An introduction to some concepts in several complex variables

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Abstract

This is a manuscript for giving an introduction to some concepts in several complex variables.

Contents

1 Preliminaries: Basic Knowledge in Several Complex Variables

Notations

- The disc $D(a_0, r) \coloneqq \{z \in \mathbb{C} \colon |z - a_0| < r\}$ for $a_0 \in \mathbb{C}$ and $r > 0$.
- The polydisc $D(a, R) \coloneqq D(a_1, R_1) \times \cdots \times D(a_n, R_n)$ for $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and $R = (R_1, \ldots, R_n) \in \mathbb{R}^n_{>0}$.
- $\Delta(r) \coloneqq D(0, r) \subset \mathbb{C}$ for $r > 0$, and $\Delta^n(r) \coloneqq \Delta(r) \times \cdots \times \Delta(r) \subset \mathbb{C}^n$ for $r > 0$. In particular, $\Delta^n \coloneqq \Delta^n(1)$.
- The ball $B(a, R) \coloneqq \{|z - a| < R\} \subset \mathbb{C}^n$ for $a \in \mathbb{C}^n$ and $R > 0$.
- $\wedge^{p,q}(\Omega)$: the set of (p, q) forms in Ω.
- $\mathscr{D}^{p,q}(\Omega)$: the set of smooth (p, q) forms with compact support in Ω.
- $\delta_\Omega(z)$: the distance from z to the boundary of Ω, i.e. $\delta_\Omega(z) = \operatorname{dist}(z, \partial\Omega)$. We also denote $\delta(z) = \delta_\Omega(z)$ for simplicity.

We recall some basic knowledge from several complex variables in this section.

1.1 Holomorphic function

Let $\Omega \subset \mathbb{C}^n$ be an open subset.

Definition 1.1. A function $f \colon \Omega \to \mathbb{C}$ is called a holomorphic function, if f is continuous and for each j,

$$z_j \longmapsto f(\ldots, z_j, \ldots)$$

is holomorphic when $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$ are fixed.

Remark 1.2. *1. (Hartogs) The continuity condition can be dropped.*

2. We usually denote the ring of holomorphic functions on Ω by $\mathscr{O}(\Omega)$ or $H(\Omega)$.

Theorem 1.3 (Cauchy integral formula on polydiscs). *Let $z_0 = (z_{0,1}, \ldots, z_{0,n}) \in \mathbb{C}^n$ and $R = (R_1, \ldots, R_n) \in \mathbb{R}^n_{>0}$. If $f \in \mathscr{O}(\Omega)$ and $\overline{D(z_0, R)} \subset \Omega$, then for all $w \in D(z_0, R)$, we have*

$$f(w) = \frac{1}{(2\pi \mathrm{i})^n} \int_{\Gamma(z_0,R)} \frac{f(z_1, \ldots, z_n)}{(z_1 - w_1) \cdots (z_n - w_n)} \, \mathrm{d}z_1 \cdots \mathrm{d}z_n,$$

where $\Gamma(z_0, R) = \partial D(z_{0,1}, R_1) \times \cdots \times \partial D(z_{0,n}, R_n)$.

Proof. See [6, p. 22]. ∎

注意这里 $\Gamma(z_0, R)$ 是所谓“特征边界”而不是实质上的边界.

Corollary 1.4 (Cauchy inequality). *Suppose $f \in \mathscr{O}(\Omega)$ and $\overline{D(z_0, R)} \subset \Omega$. Then*

$$|f^{(\alpha)}(z_0)| \leq \frac{\alpha!}{R^\alpha} \sup_{\Gamma(z_0,R)} |f|, \quad \forall\, \alpha \in \mathbb{Z}^n_{\geq 0}.$$

COROLLARY 1.5. *We can write*

$$f(z) = \sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} \frac{f^{(\alpha)}(z_0)}{\alpha!}(z-z_0)^\alpha, \quad \forall\, z\in D(z_0,R),$$

when $f\in\mathscr{O}(D(z_0,R))$*, where the expansion is compactly (absolutely) convergent.*

这也常作为全纯函数 (复解析函数) 的定义. 例如在 Ohsawa 的书 [14] 中.

COROLLARY 1.6 (Montel's theorem). *Every locally uniformly bounded sequence* (f_j) *in* $\mathscr{O}(\Omega)$ *has a convergent subsequence.*

这表明 $\mathscr{O}(\Omega)$ 为一个 Fréchet 空间.

Recall

$$\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - \mathrm{i}\frac{\partial}{\partial y_k}\right), \quad \frac{\partial}{\partial \bar z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + \mathrm{i}\frac{\partial}{\partial y_k}\right), \quad 1\leq k\leq n,$$

and

$$\begin{aligned}\partial u =& \sum_{I,J}\sum_{1\leq k\leq n}\frac{\partial u_{I,J}}{\partial z_k}\,\mathrm{d}z_k\wedge \mathrm{d}z_I\wedge \mathrm{d}\bar z_J,\\ \bar\partial u =& \sum_{I,J}\sum_{1\leq k\leq n}\frac{\partial u_{I,J}}{\partial \bar z_k}\,\mathrm{d}\bar z_k\wedge \mathrm{d}z_I\wedge \mathrm{d}\bar z_J,\\ \mathrm{d}u =& \partial u + \bar\partial u,\end{aligned}$$

for the (p,q) form

$$u = \sum_{|I|=p,\,|J|=q} u_{I,J}\,\mathrm{d}z_I\wedge \mathrm{d}\bar z_J, \quad u_{I,J}\in C^s(\Omega,\mathbb{C}).$$

Then

$$\mathrm{d}^2 = \partial^2 = \bar\partial^2 = 0, \quad \partial\bar\partial + \bar\partial\partial = 0,$$

and

$$\begin{aligned} f\colon\Omega\to\mathbb{C} \text{ holomorphic} \quad &\Leftrightarrow \quad \frac{\partial f}{\partial \bar z_k} = 0 \quad \forall\, 1\leq k\leq n, \quad \Leftrightarrow \quad \bar\partial f = 0\\ &\Rightarrow \quad \partial\bar\partial f = 0 \quad \Rightarrow \quad \mathrm{i}\partial\bar\partial\log|f| = 0 \text{ on } \Omega\setminus f^{-1}(0).\end{aligned}$$

LEMMA 1.7 (Dolbeault-Grothendieck lemma). *Let* Ω *be a neighborhood of* 0 *in* $\mathbb{C}^n$ *and* u *be a* (p,q) *form on* Ω *of* C^s *such that* $\bar\partial v = 0$*, where* $1\leq s\leq\infty$*.*

1. *If* $q=0$*, then* $v=\sum_{|I|=p} v_I(z)\,\mathrm{d}z_I$ *is a holomorphic* $p-$*form, i.e. a form whose coefficients are holomorphic functions.*
2. *If* $q\geq 1$*, there exists a neighborhood* $\omega\subset\Omega$ *of* 0 *and a* $(p,q-1)$ *form* u *of* C^s *such that* $\bar\partial u = v$ *on* ω*.*

Proof. See [6, p. 28].

Use the **Koppelman formula**:

$$v(z) = \int_{\partial\omega} K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta) + \bar\partial_z\int_\omega K^{p,q-1}_{\mathrm{BM}}(z,\zeta)\wedge v(\zeta) + \int_\omega K^{p,q}_{\mathrm{BM}}(z,\zeta)\wedge\bar\partial v(\zeta),$$

for ω with piecewise C^1 boundary and (p,q) form v of class C^1 on $\overline{\omega}$, where $K^{p,q}_{\mathrm{BM}}(z,\zeta)$ is the component of $K_{\mathrm{BM}}(z,\zeta)$ of type (p,q) in z and $(n-p,n-q-1)$ in ζ, $K_{\mathrm{BM}}(z,\zeta)$ is the pull back of k_{BM} by the map $\pi\colon \mathbb{C}^n\times\mathbb{C}^n\to\mathbb{C}^n$, $(z,\zeta)\mapsto z-\zeta$, and

$$k_{\mathrm{BM}}(z)=c_n\sum_{1\le j\le n}(-1)^j\frac{\overline{z}_j\,\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_n\wedge\mathrm{d}\bar{z}_1\wedge\ldots\widehat{\mathrm{d}\bar{z}_j}\ldots\wedge\mathrm{d}\bar{z}_n}{|z|^{2n}}$$

is the <u>Bochner-Martinelli kernel</u>,

$$c_n=(-1)^{\frac{n(n-1)}{2}}\frac{(n-1)!}{(2\pi\mathrm{i})^n}.$$

■

这个引理说明了 $\overline{\partial}-$ 方程总是局部可解的.

THEOREM 1.8 (Weierstrass preparation theorem). *Let g be a holomorphic function on a neighborhood of 0 in $\mathbb{C}^n$, such that $\dfrac{g(0,z_n)}{z_n^s}$ has a not zero finite limit at $z_n=0$. For $r'=(r'_1,\ldots,r'_{n-1})$ and r_n small enough, one can write*

$$g(u)=u(z)P(z',z_n),$$

where u is an invertible holomorphic function in a neighborhood of $\overline{\Delta^n(r',r_n)}$, and P is a Weierstrass polynomial in z_n, that is, a polynomial of the form

$$P(z',z_n)=z_n^s+a_1(z')z_n^{s-1}+\cdots+a_s(z'),\quad a_k(0)=0,$$

with holomorphic coefficients $a_k(z')$ on a neighborhood of $\overline{\Delta^{n-1}(r')}$ in $\mathbb{C}^{n-1}$.

Proof. See [6, p. 79]. ■

COROLLARY 1.9 (Weierstrass division theorem). *Every bounded holomorphic function f on $D=\Delta(r',r_n)$ can be represented in the form*

$$f(z)=g(z)q(z)+R(z',z_n),$$

where q and R are holomorphic in D, $R(z',z_n)$ is a polynomial of degree $\le s-1$ in z_n, and

$$\sup_D|q|\le C\sup|f|,\quad \sup_D|R|\le C\sup|f|$$

for some constant $C\ge 0$ independent of f. The representation is unique.

