with Colliding Poles

而 S. Nivoche 在 [Niv21] 给出了 g_∞ 与 S 的 singular degree 的如下关系:

THEOREM 3.1 ([Niv21]). *Let S be a finite set of points in $\mathbb{C}^n$. The two psh functions g_∞ and $\widetilde{g_\infty}$ satisfy several properties:*

(i) The Lelong number $\nu(g_\infty, O) = \Omega(S)$, and $(\mathrm{dd}^c g_\infty)^n = 0$ in $B(O,1) \setminus \{O\}$.

$$\Omega(S)^n \leq (\mathrm{dd}^c g_\infty)^n(O) \leq \int_{B(O,1)} (\mathrm{dd}^c g_\infty)^n \leq |S|,$$

and we have

$$g_\infty(z) \leq \Omega(S) \log \|z\|, \ \text{in } B(O,1).$$

(ii) The family $\left(\widetilde{g_1}(tS,\cdot)\right)_{t\in\mathbb{C}^*}$ converges uniformly outside the origin in $B(O,1)$ to $\widetilde{g_\infty}$ which is equal to $\Omega(S)\log\|z\|$ in $\overline{B}(O,1)$.

(iii) If $\Omega(S) = |S|^{1/n}$ then $(\mathrm{dd}^c g_\infty)^n(O) = \Omega(S)^n = |S|$ and

$$g_\infty(z) = |S|^{1/n} \log \|z\|, \ \text{in } B(O,1).$$

(iv) Conversely if g_∞ is equal to $\Omega(S)\log\|\cdot\|$ in $\overline{B}(O,1)$, then $\Omega(S) = |S|^{1/n}$.

作为应用, S. Nivoche 给出了一个 $\mathbb{C}^n$ 中的多重位势论的猜想与一个弱版本的 $\mathbb{P}^n$ 中 Nagata 猜想的等价性:

CONJECTURE 3.2 (Conjecture $\mathcal{P}_1$). *In $\mathbb{C}^n$, except for a finite number of integer values r, for any general set $S = \{p_1, \ldots, p_r\}$ of r points, the family of pluricomplex Green fucntions $\left(g_{B(O,1)}(tS,\cdot)\right)_{t\in\mathbb{C}^*}$ converges locally uniformly outside the origin of $B(O,1)$ to $r^{1/n} g_{B(O,1)}(O,\cdot)$, when t tends to 0.*

CONJECTURE 3.3 (Conjecture $\mathcal{A}_1$). *In $\mathbb{P}^n$, except for a finite number of integer values r, for any general set $S = \{p_1, \ldots, p_r\}$ of r points, $\Omega(S) = r^{1/n}$.*

特别地, $n = 2$ 时, 上述猜想与原始的 Nagata 猜想是等价的 (r 不是完全平方数时. 而完全平方数情形是已知的). 以及在 $\mathbb{P}^n$ 中, 上述猜想成立可以推出 Iarrobino's conjecture 成立.

将 $B(O,1)$ 换为一般的有界超凸 (hyperconvex) 域 D, 并将 O 换为 D 中任意一点 z_0, 也可以得到类似结果.

4 Entire psh Functions in $\mathbb{C}^n$ and Affine Invariants

令 u 为 $\mathbb{C}^n$ 上的多次调和函数, 并记

$$\gamma_u := \limsup_{\|z\|\to+\infty} \frac{u(z)}{\log\|z\|} \in [0,+\infty].$$

若 $S=\{p_1,\ldots,p_r\}\subset\mathbb{C}^n$ 为一有限点集, 对任意 $\mathbb{C}^n$ 上的多次调和函数 u, 记

$$\omega(S,u) := \frac{\sum_{j=1}^k \nu(u,p_j)}{\gamma_u}.$$

对点集 S, 定义与其相关的仿射不变量

$$\omega_{\mathrm{psh}}(S) := \sup\{\omega(S,u)\colon u\in \mathrm{PSH}(\mathbb{C}^n)\},$$