Proof. See [6, p. 80]. ■

COROLLARY 1.10. *The local ring*

$$\mathscr{O}_n=\mathscr{O}_{\mathbb{C}^n,0}=\mathbb{C}\{z_1,\ldots,z_n\}$$

of convergent power series is a Noetherian (i.e. every ideal $\mathscr{I}$ of $\mathscr{O}_n$ is finitely generated) UFD.

Proof. See [6, pp. 81–83]. ■

1.2 Analytic set

Definition 1.11. A subset $A \subset \Omega$ is said to be an analytic subset of Ω if A is closed and if for every point $x_0 \in A$ there exist a neighborhood U of x_0 and holomorphic functions $g_1, \ldots, g_N \in \mathscr{O}(U)$ such that

$$A \cap U = \{z \in U : g_1(z) = \ldots = g_N(z) = 0\}.$$

Remark 1.12. 1. *We can define the germ (A, x) by modulus of the equivalence: $A \sim B$ iff $A \cap V = B \cap V$ for $x \in V$ open neighborhood.*

2. *Let $\mathscr{I} = (g_1, \ldots, g_N)$ be an ideal of $\mathscr{O}_x$. Then we can denote by $(V(\mathscr{I}), x)$ the zero variety by*

$$V(\mathscr{I}) = \{z \in U : g_1(z) = \ldots = g_N(z) = 0\}.$$

3. *Let (A, x) be a germ of analytic set at x. Denote*

$$\mathscr{I}_{A,x} = \{f \in \mathscr{O}_x : f(z) = 0, \ \forall z \in (A, x)\}.$$

Then $\mathscr{I}_{A,x}$ is an ideal of $\mathscr{O}_x$, and $\mathscr{I}_{V(\mathscr{I}),x} \supset \mathscr{I}$.

theorem 1.13 (Cartan 50'). *For any analytic set A, the sheaf of ideals $\mathscr{I}_A$ is a coherent analytic sheaf.*

Proof. See [6, pp. 99–100].

Omit. Use the following local parametrization theorem. ■

theorem 1.14 (Local parametrization theorem). *Let $\mathscr{I}$ be a prime ideal of $\mathscr{O}$ and let $A = V(\mathscr{I})$. Then we can choose the coordinates*

$$(z'; z'') = (z_1, \ldots, z_d; z_{d+1}, \ldots, z_n)$$

such that:

The ring $\mathscr{O}_n/I$ is a finite integral extension of $\mathscr{O}_d$; let q be the degree of the extension and let $\delta(z') \in \mathscr{O}_d$ be the discriminant of the irreducible polynomial of a primitive element $u(z'') = \sum_{k>d} c_k z_k$. If Δ', Δ'' are polydiscs of sufficiently small radii r', r'' and if $r' \le r''/C$ with C large, the projection map $\pi : A \cap (\Delta' \times \Delta'') \to \Delta'$ is a ramified covering with q sheets, whose ramification locus is contained in $S = \{z' \in \Delta' : \delta(z') = 0\}$. This means that

1. *the open subset $A_S = A \cap \big((\Delta' \setminus S) \times \Delta''\big)$ is a smooth $d-$dimensional manifold, dense in $A \cap (\Delta' \times \Delta'')$;*

2. *$\pi : A_S \to \Delta' \setminus S$ is a covering;*

3. *the fibers $\pi^{-1}(z')$ have exactly q elements if $z' \notin S$ and at most q if $z' \in S$.*

Moreover, A_S is a connected covering of $\Delta' \setminus S$, and $A \cap (\Delta' \times \Delta'')$ is contained in a cone $\{|z''| \le C|z'|\}$.

Proof. See [6, pp. 95–97]. ■

这个定理可以看作为解析版本的 Noether 正规化定理.

THEOREM 1.15 (Hilbert's Nullstellensatz). *For every ideal $\mathscr{I} \subset \mathscr{O}_n$,*

$$\mathscr{I}_{V(I),0} = \sqrt{\mathscr{I}},$$

where $\sqrt{I}$ is the radical of $\mathscr{I}$, i.e. the set of germs $f \in \mathscr{O}_n$ s.t. some power $f^k \in \mathscr{I}$.

Proof. See [6, p. 97]. ∎

DEFINITION 1.16. Let $A \subset \Omega$ be an analytic subset and $x \in A$. Then $x \in A$ is called a regular point of A if $A \cap U$ is a $\mathbb{C}-$analytic submanifold of U for some neighborhood U of x. Otherwise, x is said to be singular.

The corresponding subsets of A are denoted by A_{reg} and A_{sing} respectively.

DEFINITION 1.17. For an irreducible germ of analytic set (A, x), define the dimension by

$$\dim(A, x) := \dim(A_{\text{reg}}, x).$$

If (A, x) has several irreducible components (A_l, x), then

$$\dim(A, x) := \max\{\dim(A_l, x)\}.$$

If all the irreducible components of (A, x) have the same dimension, then (A, x) is said to be of pure dimension. Set $\operatorname{codim}(A, x) := n - \dim(A, x)$.

THEOREM 1.18. *A_{sing} is an analytic subset of A.*

Proof. See [6, pp. 100–101].

(A Irreducible:) Let $g_1, \ldots, g_N$ be local generators of A in a neighborhood U of regular point. Then $A_{\text{sing}} \cap U$ is defined by

$$\det\left(\frac{\partial g_j}{\partial z_k}\right)_{j\in J,\, k\in K} = 0, \quad J \subset \{1, \ldots, N\},\ K \subset \{1, \ldots, n\},\ |J| = |K| = n - d,$$

where $d = \dim A$. ∎

EXAMPLE 1.19. Let

$$A = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \colon z_1 z_2^2 = z_2 z_3^3 = 0\}.$$

Then $A = A_1 \cup A_2$, where

$$A_1 = \{z_1 = z_3 = 0\}, \quad A_2 = \{z_2 = 0\}.$$

Then $\dim A_1 = 1$, $\dim A_2 = 2 \Rightarrow \dim A = 2$. We also can see $A_{\text{sing}} = \{0\}$, and

$$\mathscr{I}_{A,0} = (z_1 z_2, z_2 z_3) = \sqrt{(z_1 z_2^2, z_2 z_3^3)}.$$

1.3 Plurisubharmonic function

Definition 1.20. A function $u\colon \Omega \to [-\infty, +\infty)$ is said to be plurisubharmonic (psh for short) if

1. u is upper semicontinuous;
2. for every complex line $L \subset \mathbb{C}^n$, $u\big|_{\Omega\cap L}$ is subharmonic on $\Omega \cap L$.

We denote the set of psh functions on Ω by $\mathrm{Psh}(\Omega)$.

Proposition 1.21.
- *If $u_k \in \mathrm{Psh}(\Omega)$ is decreasing, then $u = \lim u_k \in \mathrm{Psh}(\Omega)$.*
- *If $u_1, \dots, u_p \in \mathrm{Psh}(\Omega)$, and $\chi\colon \mathbb{R}^p \to \mathbb{R}$ is convex function s.t. $\chi(t_1, \dots, t_p)$ is non-decreasing in each t_j, then $\chi(u_1, \dots, u_p) \in \mathrm{Psh}(\Omega)$. In particular,*

$$u_1 + \cdots + u_p,\ \max\{u_1, \dots, u_p\},\ \log(\mathrm{e}^{u_1} + \cdots + e^{u_p}) \in \mathrm{Psh}(\Omega).$$

- *Let $u \in \mathrm{Psh}(\Omega)$, and $u \not\equiv -\infty$ on each component of Ω. If (ρ_ε) is a family of smoothing kernels, then $u \star \rho_\varepsilon \in C^\infty \cap \mathrm{Psh}(\Omega_\varepsilon)$, the family $(u \star \rho_\varepsilon)$ is non-decreasing in ε, and $\lim_{\varepsilon\to 0} u \star \rho_\varepsilon = u$ (from above).*

Proof. See [6, p. 39]. ∎

Proposition 1.22. *Let $(u_\alpha)_{\alpha\in I}$ be a family of upper semicontinuous functions $\Omega \to [-\infty, +\infty)$ which is locally uniformly bounded above, and $u := \sup_\alpha u_\alpha$.*

The upper semicontinuous regularization of u is defined by:

$$u^\star(z) := \lim_{\varepsilon\to 0} \sup_{B(z,\varepsilon)} u.$$

*(**Choquet's lemma**) Then the family (u_α) has a countable subfamily $(v_j) = (u_{\alpha(j)})$ s.t. $v = \sup_\alpha v_j$ satisfies*

$$v \le u \le u^\star \le u^\star = v^\star.$$

If all $u_\alpha \in \mathrm{Psh}(\Omega)$, then $u^\star \in \mathrm{Psh}(\Omega)$ and $u = u^\star$ a.e.

Proof. See [6, p. 38]. ∎

Example 1.23. If $f_1, \dots, f_q \in \mathscr{O}(\Omega)$ and $\alpha_1, \dots, \alpha_q \ge 0$, then

$$\log\left(|f_1|^{\alpha_1} + \cdots + |f_q|^{\alpha_q}\right) \in \mathrm{Psh}(\Omega).$$

$u \in \mathrm{Psh}(\Omega)$ is called as having analytic singularities near $z \in \Omega$ if there exists a constant $c > 0$, holomorphic functions $f_1, \dots, f_q$ near z, and a bounded function v near z, such that

$$u = \frac{c}{2}\log\left(|f_1|^2 + \cdots + |f_q|^2\right) + v \quad \text{near } z.$$

Remark 1.24. *If $u \in C^2(\Omega, \mathbb{R})$, then*

$$u \in \mathrm{Psh}(\Omega) \quad \Longleftrightarrow \quad \left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)_{1\le j,k\le n} \text{ is semi-positive.}$$

If the matrix is (strictly) positive definite at every point, then u is called strictly psh and denote by $u \in \mathrm{SPsh}(\Omega)$.

Proposition 1.25. *If $\Omega = \omega + \mathrm{i}\omega'$, where ω, ω' are open subsets of $\mathbb{R}^n$, and $u \in \mathrm{Psh}(\Omega)$ which depends only on $x = \mathrm{Re}\, z$, then $\omega \ni x \longmapsto u(x)$ is convex.*

As a corollary, if $u \in \mathrm{Psh}(D(a,R)) \subset \mathbb{C}^n$, then

$$\begin{aligned}\mu(u; r_1, \dots, r_n) &= \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} u(a_1 + r_1 \mathrm{e}^{\mathrm{i}\theta_1}, \dots, a_n + r_n \mathrm{e}^{\mathrm{i}\theta_n})\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n,\\ m(u; r_1, \dots, r_n) &= \sup_{z\in D(a,r)} u(z_1, \dots, z_n),\end{aligned}$$

are convex functions of $(\log r_1, \dots, \log r_n)$ that are non-decreasing in each variable.

Proof. See [6, p. 41].

Let

$$\begin{aligned}\widetilde{\mu}(u; z_1, \dots, z_n) &= \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi} u(a_1 + \mathrm{e}^{z_1} \mathrm{e}^{\mathrm{i}\theta_1}, \dots, a_n + \mathrm{e}^{z_n} \mathrm{e}^{\mathrm{i}\theta_n})\,\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n,\\ \widetilde{m}(u; z_1, \dots, z_n) &= \sup_{|w_j|\le 1} u(a_1 + \mathrm{e}^{z_1} w_1, \dots, a_n + \mathrm{e}^{z_n} w_m).\end{aligned}$$

■

1.4 Pseudoconvexity

Definition 1.26. Let Ω be an domain of $\mathbb{C}^n$. Ω is said to be

- a domain of holomorphy, if every domain Ω' with $\Omega \subsetneqq \Omega'$, there exists $F \in \mathscr{O}(\Omega)$, which can not be extended to be a holomorphic function on Ω';
- holomorphically convex, if $\widehat{K}_\Omega \Subset \Omega$ for every compact subset $K \subset X$, where the holomorphic hull

$$\widehat{K}_\Omega \coloneqq \{z \in \Omega \colon |f(z)| \le \sup_K |f|,\ \forall\, f \in \mathscr{O}(\Omega)\}.$$

如果 Ω 是全纯凸域, 那我们可以取 Ω 的紧穷竭:

$$K_1 \Subset K_2 \Subset \dots \Subset K_j \Subset \dots \Subset \Omega, \quad \bigcup_{j=1}^{\infty} K_j = \Omega,$$

并使得每个 K_j 满足 $\widehat{(K_j)}_\Omega = K_j$.

Remark 1.27.
- *Ang domain in $\mathbb{C}$ is a domain of holomorphy.*
- *(Hartogs phenomenon) There exists a domain in $\mathbb{C}^n$ which is not a domain of holomorphy when $n > 1$.*

$$\Omega = \Big\{(z_1, z_2) \colon |z_1| < 1,\ \frac{1}{2} < |z_2| < 1\Big\} \cup \Big\{(z_1, z_2) \colon \frac{1}{2} < |z_1| < 1,\ |z_2| < 1\Big\}.$$

In fact, when $n > 1$, every holomorphic function $f \in \mathscr{O}(\Omega \setminus K)$, can be extended to be an element in $\mathscr{O}(\Omega)$, where K is a compact subset of Ω s.t. $\Omega \setminus K$ is connected (Hartogs).

THEOREM 1.28 (Cartan-Thullen theorem). *TFAE:*

1. *Ω is a domain of holomorphy;*
2. *Ω is holomorphically convex;*
3. *There exists a function $F \in \mathscr{O}(\Omega)$ which is unbounded on any neighborhood of any point of $\partial\Omega$.*

Proof. See [6, pp. 48–49]. ■

DEFINITION 1.29. $\psi\colon \Omega \to [-\infty, +\infty)$ is said to an exhaustion if every sublevel set

$$\Omega_c \coloneqq \{z \in \Omega\colon \psi(z) < c\} \Subset \Omega, \quad \forall\, c \in \mathbb{R}.$$

Ω is said to be

1. weakly pseudoconvex if there exists a smooth psh exhaustion function $\psi \in C^\infty(\Omega) \cap \mathrm{Psh}(\Omega)$;
2. strongly pseudoconvex if there exists a smooth strictly psh exhaustion function $\psi \in C^\infty(\Omega) \cap \mathrm{SPsh}(\Omega)$.

THEOREM 1.30. *TFAE:*

1. *Ω is weakly pseudoconvex;*
2. *Ω is strongly pseudoconvex;*
3. *Ω has a psh exhaustion function ψ;*
4. *$-\log d(z, \Omega^c)$ is psh on Ω.*

Proof. See [6, pp. 54–55]. ■

THEOREM 1.31. *$\Omega \subset \mathbb{C}^n$ domain. Then*

$$\Omega \text{ holomorphically convex} \iff \Omega \text{ pseudoconvex.}$$

Proof. See [6, pp. 49–50] and [6, Chapter VIII] (by the theory of L^2 estimates for $\overline{\partial}$). ■

拟凸推全纯凸即是经典的 Levi 问题 (Oka, Norguet, Bremermann). 这些等价性对一般的复流形并不对, 不过全纯凸推弱拟凸是直接的, 而强拟凸实际上等价于为 Stein 流形 (全纯凸 + 局部可分离), 其中强拟凸推 Stein 也被称为 Levi 问题 (由 Grauert 完全解决).

EXAMPLE 1.32. Let $\Omega = \Delta^* \times \Delta \subset \mathbb{C}^2$, where $\Delta^* = \Delta \setminus \{0\}$. Then

$$\psi(z) = \max\big\{-\log|z_1|, -\log(1-|z_1|^2), -\log(1-|z_2|^2)\big\} \in \mathrm{Psh}(\Omega),$$

and $\psi(z) \to +\infty$ as $z \to \partial\Omega$.

THEOREM 1.33 (Kiselman's Minimum Principle). *Let $\Omega \subset \mathbb{C}^p \times \mathbb{C}^n$ be a pseudoconvex open set s.t. each slice*

$$\Omega_\zeta = \{z \in \mathbb{C}^n : (\zeta, z) \in \Omega\}, \quad \zeta \in \mathbb{C}^p,$$

is a convex tube $\omega_\zeta + \mathrm{i}\mathbb{R}^n$, $\omega_\zeta \subset \mathbb{R}^n$. If $v(\zeta, z) \in \mathrm{Psh}(\Omega)$ does not depend on $\operatorname{Im} z$, then

$$u(\zeta) = \inf_{z \in \Omega_\zeta} v(\zeta, z) \in \mathrm{Psh}(\Omega'), \quad \Omega' := \mathrm{pr}_{\mathbb{C}^n}(\Omega),$$

or locally $\equiv -\infty$.

Proof. See [6, p. 57]. ■

如果没有 v 与虚部无关的条件, 这个结论是不对的.

1.5 Pluripolar set

DEFINITION 1.34.

- A set $A \subset \Omega$ is said to be (locally) pluripolar if for every $z \in \Omega$, there exists a neighborhood U of x and $u \in \mathrm{Psh}(U)$, $u \not\equiv -\infty$, such that $A \cap U \subset \{x \in U : u(x) = -\infty\}$;
- A set $A \subset \Omega$ is said to be complete pluripolar if for every $z \in \Omega$, there exists a neighborhood U of x and $u \in \mathrm{Psh}(\Omega) \cap L^1_{\mathrm{loc}}(U)$, such that $A \cap U = \{x \in U : u(x) = -\infty\}$;
- $A \subset \mathbb{C}^n$ is globally pluripolar if $A \subset u^{-1}(-\infty)$ for some $u \in \mathrm{Psh}(\mathbb{C}^n)$;
- $A \subset \Omega$ is negligible if there is a family of psh functions $\{u_\alpha\}$ on Ω locally uniformly bounded from above such that $A \subset \{u < u^\star\}$, where $u = \sup_\alpha u_\alpha$.

REMARK 1.35.

- *Any analytic subset is a (complete) pluripolar set. If we take $A \cap U = \{g_1 = \ldots = g_N = 0\}$, then $A \cap U = u^{-1}(-\infty)$, where $u = \log(|g_1|^2 + \ldots + |g_N|^2)$.*

注意 analytic set 一定是 pluripolar set, 但反过来不一定, 实际上 analytic set 一定是无处稠密的, 但 pluripolar set 却可能是稠密的.

EXAMPLE 1.36. Let $\Omega = \Delta \subset \mathbb{C}$, and $\psi \in \mathrm{Psh}(\Delta)$ as

$$\psi(z) = \sum_{k=1}^{\infty} \frac{1}{k^2} \log\left|z - \frac{1}{2^k}\right|.$$

Then $A = \psi^{-1}(-\infty) = \{0\} \cup \{1/2^k : k \geq 1\}$ is a (complete) pluripolar set, but A is not an analytic subset since 0 is a cluster point in A.

If we let

$$\phi(z) = \sum_{k=1}^{\infty} \frac{1}{k^3} \log\left|z - \frac{1}{2^k}\right|,$$

then $A = \phi^{-1}(-\infty) = \{1/2^k : k \geq 1\}$ is a pluripolar set which is not closed.

PROPOSITION 1.37. *Let $A \subset \Omega$ be a closed pluripolar subset. Then:*

1. *Every $v \in \mathrm{Psh}(\Omega \setminus A)$ that is locally bounded above near A extends uniquely into a function $\widetilde{v} \in \mathrm{Psh}(\Omega)$;*
2. *Every $f \in \mathscr{O}(X \setminus A)$ that is locally bounded near A extends to a function $f \in \mathscr{O}(\Omega)$.*

Proof. See [6, pp. 44–45].

May assume $u \leq 0$ on W where $A \cap W \subseteq u^{-1}(-\infty)$. Let $v_\varepsilon = v + \varepsilon u$ and $v_\varepsilon = -\infty$ on $A \cap W$. Then $\widetilde{v} = (\sup v_\varepsilon)^*$.

Apply (1) to $\pm\mathrm{Re}\, f$ and $\pm\mathrm{Im}\, f$. ■

THEOREM 1.38. *Let $A \subset \mathbb{C}^n$. Then A is*

$$\textit{locally pluripolar} \quad \Longleftrightarrow \quad \textit{globally pluripolar} \quad \Longleftrightarrow \quad \textit{negligible.}$$

Proof. See [3, Chapter 3.1]. ■

2 L^2 Methods and Multiplier Ideal Sheaf

2.1 L^2 estimate of $\overline{\partial}-$equation

We only focus on $(n,0)$ and $(n,1)$ forms in this section.