以及

$$\omega_{\mathrm{psh}}^{+}(S) := \sup\{\omega(S,u)\colon u\in \mathrm{PSH}(\mathbb{C}^n)\cap L^\infty_{\mathrm{loc}}(\mathbb{C}^n\setminus S)\}.$$

Conjecture 4.1 (Conjecture $\mathcal{P}_2$). *In $\mathbb{C}^n$, except for a finite number of integer values r, for any general set $S=\{p_1,\ldots,p_r\}$ of r points,*

$$\omega_{\mathrm{psh}}(S) = \omega_{\mathrm{psh}}^{+}(S).$$

此问题与 Chudnovsky [Chud79] 定义的 "the very singular degree" of S, 即 $|S|/\omega(S)$ 相关, 其中:

$$\omega(S) := \sup\left\{\frac{\sum_{j=1}^r \mathrm{ord}(P,p_j)}{\deg P}\colon P\in\mathbb{C}[z]\right\}.$$

注意到 $\omega(S)\geq \dfrac{|S|l}{\Omega(S,l)}$, 我们有

$$\omega(S)\geq\frac{|S|}{\Omega(S)}.$$

Conjecture 4.2 (Conjecture $\mathcal{A}_2$). *In $\mathbb{P}^n$, except for a finite number of integer values r, for any general set $S=\{p_1,\ldots,p_r\}$ of r points, $\omega(S)=|S|^{1-\frac{1}{n}}$.*

显然 Conjecture $\mathcal{A}_1$ 与 $\mathcal{A}_2$ 等价.

Conjecture 4.3 (Conjecture $\mathcal{P}_3$). *In $\mathbb{C}^n$, except for a finite number of integer values r, for* *any general set S of r points, we have:*

for any $\epsilon > 0$, there exists an entire continuous psh function $v \in L^\infty_{\mathrm{loc}}(\mathbb{C}^n \setminus S)$, such that $\nu(u,p) \geq 1$ for any $p \in S$ and $\gamma_v \leq (1+\epsilon)|S|^{1/n}$.

S. Nivoche 证明了上述所有猜想的等价性.

THEOREM 4.4 ([Niv21]). *Each conjecture $\mathcal{P}_1$, $\mathcal{P}_2$, $\mathcal{P}_3$ is equivalent to $\mathcal{A}_1$ and $\mathcal{A}_2$.*

5 Proof of Theorem 3.1

首先可以看下面这个简单的例子:

EXAMPLE 5.1. 令 $S = \{(1/2, 0), (-1/2, 0)\}$ 为 $\mathbb{C}^2$ 中两点组成的点集. 根据前述结果 $\Omega(S) = 1$. 考虑单位多圆盘 Δ^2, 对任意充分小 $t \in \mathbb{C}^*$, 可知

$$g_1(tS, z) = \max\left\{\log\left|\frac{(z_1 - t/2)(z_1 + t/2)}{(1 - \bar{t}z_1/2)(1 + \bar{t}z_1/2)}\right|, \log|z_2|\right\}$$

为 Δ^2 上的在 tS 中的点处分别具有对数极性的多复 Green 函数. 计算可得

$$g_\infty(z) = \max\left\{2\log|z_1|, \log|z_2|\right\},$$

并且由此可得 $\widetilde{g_\infty}(z) = \max\{\log|z_1|, \log|z_2|\} = \log|z|$.

下面简要叙述一下定理 3.1 的证明.

5.1 Schwarz Lemmas

LEMMA 5.2. *Let $Q \in \mathbb{C}[z_1, \ldots, z_n]$ be a holomorphic polynomial, and $0 < \varrho \leq R$ be positive real numbers. Then*

$$\log\|Q\|_\varrho - \log\|Q\|_R \geq \deg(Q) \cdot \log\left(\frac{\varrho}{R}\right),$$

where $\|f\|_r := \sup_{z \in B(O,r)} |f(z)|$.