Let Ω be a domain in $\mathbb{C}^n$, and $\phi \in C^2 \cap \mathrm{Psh}(\Omega)$. We can see that $(L,h) := (\Omega \times \mathbb{C}, \mathrm{e}^{-\phi})$ is a trivial line bundle with a singular hermitian metric $h := \mathrm{e}^{-\phi}$ on Ω. Let $\omega = \frac{\mathrm{i}}{2}\sum_i \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i$ and $\mathrm{d}V = \frac{\omega^n}{n!}$. Then

$$L^{n,q}_{(2)}(\Omega,h) := \left\{u \in \wedge^{n,q}(\Omega) \colon \int_\Omega |u|_h^2 \,\mathrm{d}V < +\infty\right\}$$

is a Hilbert space, where

$$|u|_h^2 \,\mathrm{d}V := 2^q \Big(\sum_{|J|=q} |u_J|^2 \mathrm{e}^{-\phi}\Big)\,\mathrm{d}z \wedge \mathrm{d}\bar{z},$$

for

$$u(z) = \sum_{|J|=q} u_J \,\mathrm{d}z \wedge \mathrm{d}\bar{z}_J, \quad \text{where} \quad \mathrm{d}z = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n.$$

我们也可以将其视为光滑紧支形式在 (加权) L^2 范数下的完备化.

It can be seen that the operator:

$$\overline{\partial}\colon \quad L^{n,q-1}_{(2)}(\Omega,h) \supset \mathscr{D}^{n,q-1}(\Omega) \longrightarrow \mathscr{D}^{n,q}(\Omega) \subset L^{n,q}_{(2)}(\Omega,h)$$

is a densely defined closed operator, which makes it possible to extend $\overline{\partial}$, and define its adjoint. Let $\mathrm{Dom}(\overline{\partial}) \subset L^{n,q-1}_{(2)}(\Omega,h)$ be its domain and $\overline{\partial}^*_h$ be its adjoint operator.

注意如果考虑的是 (n,q) 形式, 所以 ∂ 算子作用上去都是 0. 我们计算一下 $\overline{\partial}^*_h$ 的表达式: ($q=1$ 情形) 设

$$U = u\,\mathrm{d}z, \quad V = \sum_j v^j \,\mathrm{d}\bar{z}^j \wedge \mathrm{d}z,$$

则

$$\overline{\partial} U = \sum_j u_{\bar{j}} \,\mathrm{d}\bar{z}^j \wedge \mathrm{d}z.$$

可得

$$(\overline{\partial}U, V) = 2\int_\Omega \sum_j u_{\bar{j}} \overline{v^j} \mathrm{e}^{-\phi} = -2\int_\Omega \sum_j u \left(\overline{v^j}\mathrm{e}^{-\phi}\right)_{\bar{j}} = -2\int_\Omega \sum_j u\left(\overline{v^j}_{\bar{j}}\mathrm{e}^{-\phi} - \overline{v^j}\phi_{\bar{j}}\mathrm{e}^{-\phi}\right),$$

$\Rightarrow$

$$\overline{\partial}^*_h V = \sum_j (\phi_j v^j - v^j_j)\,\mathrm{d}z = \sum_j \left(\frac{\partial \phi}{\partial z_j} v^j - \frac{\partial v^j}{\partial z_j}\right)\mathrm{d}z.$$

THEOREM 2.1 (Bochner-Kodaira-Nakano inequality). *For $u \in \mathscr{D}^{n,q-1}(\Omega)$, one has*

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}^*_h u\|^2 \geq (\mathrm{i}\Theta_h \Lambda u, u),$$

where Θ_h is the curvature form $\mathrm{i}\Theta_h = \mathrm{i}\partial\overline{\partial}\phi$, and $\Lambda = \frac{-\mathrm{i}}{2}\sum_j (\mathrm{d}\bar{z}_j)^*(\mathrm{d}z_j)^*$:

$$(\mathrm{i}\Theta_h\Lambda u, u) = 2^q \int_\Omega \sum_{|K|=q-1}\sum_{1\leq i,j\leq n} \phi_{i\bar{j}} u_{K,i}\overline{u_{K,j}}\mathrm{e}^{-\phi}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}, \quad \phi_{i\bar{j}} = \frac{\partial^2\phi}{\partial z_j \partial \bar{z}_j}.$$

Proof. See [6, pp. 329–332]. ■

特别地, 如果 $q=1$:

$$(\mathrm{i}\Theta_h\Lambda u, u) = 2\int_\Omega \sum_{1\le i,j\le n} \phi_{i\bar{j}} u_i \overline{u_j} \mathrm{e}^{-\phi}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}.$$

在一般的 Kähler 流形上的向量丛 (F,h) 上, 对任意 $F-$ 值的光滑紧支 (p,q) 形式, 有 Bochner-Kodaira-Nakano 等式:

$$\overline{\Box} = \Box + [\mathrm{i}\Theta_{F,h},\Lambda].$$

并由此可得

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}_h^* u\|^2 - \|\partial^* u\|^2 - \|\partial_h u\|^2 = ([\mathrm{i}\Theta_h,\Lambda]u,u).$$

为了计算方便, 我们再考虑一个更简单的情形: $n=1$. 这时若 $V = v\,\mathrm{d}\bar{z}\wedge\mathrm{d}z$,

$$\overline{\Box}V = \overline{\partial}\overline{\partial}_h^* V = \overline{\partial}\left((\phi_z v - v_z)\,\mathrm{d}z\right) = \left(\phi_{z\bar{z}}v + \phi_z v_{\bar{z}} - v_{\bar{z}z}\right)\mathrm{d}\bar{z}\wedge\mathrm{d}z.$$

$\Rightarrow$

$$\|\overline{\partial}_h^* V\|^2 = (\overline{\Box}V,V) = 2\int_\Omega \left(\phi_{z\bar{z}}v\bar{v} + \phi_z v_{\bar{z}}\bar{v} - v_{\bar{z}z}\bar{v}\right)\mathrm{e}^{-\phi}\,\mathrm{d}z\wedge\mathrm{d}\bar{z},$$

其中

$$\begin{aligned}\int_\Omega -v_{\bar{z}z}\bar{v}\mathrm{e}^{-\phi}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} &= \int_\Omega v_{\bar{z}}\left(\bar{v}\mathrm{e}^{-\phi}\right)_z\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\\ &= \int_\Omega v_{\bar{z}}\left(\overline{v_{\bar{z}}} - \phi_z\bar{v}\right)\mathrm{e}^{-\phi}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\\ &= \|v_{\bar{z}}\|^2 - \int_\Omega v_{\bar{z}}\phi_z\bar{v}\mathrm{e}^{-\phi}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}.\end{aligned}$$

从而

$$\|\overline{\partial}_h^* V\|^2 = 2\int_\Omega \phi_{z\bar{z}}v\bar{v}\mathrm{e}^{-\phi}\,\mathrm{d}z\wedge\mathrm{d}\bar{z} + \|v_{\bar{z}}\|^2 = 2\big([\mathrm{i}\Theta_h,\Lambda]V,V\big) + \|\partial^* V\|^2.$$

值得注意的是, 以上计算过程实际上并不要求 $\mathrm{i}\partial\overline{\partial}\phi$ 的正性.

Lemma 2.2. *If Ω is pseudoconvex, then $\mathscr{D}^{n,q}(\Omega)$ is dense in $\mathrm{Dom}(\overline{\partial}^*)$ for the graph norm*

$$u \longmapsto \|u\| + \|\overline{\partial}_h^* u\|.$$

Proof. See [6, pp. 368–370]. ■

Consider two closed and densely defined operators T,S among Hilbert spaces:

$$\mathscr{H}_1 \xrightarrow{T} \mathscr{H}_2 \xrightarrow{S} \mathscr{H}_2$$

with $S\circ T = 0$.

Lemma 2.3. *In order that $\operatorname{Im} T = \operatorname{Ker} S$, it suffices that*

$$\|T^* x\|_1^2 + \|Sx\|_3^2 \ge C\|x\|_2^2, \quad \forall\, x\in \mathrm{Dom}\, S\cap \mathrm{Dom}\, T^*,$$

for some constant $C>0$. In that case, for every $v\in\mathscr{H}_2$ s.t. $Sv=0$, there exists $u\in\mathscr{H}_1$ s.t. $Tu=v$ and

$$\|u\|_1^2 \le \frac{1}{C}\|v\|_2^2.$$

Proof. See [6, pp. 364–365].

Write $x \in \operatorname{Dom} T^*$ as $x = x' + x''$ with $x' \in \operatorname{Ker} S$ and $x'' \in (\operatorname{Ker} S)^{\perp} \subset \operatorname{Ker} T^*$. If $Sv = 0$, then $\langle v, x''\rangle_2 = 0$. Note that $Sx' = T^*x'' = 0$. We can get

$$|\langle v, x\rangle_2|^2 = |\langle v, x'\rangle_2|^2 \le \|v\|_2^2\|x'\|_2^2 \le \frac{1}{C}\|v\|_2^2\|T^*x'\|_2^2 = \frac{1}{C}\|v\|_2^2\|T^*x\|_1^2.$$

Finally, use the Hahn-Banach theorem and Riesz representation theorem to the operator:

$$\begin{aligned} \mathcal{L}\colon \operatorname{Im}(T^*) &\longrightarrow \mathbb{C} \\ y = T^*x &\longmapsto \langle x, v\rangle_2. \end{aligned}$$

■

Corollary 2.4. *Let $\Omega \subseteq \mathbb{C}^n$ be a pseudoconvex domain. If the operator $A := \mathrm{i}\Theta_h\Lambda$ is positive definite everywhere on $\wedge^{n,q}(\Omega)$, $q \ge 1$, then for every form $v \in L^{n,q}_{(2)}(\Omega, h)$ satisfying $\overline{\partial} v = 0$ and $\int_\Omega (A^{-1}v, v)\,\mathrm{d}V < +\infty$, there exists $u \in L^{n,q-1}_{(2)}(\Omega, h)$ such that $\overline{\partial} u = v$ and*

$$\int_\Omega |u|_h^2\,\mathrm{d}V \le \int_\Omega (A^{-1}v, v)\,\mathrm{d}V.$$

Proof. See [6, p. 371]. ■

特别地, 如果 $q = 1$:

$$(A^{-1}v, v) = 2\int_\Omega \sum_{1\le i,j\le n} \phi^{i\bar{j}} v_j \overline{v_j} \mathrm{e}^{-\phi}\,\mathrm{d}z \wedge \mathrm{d}\bar{z},$$

其中 $(\phi^{i\bar{j}}) = (\phi_{i\bar{j}})^{-1}$ 是 ϕ 的 Hessian 矩阵的逆矩阵.

如果 $\mathrm{i}\partial\overline{\partial}\phi \ge \varepsilon\omega$ 对某个 $\varepsilon > 0$ 成立, 也就是 $\phi - \frac{\varepsilon}{2}|z|^2 \in \operatorname{Psh}(\Omega)$ 或者说

$$\sum_{i,j} \phi_{i\bar{j}} v_j \overline{v_j} \ge \varepsilon \sum_j |v_j|^2,$$

则

$$\int_\Omega |u|_h^2\,\mathrm{d}V \le \frac{1}{q\varepsilon}\int_\Omega |v|_h^2\,\mathrm{d}V.$$

Corollary 2.5. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded pseudoconvex domain, and $\varphi \in L^1_{\mathrm{loc}}\big(\Omega \to [-\infty, +\infty)\big)$ satisfying $\mathrm{i}\partial\overline{\partial}\varphi \ge -A\omega$ on Ω for some constant $A > 0$. Then for every $(0,1)$ form $v = \sum_{j=1}^n v_j\,\mathrm{d}\bar{z}_j$ satisfying $\overline{\partial} v = 0$ and*

$$\int_\Omega \left(\sum_{j=1}^n |v_j|^2\right) \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda < +\infty,$$

there exists a function $u \in \operatorname{Dom}(\overline{\partial})$ on Ω such that $\overline{\partial} u = v$ and

$$\int_\Omega |u|^2 \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda \le C_{A,\Omega}\int_\Omega \left(\sum_{j=1}^n |v_j|^2\right) \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda,$$

where $C_{A,\Omega} > 0$ only depends on A and the diameter of Ω.

If v is smooth, then u can be chosen as smooth function.

Proof. Take $h = \mathrm{e}^{-\varphi - A'|z|^2}$ for $A' > A$, and transform (n, q) forms into $(0, q)$ forms. ∎

THEOREM 2.6. *Let $\varphi \in \mathrm{Psh}(\Omega)$, where $\Omega \subset \mathbb{C}^n$ is bounded and pseudoconvex. If $z_0 \in \Omega$ and $\mathrm{e}^{-\varphi}$ is integrable near z_0 (e.g. $\varphi(z_0) > -\infty$), then there exists $f \in \mathscr{O}(\Omega)$ such that $f(z_0) = 1$ and*

$$\int_\Omega |f|^2 \mathrm{e}^{-\varphi}\, \mathrm{d}\lambda < +\infty.$$

Proof. Let θ be a cut-off function such that $\theta \equiv 1$ and supported near z_0. Let $v = \overline{\partial}(1-\theta)$, and since $\mathrm{e}^{-\varphi}$ is integral near z_0 we can take θ such that

$$\int_\Omega |v|^2 \mathrm{e}^{-\varphi - 2n\log|z-z_0| - |z|^2}\, \mathrm{d}\lambda < +\infty.$$

Then we can get a solution u of $\overline{\partial} u = v = -\overline{\partial}\theta$ with (Ω is bounded)

$$\int_\Omega |u|^2 \mathrm{e}^{-\varphi - 2n\log|z-z_0|}\, \mathrm{d}\lambda < +\infty.$$

Let $f = u + \theta$. Then we have $f \in \mathscr{O}(\Omega)$, $u(z_0) = 0 \Rightarrow f(z_0) = \theta(z_0) = 1$, and $\int_\Omega |f|^2 \mathrm{e}^{-\varphi}\, \mathrm{d}\lambda < +\infty$.
∎

COROLLARY 2.7. *Let $\varphi \in \mathrm{Psh}(\Omega)$. Then*

$$V(\mathscr{I}(\varphi)) \coloneqq \{z \in \Omega \colon \mathrm{e}^{-\varphi} \text{ is integrable near } z\}$$

is an analytic subset.

Proof. See [6, p. 384].

For every $x \in \Omega$, let U be a small bounded pseudoconvex neighborhood of x in Ω. Denote $A^2(U, \mathrm{e}^{-\varphi}) \coloneqq L^2(U, \mathrm{e}^{-\varphi}) \cap \mathscr{O}(U)$. Then

$$V(\mathscr{I}(\varphi)) = \bigcap_{f \in A^2(U, \mathrm{e}^{-\varphi})} f^{-1}(0).$$

Then the result follows from the coherence of $\mathscr{I}(\varphi)$ or the fact that *the intersection of a family of analytic subsets is an analytic subset* (Noetherian). ∎

COROLLARY 2.8 (Levi's problem). *A bounded pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$ is a domain of holomorphy.*

Proof. See [6, pp. 373–374]

Since Ω is pseudoconvex, we have $-\log\delta(z) \in \mathrm{Psh}(\Omega)$.

For every $z_0 \in \partial\Omega$, select a sequence of pts $\{z_j\} \subset \Omega$ s.t. $z_j \to z_0$. For each j, let θ_j be a cut-off function with $\operatorname{Supp}\theta_j \Subset V_j$, where V_j is a sufficiently small neighborhood of z_j and disjoint with each other.

Let

$$\psi(z) = \chi\Big(-\log\delta(z) + |z|^2\Big) + \sum_{j=1}^{\infty} 2n\theta_j \log|z - z_j| \in \mathrm{Psh}(\Omega), \tag{2.1}$$

where $\delta(z) = \mathrm{dist}(z, \Omega^c)$, and χ is an increasing convex function undetermined.

Let

$$v = \overline{\partial}\Big(\sum_{j=1}^{\infty} j\theta_j\Big).$$

Suppose

$$\int_\Omega |v|^2 \mathrm{e}^{-\psi}\,\mathrm{d}\lambda < +\infty. \tag{2.2}$$

We solve $\overline{\partial}u = v$ with L^2 estimates

$$\int_\Omega |u|^2 \mathrm{e}^{-\psi}\,\mathrm{d}\lambda < +\infty,$$

and let $f = -u + \sum_{j=1}^{\infty} j\theta_j \in \mathscr{O}(\Omega)$. Then $u(z_j) = 0 \Rightarrow f(z_j) = j \Rightarrow f$ can not be extended across z_0.

Then we only need to find χ s.t. (2.1) and (2.2) hold. Note that $\theta_j \equiv 1$ near z_j and $\mathrm{i}\partial\overline{\partial}(\theta_j \log|z - z_j|)$ is bounded outside $\{\theta_j \equiv 1\}$. If we take χ increasing so fast, then (2.1) is OK. It also works on (2.2). ∎

2.2 L^2 extension theorem

Cartan B theorem tells us that the extension of a holomorphic function from an analytic subset to a Stein manifold always exists.

THEOREM 2.9. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ s.t. $\sup_{z\in\Omega}|z_n| \le 1$, let $\varphi \in \mathrm{Psh}(\Omega)$, and let $H = \{z_n = 0\}$. Then for every $f \in \mathscr{O}(\Omega\cap H)$ with*

$$\int_{H\cap\Omega} |f|^2 \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda_H < +\infty,$$

one can find a holomorphic extension F of f to Ω satisfying

$$\int_\Omega |F|^2 \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda \le \mathbf{C}\pi \int_{\Omega\cap H} |f|^2 \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda_H.$$

Here $\mathbf{C}$ can be chosen as: $\mathbf{C} = 1620$ (Takegoshi-Ohsawa), ... , $\mathbf{C} = 1$ (Błocki, Guan-Zhou).

THEOREM 2.10 (twisted BKN identity, Ohsawa–Takegoshi 87'). *For $u \in \mathscr{D}^{n,q}(\Omega)$,*

$$\|\sqrt{\eta}\overline{\partial}u\|^2 + \|\sqrt{\eta + c^{-1}}\,\overline{\partial}_h^* u\|^2 \ge \Big(\mathrm{i}(-\partial\overline{\partial}\eta - c\partial\eta\wedge\overline{\partial}\eta + \eta\Theta_h)\Lambda u, u\Big),$$

where $0 < \eta \in C^\infty(\Omega)$ and $0 < c \in C(\Omega)$ are bounded.

Therefore, if we assume the operator

$$\mathbf{B} := \mathrm{i}(-\partial\overline{\partial}\eta - c_1\partial\eta\wedge\overline{\partial}\eta + \eta\Theta_h)\Lambda$$

is positive definite everywhere on $\wedge^{n,q}(\Omega)$, Then, for every $v = \mathscr{D}^{n,q}(\Omega)$ with $\overline{\partial}v = 0$ and

$$\Big(\mathbf{B}^{-1}v, v\Big) < +\infty,$$

we can get a solution of $\overline{\partial}w = v$ satisfying

$$\Big\|(\eta + c_1^{-1})^{-1/2} w\Big\|^2 \le \Big(\mathbf{B}^{-1}v, v\Big).$$

Proof. See [15, pp. 115–117].