证明. Let $z_0 \in \mathbb{C}^n$ with $\|z_0\| = R$ and $|Q(z_0)| = \|Q\|_R$. For $w \in \mathbb{C}$, set

$$\widetilde{Q}(w) := w^{\deg Q} Q\left(\frac{z_0}{w}\right) \in \mathbb{C}[w].$$

Then the maximum principle gives

$$\|\widetilde{Q}\|_1 \leq \|\widetilde{Q}\|_{R/\varrho},$$

which implies

$$\|Q\|_R = |Q(z_0)| \leq \|\widetilde{Q}\|_1 \leq \|\widetilde{Q}\|_{R/\varrho} \leq \left(\frac{R}{\varrho}\right)^{\deg Q} \sup_{|w|=R/\varrho}\left|Q\left(\frac{z_0}{w}\right)\right| \leq \left(\frac{R}{\varrho}\right)^{\deg Q} \|Q\|_\varrho.$$

∎

Lemma 5.3. *Let S be a finite set of distinct points in $\mathbb{C}^n$. Let $\varepsilon > 0$. There exists $r_0(S,\varepsilon) > 0$, such that for any $l \in \mathbb{N}_+$, any $R > \varrho > r_0$ with $2e^n \cdot \varrho < R$, and any polynomial Q with $\operatorname{ord}(Q,p) \geq l$ for all $p \in S$, we have*

$$\begin{aligned}\log\|Q\|_\varrho - \log\|Q\|_R &\leq \big(\Omega(S,l) - l\varepsilon\big)\log\left(\frac{2e^n\varrho}{R}\right)\\ &\leq -l\big(\Omega(S)-\varepsilon\big)\log\left(\frac{R}{2e^n\varrho}\right).\end{aligned} \tag{5.1}$$

证明. The proof is quite complicated, so we omit it here, where the second inequality is directly from the definition of $\Omega(S)$. For the details, see [Moreau80, Wald87(79)]. ■

我们可以不妨加上 $r_0(S,\varepsilon) \geq \|S\| \coloneqq \sup_{p\in S}\|p\|$ 这样的假设.

5.2 Approximations of Pluricomplex Green Functions

Lemma 5.4. *We have*

$$g_R(S,\cdot) = \sup_{l\geq 1} H_{S,R,l} = \lim_{l\to\infty} H_{S,R,l},$$

where

$$\begin{aligned}H_{S,R,l} &= \sup\left\{\frac{1}{l}\log|f| \colon f\in\mathcal{O}(B(O,R)),\ \|f\|_R\leq 1,\ \operatorname{ord}(f,s)\geq l \text{ for any } s\in S\right\}\\ &= \sup\left\{\frac{1}{l}\log|Q| \colon Q\in\mathbb{C}[z],\ \|Q\|_R\leq 1,\ \operatorname{ord}(Q,s)\geq l \text{ for any } s\in S\right\}.\end{aligned}$$

证明. The last equality can be induced by the fact that $B(O,R)$ is a Runge domain. The approximation result can be proved by Demailly's approximation theorem:

Approximate $g_R(S,\cdot)$ by $g_r(S,\cdot)$ for $r\to R-$ and $r\to R+$ (inside and outside) respectively, and use Demailly's approximation theorem to each $g_r(S,\cdot)$ with $r > R$.