我们来算一下 $n=q=1$ 情形: 令 $V=v\,\mathrm{d}\bar{z}\wedge\mathrm{d}z\in\mathscr{D}^{1,1}(\Omega)$, 其中 $\Omega\subset\mathbb{C}$, 回忆 $\overline{\partial}_h^*V=\phi_z v-v_z$ 在这种简单情形下, 则

$$\|\sqrt{\eta}\overline{\partial}_h^*V\|^2=\left(\overline{\partial}(\eta\overline{\partial}_h^*V),V\right)_h=2\int_\Omega\left(\eta_{\bar{z}}\phi_z v\bar{v}+\eta\phi_{z\bar{z}}v\bar{v}+\eta\phi_z v_{\bar{z}}\bar{v}-\eta_{\bar{z}}v_z\bar{v}-\eta v_{z\bar{z}}\bar{v}\right)\mathrm{e}^{-\phi},$$

其中

$$\begin{aligned}\int_\Omega-\eta v_{z\bar{z}}\bar{v}\mathrm{e}^{-\phi}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}&=\int_\Omega\left(v_{\bar{z}}\eta_z\bar{v}+\eta v_{\bar{z}}\overline{(v_{\bar{z}})}-v_{\bar{z}}\eta\phi_z\bar{v}\right)\mathrm{e}^{-\phi}\\&=\int_\Omega\left(-\eta\phi_z v_{\bar{z}}\bar{v}+\eta_z v_{\bar{z}}\bar{v}\right)\mathrm{e}^{-\phi}+\|\sqrt{\eta}\partial^\star V\|^2,\end{aligned}$$

以及

$$\int_\Omega\eta_z v_{\bar{z}}\bar{v}\mathrm{e}^{-\phi}=\int_\Omega\Big(-\eta_{z\bar{z}}v\bar{v}-\eta_z v\overline{(v_z)}+\eta_z\phi_{\bar{z}}v\bar{v}\Big)\mathrm{e}^{-\phi}.$$

从而

$$\begin{aligned}\|\eta\overline{\partial}_h^*V\|^2\geq&2\int_\Omega\left(\eta\phi_{z\bar{z}}v\bar{v}-\eta_{z\bar{z}}v\bar{v}\right)\mathrm{e}^{-\phi}\\&+2\int_\Omega\left(\eta_{\bar{z}}\phi_z v\bar{v}+\eta_z\phi_{\bar{z}}v\bar{v}-\eta_{\bar{z}}v_z\bar{v}-\eta_z v\overline{(v_z)}\right)\mathrm{e}^{-\phi}\\=&\ldots+4\cdot\mathrm{Re}\int_\Omega\Big(\eta_{\bar{z}}\bar{v}(\phi_z v-v_z)\Big)\mathrm{e}^{-\phi}\\=&\ldots+4\cdot\mathrm{Re}\left(\overline{\partial}_h^*V,\partial\eta\wedge(\Lambda V)\right)_h\\\geq&\ldots-\|\sqrt{c^{-1}}\overline{\partial}_h^*V\|^2-\|\sqrt{c}\partial\eta\wedge(\Lambda V)\|^2,\end{aligned}$$

这里用到了很简单的不等式:

$$2\mathrm{Re}\,(AB)\geq-\frac{1}{C}|A|^2-C|B|^2,\quad C>0,\ A,B\in\mathbb{C}.$$

最后我们得到

$$\|\sqrt{\eta+c^{-1}}\overline{\partial}_h^*V\|^2\geq\Big(\mathrm{i}\big(\eta\Theta_h-\partial\overline{\partial}\eta-c\partial\eta\wedge\overline{\partial}\eta\big)\Lambda V,V\Big).$$

■

Proof of Theorem 2.9. We may assume Ω is bounded and φ is smooth up to the boundary of D first (This is because we can approximate Ω with relatively compact pseudoconvex Ω_j and approximate φ with smooth psh φ_k).

Let $\widetilde{f}=\mathrm{pr}^*(f\,\mathrm{d}z_1\wedge\ldots\wedge\mathrm{d}z_{n-1})$ be the constant extension, and θ a cut-off function such that $\theta=1$ near $\Omega\cap H$. For example, $\theta=\alpha(\log|z_n|^2)$, where (fix $t<0$ first)

$$\alpha(x)=\begin{cases}1&x\in(-\infty,-t-1),\\0&x\in(-t,0).\end{cases}$$

Let $v=\overline{\partial}(-\theta\widetilde{f})=-\widetilde{f}\overline{\partial}\theta$ and let $\eta,c>0$ and $\phi\in\mathrm{Psh}(\Omega)$ with

$$\eta(z)=s\big(-\beta(\psi)\big),\quad\phi(z)=u\big(-\beta(\psi)\big),\quad c(z)=r\big(-\beta(\psi)\big),$$

where $\psi = \log|z_n|^2$, such that

1. β is smooth, increasing and convex, $\beta(x) = x$ for x far from $-\infty$, and $\beta(x)$ =constant near $-\infty$, that is
$$\beta(x) = \begin{cases} x & x \in (-t, 0), \\ \text{constant} & x \in (-\infty, -t-1). \end{cases}$$
For example
$$\beta(x) = \int_0^x (1-\alpha(y))\,\mathrm{d}y;$$
2. and ($h = \mathrm{e}^{-\psi-\phi-\varphi}$ here need not be positive definite) for some μ,
$$\mathrm{i}(\eta\Theta_h - \partial\overline{\partial}\eta - c\partial\eta \wedge \overline{\partial}\eta) \geq \mathrm{i}c_2\partial\mu \wedge \overline{\partial}\mu > 0.$$

Then we can solve $\overline{\partial}w = v = \overline{\partial}(-\theta\widetilde{f})$ with L^2 estimates and let $\widetilde{F} = w + \theta\widetilde{f} \in \mathscr{O}(\Omega)$.

Direct calculation: ($\triangle = \beta(\psi)$ for simplicity)
$$\partial\overline{\partial}\eta = -s'(-\triangle)\partial\overline{\partial}(\triangle) + s''(-\triangle)\partial(\triangle) \wedge \overline{\partial}(\triangle),$$
and
$$\partial\overline{\partial}\phi = -u'(-\triangle)\partial\overline{\partial}(\triangle) + u''(-\triangle)\partial(\triangle) \wedge \overline{\partial}(\triangle),$$
($\mathrm{i}\partial\overline{\partial}(\psi+\varphi) \geq 0$) $\Rightarrow$
$$\begin{aligned}&\mathrm{i}(\eta\Theta_h - \partial\overline{\partial}\eta - c\partial\eta \wedge \overline{\partial}\eta)\\ \geq&\mathrm{i}(u''s - s'' - rs'^2)(-\triangle)\partial(\triangle) \wedge \overline{\partial}(\triangle) + \mathrm{i}(s' - su')(-\triangle)\partial\overline{\partial}(\triangle)\\ =& \ldots + (s' - su')(-\triangle)\Big(\beta'(\psi)\mathrm{i}\partial\overline{\partial}\psi + \beta''(\psi)\mathrm{i}\partial\psi \wedge \overline{\partial}\psi\Big).\end{aligned}$$

We hope:

(a) $u''s - s'' - rs'^2 = 0$;

(b) $s' - su' = 1$.

May assume $\beta'' \approx 1$ and $-\alpha' \approx 1$ on an interval with length 1 (e.g. $\alpha'(x) = -1$ on $x \in (-t-1+\varepsilon, -t-\varepsilon)$ for ε small enough). We then have $-\overline{\partial}\theta \approx \overline{\partial}\psi$ and thus (let $\varepsilon \to 0^+$)
$$\begin{aligned}\int_\Omega (\eta + c^{-1})^{-1}|\widetilde{F} - \theta\widetilde{f}|^2\mathrm{e}^{-\psi-\phi-\varphi} &\leq \int_{\{-t-1<\psi<-t\}} |\widetilde{f}|^2\mathrm{e}^{-\psi-\phi-\varphi}\\ &\leq \left(\sup_{\{-t-1<\psi<-t\}} \mathrm{e}^{-\phi}\right)\int_{\{-t-1<\psi<-t\}} |\widetilde{f}|^2\mathrm{e}^{-\psi-\varphi}.\end{aligned} \tag{2.3}$$

We hope:

(c) $(\eta + c^{-1})^{-1}\mathrm{e}^{-\phi} = \mathrm{e}^{\beta(\psi)} \geq \mathrm{e}^\psi = |z_n|^2$, i.e.
$$(s + r^{-1})^{-1}\mathrm{e}^{-u} = \mathrm{e}^{-t};$$

(d) $\lim_{t\to+\infty} u(t) = 0$.

Then do approximations (let $t \to +\infty$ next) to get

$$\int_\Omega |F|^2 \mathrm{e}^{-\varphi} \le \mathbf{C} \liminf_{t\to+\infty} \int_{\{-t-1<\psi<-t\}} |\widetilde{f}|^2 \mathrm{e}^{-\psi-\varphi} = \mathbf{C}\pi \int_{\Omega\cap H} |f|^2 \mathrm{e}^{-\varphi}\, \mathrm{d}\lambda_H,$$

where

$$\mathbf{C} := \lim_{t\to+\infty} \left(\sup_{\{-t-1<\psi<-t\}} \mathrm{e}^{-\phi} \right) = 1.$$

Finally, we solve the ODE system

$$\begin{cases} u''s - s'' - rs'^2 = 0, \\ s' - su' = 0, \\ (s + r^{-1})^{-1}\mathrm{e}^{-u} = \mathrm{e}^{-t}. \end{cases}$$

In fact:

$$s(t) = \frac{t}{1-\mathrm{e}^{-t}} - 1, \quad u(t) = -\log(1-\mathrm{e}^{-t}), \quad r = \frac{u''s - s''}{s'^2} = \frac{1}{\mathrm{e}^{t-u} - s} = \frac{1-\mathrm{e}^{-t}}{\mathrm{e}^t - t - 1}.$$

The constant $\mathbf{C} = 1$ is the optimal constant.

实际上严格来说需要很多个逼近步骤: 例如区域的逼近, 权函数的光滑化逼近, ε 的趋于 0, 以及 t 的趋于无穷. 所以 Stein 流形上的证明里出现过四个指标依次取极限, 而弱拟凸 Kähler 流形上甚至在五个以上. ■

For general $\psi \in \mathrm{Psh}(\Omega)$ with pole:

THEOREM 2.11 (L^2 estimate part). *Fix $B > 0$. Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with $o \in \Omega$, and $\psi \in \mathrm{Psh}^-(\Omega)$ with $\psi(o) = -\infty$. Let $F \in A^2(\{\psi < -t_0\})$, where $t_0 \ge 0$. Then there exists a holomorphic function $\widetilde{F} \in \mathscr{O}(\Omega)$, s.t.*

$$|\widetilde{F} - F|^2 \mathrm{e}^{-\psi} \text{ is locally } L^1 \text{ integrable near } o,$$

and

$$\int_\Omega \left| \widetilde{F} - (1 - b_{t_0,B}(\psi))F \right|^2 \le \mathbf{C} \int_\Omega \frac{1}{B} \mathbb{I}_{\{-t_0-B<\psi<-t_0\}} |F|^2 \mathrm{e}^{-\psi}, \quad \mathbf{C} = 1 - \mathrm{e}^{-(t_0+B)},$$

where $\mathbb{I}_{\{-t_0-B<\psi<-t_0\}}$ is the character function of the set $\{-t_0 - B < \psi < -t_0\}$, and $b_{t_0,B}(t) = \int_{-\infty}^t \frac{1}{B} \mathbb{I}_{\{-t_0-B<s<-t_0\}}\, \mathrm{d}s$.

Proof. ($B = 1$ case) Replace $\log|z_n|^2$ by ψ in the above proof. We will get

$$\int_\Omega |\widetilde{F} - (1 - b_{t_0}(\psi))F|^2 \mathrm{e}^{-\psi+\beta(\psi)} \le \left(\sup_\Omega \mathrm{e}^{-\phi} \right) \int_{\{-t_0-1<-\psi<-t_0\}} |F|^2 \mathrm{e}^{-\psi}.$$

Then $|\widetilde{F} - F|^2 \mathrm{e}^{-\psi}$ is locally L^1 integerable near o, and the estimate holds by $\beta(x) \ge x$. ■

THEOREM 2.12 (A twisted version). *Let D be a pseudoconvex domain in $\mathbb{C}^n$ s.t. $\sup_{z\in D}|z_n| \le 1$, let $\varphi \in \mathrm{Psh}(D)$, and let $H = \{z_n = 0\}$.*

Suppose $c(t)\colon [0,+\infty) \to (0,+\infty)$ is smooth s.t. $c(t)\mathrm{e}^{-t}$ is decreasing in t, and $\int_0^{+\infty} c(t)\mathrm{e}^{-t}\,\mathrm{d}t < +\infty$.

Then for every $f \in \mathscr{O}(D\cap H)$ with

$$\int_{D\cap H} |f|^2 \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda < +\infty,$$

one can find a holomorphic extension F of f to D satisfying

$$\int_D c(-2\log|z_n|)|F|^2\mathrm{e}^{-\varphi}\,\mathrm{d}\lambda \le \pi\int_0^{+\infty} c(t)\mathrm{e}^{-t}\,\mathrm{d}t\int_{D\cap H}|f|^2\mathrm{e}^{-\varphi}\,\mathrm{d}\lambda_H.$$

Proof. We need

$$(\eta + c^{-1})^{-1}\mathrm{e}^{-\phi} = c(\beta(\psi))\mathrm{e}^{\beta(\psi)}, \quad \text{that is} \quad (s + r^{-1})^{-1}\mathrm{e}^{-u} = c(t)\mathrm{e}^{-t}.$$

See [10]. Solve the ODE system:

$$\begin{cases} \left(s + \dfrac{s'^2}{u''s - s''}\right)\mathrm{e}^{u-t} = \dfrac{1}{c(t)}, \\ s' - su' = 1. \end{cases}$$

We get

$$u(t) = -\log\left(\int_0^t c(t_1)\mathrm{e}^{-t_1}\,\mathrm{d}t_1\right),$$

and

$$s(t) = \frac{\int_0^t \left(\int_0^{t_2} c(t_1)\mathrm{e}^{-t_1}\,\mathrm{d}t_1\right)\mathrm{d}t_2}{\int_0^t c(t_1)\mathrm{e}^{-t_1}\,\mathrm{d}t_1}.$$

Also, we let $r = (u''s - s'')/s'^2$.

实际上 c 只需要满足正, 连续, 积分有限, 以及

$$\left(\int_0^t c(t_1)\mathrm{e}^{-t_1}\,\mathrm{d}t_1\right)^2 > c(t)\mathrm{e}^{-t}\left(\int_0^t\left(\int_0^{t_2} c(t_1)\mathrm{e}^{-t_1}\,\mathrm{d}t_1\right)\mathrm{d}t_2\right), \quad \forall\, t > 0.$$

∎

2.2.1 Applications of the L^2 extension theorem

Let $D = \{z\colon \rho(z) < 0\} \subset \mathbb{C}^n$ be a bounded strongly pseudoconvex domain with C^2 boundary, where ρ is C^2 and strictly psh on an open neighborhood Ω of $\overline{D}$. Let $\delta(z) = \mathrm{dist}(z, \partial D)$ for $z \in D$.

Let $\varphi \in \mathrm{Psh}(D) \cap C^0(\overline{D})$. Let

$$A^2_{\alpha,\varphi}(D) = \left\{ f \in \mathscr{O}(D)\colon \int_D \mathrm{e}^{-\varphi}\delta^\alpha |f|^2\,\mathrm{d}\lambda < +\infty \right\},$$

and

$$\|f\|^2_{\alpha,\varphi} = (\alpha+1)\int_D \mathrm{e}^{-\varphi}\delta^\alpha |f|^2 \,\mathrm{d}\lambda.$$

Let

$$A^2_{-1,\varphi}(D) = \Big\{ f\in \mathscr{O}(D)\colon \lim_{\alpha\to -1^+}(1+\alpha)\int_D \mathrm{e}^{-\varphi}\delta^\alpha |f|^2\,\mathrm{d}\lambda < +\infty \Big\},$$

and

$$\|f\|^2_{-1,\varphi} = \lim_{\alpha\to -1^+}(1+\alpha)\int_D \mathrm{e}^{-\varphi}\delta^\alpha |f|^2\,\mathrm{d}\lambda.$$

Let $H=\{h=0\}$ be a smooth complex hypersurface in Ω, and $\mathrm{d}h|_z\neq 0$ for every $z\in H\cap D$.

Assume ∂D intersects with H transversally ([10] says this condition can be dropped, but I do not think so).

THEOREM 2.13 (Beatrous, Diederich–Herbot, Ohsawa, Guan–Zhou). *The extension operator:*

$$A^2_{\alpha+1,\varphi}(D\cap H) \longrightarrow A^2_{\alpha,\varphi}(D)$$

for every $\alpha>-1$ has a bound $C_0\max\{C_1^\alpha, C_2^\alpha\}$, where $C_0, C_1, C_2>0$ are independent of α.

Consequently, the extension operator from $A^2_{0,\varphi}(D\cap H)$ to $A^2_{-1,\varphi}(D)$ is bounded.

Proof. See [10, p. 60] or [8].

Since $\rho\in C^2(\Omega)\cap \mathrm{SPsh}(\Omega)$, and ∂D is C^2, $-\rho/\delta$ has uniform positive upper and lower bound on D.

Replace δ by $-\rho$. Let

$$\psi = -\log\left(-\frac{\rho}{\varepsilon_0|h|^2}+1\right) < 0,$$

where $\varepsilon_0>0$ is sufficiently small s.t. $\rho-\varepsilon_0|h|^2$ is strictly psh in a neighborhood of D. Since $-\log(-t)$ is increasing and convex for $t<0$, we have $-\log(-\rho+\varepsilon_0|h|^2)\in \mathrm{Psh}(D)\Rightarrow \psi\in\mathrm{Psh}(D)$.

Let

$$c(t) = \begin{cases} t^\alpha & 0<t<1,\\ 1 & t\geq 1.\end{cases}$$

Then $\int_0^{+\infty} c(t)\mathrm{e}^{-t}\,\mathrm{d}t < \dfrac{1}{1+\alpha}+1$. Let

$$\phi = -\alpha\log(-\rho+\varepsilon_0|h|^2) \quad\Rightarrow\quad \phi+\psi\in\mathrm{Psh}(D).$$

For this case we can still use the twisted L^2 extension theorem. We may see h as z_n in the above theorem. The functions

$$\mathrm{e}^{-\psi} \approx \begin{cases} (\varepsilon_0|h|^2)^{-1} & z\to H\setminus\partial D,\\ -\rho/\varepsilon_0|h|^2 & z\to\partial D,\end{cases}$$

and

$$\mathrm{e}^{-\phi} = \big(-\rho+\varepsilon_0|h|^2\big)^\alpha \approx \begin{cases} (\varepsilon_0|h|^2)^\alpha & z\to H\setminus\partial D,\\ (-\rho)^\alpha & z\to \partial D\setminus H.\end{cases}$$

If ∂D intersects with H transversally, then we are able to omit the areas near $\partial D\cap H$.

We will have the following estimate of the extension $F|_{D\cap H} = f$ (denote a local holomorphic extension of f by f itself):

$$\int_D c(-\psi)|F|^2 \mathrm{e}^{-\phi-\varphi}\,\mathrm{d}\lambda \le C_{D,H}\frac{2+\alpha}{1+\alpha}\liminf_{t\to+\infty}\int_{\{-t-1<\psi<-t\}}|f|^2\mathrm{e}^{-\psi-\phi-\varphi}\,\mathrm{d}\lambda$$
$$\le C_{D,H}C_0^\alpha\frac{1}{1+\alpha}\int_{D\cap H}\mathrm{e}^{-\varphi}(-\rho)^{\alpha+1}|f|^2\,\mathrm{d}\lambda_H.$$

Finally, note that

$$\begin{aligned}\frac{c(-\psi)\mathrm{e}^{-\phi}}{(-\rho)^\alpha}\Big|_{\{\psi>-1\}} &= \left(\log\left(1+\frac{-\rho}{\varepsilon_0|h|^2}\right)\right)^\alpha\cdot\left(1+\frac{\varepsilon_0|h|^2}{-\rho}\right)^\alpha\\ &\ge\begin{cases}C_3^{-\alpha} & z\to\partial D\setminus H,\\ C_4^{-\alpha} & z\to\partial D\cap H,\end{cases}\\ &\ge\min\{C_3^{-\alpha},C_4^{-\alpha}\},\end{aligned}$$

Thus, we have

$$\int_D \mathrm{e}^{-\varphi}(-\rho)^\alpha|F|^2\,\mathrm{d}\lambda\le C\max\{C_1^\alpha,C_2^\alpha\}\frac{1}{1+\alpha}\int_{D\cap H}\mathrm{e}^{-\varphi}(-\rho)^{\alpha+1}|f|^2\,\mathrm{d}\lambda_H.$$

Finally, we can replace $-\rho$ by δ. ■

Let $\Gamma\subseteq\mathbb{C}$ be a discrete set of points such that

$$\inf\{|\gamma-\gamma'|\colon \gamma,\gamma'\in\Gamma,\ \gamma\ne\gamma'\}>0. \tag{2.4}$$

We say that the interpolation problem is solvable for Γ if for any collection $\{a_\gamma\in\mathbb{C}\colon \gamma\in\Gamma\}$ of complex numbers such that

$$\sum_{\gamma\in\Gamma}|a_\gamma|^2\mathrm{e}^{-|\gamma|^2}<+\infty,$$

there is an entire function f such that

$$f(\gamma)=a_\gamma \quad\text{and}\quad \int_{\mathbb{C}}|f(z)|^2\mathrm{e}^{-|z|^2}\,\mathrm{d}\lambda(z)<+\infty.$$

THEOREM 2.14 (Seip). *The followings are equivalent.*

1. *The discrete set* Γ *satisfies* (2.4), *and*

$$D(\Gamma)\coloneqq\limsup_{R\to+\infty}\sup_{z\in\mathbb{C}}\frac{\#\{\gamma\in\Gamma\colon |z-\gamma|<R\}}{R^2}<1.$$

2. *The interpolation problem is solvable for* Γ.