For S being a single point, see [Niv98]. Similar results also hold for global Zhou weights, and even tame maximal weights (see [BGMY23], 私货私货). ■

5.3 Calculating the Lelong Number

For every l, there exists a polynomial Q s.t. $\operatorname{ord}(Q,p)\geq l$ for every $p\in S$, $\deg(Q) = \Omega(S,l)$ and $\|Q\|_R = 1$. According to Lemma 5.2 and Lemma 5.3, we obtain

$$\frac{\Omega(S,l)}{l}\log\left(\frac{\varrho}{R}\right) \leq \sup_{z\in B(O,\varrho)} H_{S,R,l}(z) \leq \left(\frac{\Omega(S,l)}{l} - \varepsilon\right)\log\left(\frac{2\varrho e^n}{R}\right).$$

Since $H_{S,1/|t|,l}(z) = H_{tS,1,l}(tz)$ when $|z| \leq 1/|t|$ and $t \in \mathbb{C}^*$, we have

$$\frac{\Omega(S,l)}{l}\log(|t|\varrho) \leq \sup_{z\in B(O,|t|\varrho)} H_{tS,1,l}(z) \leq \left(\frac{\Omega(S,l)}{l} - \varepsilon\right)\log(2\varrho|t|e^n),$$

or equivalently: for any $t \in \mathbb{C}^*$ and $\varrho' > 0$ such that $|t|r_0(S,\varepsilon) \leq \varrho' \leq 1$,

$$\frac{\Omega(S,l)}{l}\log\varrho' \leq \sup_{z\in B(O,\varrho')} H_{tS,1,l}(z) \leq \left(\frac{\Omega(S,l)}{l} - \varepsilon\right)\log(2\varrho' e^n).$$

Let t be sufficiently small s.t. $tS \subset B(O,\varrho')$. Since $(H_{tS,1,l})_l$ converges uniformly to $g_1(tS,\cdot)$ (Lemma 5.4 and more) as $l \to \infty$, we deduce

$$\Omega(S)\log\varrho' \leq \sup_{z\in B(O,\varrho')} g_1(tS,z) \leq (\Omega(S) - \varepsilon)\log(2\varrho' e^n),$$

where $0 < \varrho' \leq 1$, $t \in \mathbb{C}^*$ with $|t|r_0(S,\varepsilon) \leq \varrho'$ and $tS \subset B(O,\varrho')$.

Thus, for every fixed $\rho \in (0,1)$ and $\varepsilon_1 > 0$ with $\varepsilon_1 r_0(S,\varepsilon) \leq \rho$,

$$\Omega(S)\log\rho \leq \sup_{z\in B(O,\rho)}\left(\sup_{|t|\leq\varepsilon_1} g_1(tS,z)\right)^* \leq (\Omega(S)-\varepsilon)\log(2\rho e^n).$$

Since

$$g_\infty = \lim_{\varepsilon_1\to 0}\left(\sup_{|t_1|\leq\varepsilon_1} g_1(tS,\cdot)\right)^*,$$

where the limit decreases when ε_1 decreases, and according to the fact that $(g_1(tS,\cdot))_{t\in\mathbb{C}^*}$ converges locally uniformly outside the origin in $B(O,1)$ to g_∞ ([RT14, Theorem 1.1]), we obtain

$$\Omega(S)\log\rho \leq \sup_{z\in B(O,\rho)} g_\infty(z) \leq (\Omega(S)-\varepsilon)\log(2\rho e^n).$$

Finally, we get

$$\nu(g_\infty, O) = \Omega(S).$$

5.4 The Other Results

Left to the readers. Or see [Niv21].

For example, using the comparison principle for the complex Monge-Ampère operator:

$$\int_{\{u<v\}} (\mathrm{dd}^c v)^n \leq \int_{\{u<v\}} (\mathrm{dd}^c u)^n,$$

and the fact

$$\big(\mathrm{dd}^c g_1(tS,\cdot)\big)^n = \sum_{p\in S} \delta_{tp},$$

one can deduce that

$$\int_{B(O,1)} (\mathrm{dd}^c g_\infty)^n \leq |S|.$$

6 Equivalence of the Conjectures

The equivalence of Conjecture $\mathcal{P}_1$ and $\mathcal{A}_1$ is a consequence of Theorem 3.1.
For the rest, see [Niv21].

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