Proof. (Berndtsson–Ortega Cerdà) We prove $1\Rightarrow 2$. Let $R>0$, and

$$\psi(z)=V_R(z)\coloneqq\sum_{\gamma\in\Gamma}\left(\log|z-\gamma|^2-\frac{1}{\pi R^2}\int_{\{|\zeta-z|<R\}}\log|\zeta-\gamma|^2\,\mathrm{d}\lambda(\zeta)\right).$$

Then

1. V_R is well-defined and negative;
2. In the sense of currents,
$$\mathrm{i}\partial\overline{\partial}V_R(z) \geq -\frac{\#\{\gamma \in \Gamma \colon |z-\gamma| < R\}}{R^2}\,\mathrm{d}\lambda(z);$$
3. e^{-V_R} is not locally integrable at each $\gamma \in \Gamma$;
4. There is a constant $C_{R,\varepsilon} > 0$ such that if $\gamma \in \Gamma$ and $\varepsilon/2 < |z-\gamma| < \varepsilon$ then $V_R(z) \geq -C_{R,\varepsilon}$.

■

THEOREM 2.15 (Guan 19'). *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with $o \in \Omega$, and $\psi \in \mathrm{Psh}^-(\Omega)$ with $\psi(o) = -\infty$. Assume the holomorphic germ $(F, o) \in \mathscr{O}_o \setminus \mathscr{I}(\psi)_o$. The minimal L^2 integral on any domain $D \ni o$ is defined as*
$$C_{F,\psi}(D) = C_{F,\mathscr{I}(\psi)_o}(D) \coloneqq \Big\{ \int_D |\widetilde{F}|^2 \colon (\widetilde{F} - F, o) \in \mathscr{I}(\psi)_o \ \&\ \widetilde{F} \in \mathscr{O}(D) \Big\}.$$

Then the function
$$(0, +\infty) \ni t \longrightarrow G(t) \coloneqq C_{F,\psi}\big(\Omega \cap \{\psi < -t\}\big)$$
satisfies that $G(-\log r)$ is concave in $r \in [0, 1)$.

Proof. See [7].

Use only mathematical analysis and real analysis. However, the computation is a bit complicated, so we introduce Berndtsson's log-psh of fiberwise Bergman kernels, which can be directly implied by the optimal L^2 extension theorem, fewer calculations are used, and usually have similar efforts. ■

COROLLARY 2.16 (Berndtsson 06'). *Let Ω be a pseudoconvex domain in $\mathbb{C}^{n+m}$ over a pseudoconvex domain $D \subset \mathbb{C}^m$, and pr is the projection. Let $\varphi \in \mathrm{Psh}(\Omega)$. Denote the Bergman kernel on each fiber $\Omega_w = \mathrm{pr}^{-1}(w) \cap \Omega$ w.r.t. the weight $\varphi|_{\Omega_w}$ by*
$$K_\varphi(z, w) \coloneqq K_{\Omega_w, \mathrm{e}^{-\varphi|_{\Omega_w}}}(z) = \sup_{f \in \mathscr{O}(\Omega_w)} \frac{|f(z)|^2}{\int_{\Omega_w} |f|^2 \mathrm{e}^{-\psi|_{\Omega_w}}}.$$

Then $\log K_\varphi(z, w) \in \mathrm{Psh}(\Omega)$.

Proof. See [15, pp. 191–192].

(Guan–Zhou method) We only prove the mean value inequality on the direction of w. Assume $D = \Delta$ the unit disc in $\mathbb{C}$. Let $f \in \mathscr{O}(\Omega_0)$ such that
$$K_\varphi(z_0, 0) = \frac{|f(z_0)|^2}{\int_{\Omega_0} |f(z)|^2 \mathrm{e}^{-\psi(z)}}.$$

Using the optimal L^2 extension theorem, we can find some $F(z, w) \in \mathscr{O}(\Omega)$, such that $F(z, 0) = f(z)$ and
$$\int_\Omega |F|^2 \mathrm{e}^{-\varphi} \leq \pi \int_{\Omega_0} |f|^2 \mathrm{e}^{-\varphi}.$$

Then apply Jensen's inequality to the increasing convex function $y = \log x$:

$$
\begin{aligned}
\log\left(\int_{\Omega_0} |f|^2 \mathrm{e}^{-\varphi}\right) &\geq \log\left(\frac{1}{\pi}\int_{\Omega} |F|^2 \mathrm{e}^{-\varphi}\right) \\
&= \log\left(\frac{1}{\pi}\int_{w\in\Delta}\left(\int_{\Omega_w} |F_w|^2 \mathrm{e}^{-\varphi}\right)\mathrm{d}\lambda_w\right) \\
&\geq \frac{1}{\pi}\int_{w\in\Delta}\log\left(\int_{\Omega_w} |F_w|^2 \mathrm{e}^{-\varphi}\right)\mathrm{d}\lambda_w \\
&\geq \frac{1}{\pi}\int_{w\in\Delta}\Big(\log|F_w(z_0)|^2 - \log K_\varphi(z_0,w)\Big)\mathrm{d}\lambda_w \\
&= \frac{1}{\pi}\int_{w\in\Delta}\log|F(z_0,w)|^2 - \frac{1}{\pi}\int_{w\in\Delta}\log K_\varphi(z_0,w) \\
&\geq \log|F(z_0,0)|^2 - \frac{1}{\pi}\int_{w\in\Delta}\log K_\varphi(z_0,w) \\
&= \log|f(z_0)|^2 - \frac{1}{\pi}\int_{w\in\Delta}\log K_\varphi(z_0,w),
\end{aligned}
$$

where $F_w(\,\cdot\,) = F(\,\cdot\,,w)$, and $\log|F_w(z_0)|^2$ is subharmonic in w. Thus,

$$
\log K_\varphi(z_0,0) \leq \frac{1}{\pi}\int_{w\in\Delta}\log K_\varphi(z_0,w).
$$

Berndtsson 最初的证明方法是先在光滑情形直接使用再生核公式计算二阶导, 最后结合 Hörmander L^2 估计. 之后这套方法发展为了他所提出的复 Brunn–Minkowski 理论, 并在 direct image of relative canonical bundle 的正性上有重要应用. ■

Let D be a domain in the complex plane containing the origin. Let $G(z) \coloneqq G(z,0)$ be the Green function on D with pole at 0. Then $2G(z) = \log|z|^2 - h(z)$, where h is a harmonic function such that G vanishes on the boundary of D. Then $c_D \coloneqq h(0)$ is the Robin constant at 0, and

$$
c_\beta^2(0) = \mathrm{e}^{-c_D} \coloneqq \exp\lim_{z\to 0}\Big(2G(z) - \log|z|^2\Big),
$$

where c_β is called the logarithm capacity.

Corollary 2.17 (Błocki, Guan–Zhou). *The inequality part of Suita's conjecture says that*

$$
\pi K(0) \geq \mathrm{e}^{-c_D},
$$

where $K(0)$ is the Bergman kernel at 0.

Proof. See [1, p. 2].

(Berndtsson–Lempert) Let $D_t = \{z\in D\colon 2G(z) < -t\}$, and

$$
\Omega \coloneqq \Big\{(z,w)\in D\times[0,+\infty)+\mathrm{i}\mathbb{R}\colon 2G(z)+\operatorname{Re}w<0\Big\}\subseteq\mathbb{C}^2,
$$

which is a pseudoconvex domain. Then $\log K(0,w)$ is subharmonic in w and only depends on $\operatorname{Re} w$ $\Rightarrow \log K_{D_t}(0) = \log K(0, t+\mathrm{i}\,\cdot\,)$ is convex in $t\in[0,+\infty)$. Using the definition of c_D, we can find

$$
\Delta_{r_1}\subseteq D_t\subseteq\Delta_{r_2}
$$

for $r_1 = \mathrm{e}^{(-t+c_D-\varepsilon)/2}$ and $r_2 = \mathrm{e}^{(-t+c_D+\varepsilon)/2}$ if t is sufficiently large. It follows that

$$K_t \sim \frac{\mathrm{e}^{t-c_D}}{\pi}, \quad t \to +\infty.$$

Thus, $\log K_{D_t}(0) - t$ is convex and bounded from above in $t \in [0, +\infty)$, which implies the function is decreasing $\Rightarrow$

$$K_D(0) \geq \lim_{t\to+\infty} \mathrm{e}^{-t} K_{D_t}(0) + t = \frac{\mathrm{e}^{-c_D}}{\pi}.$$

Suita 猜想在开黎曼面上也是成立的, 并且等号成立当且仅当 D 全纯同构于 Δ 去掉某个容度为 0 的集合 (Guan–Zhou). 这里等号成立条件是更困难得多的, 需要用到后面提到的 twisted version 的 optimal L^2 延拓定理. 而更广范围的 optimal L^2 延拓定理的等号成立条件也是后续利用极小 L^2 积分凹性研究的主要内容之一. ■

Let $D \subseteq \mathbb{C}$ be a planar regular region bounded with n boundary components which are analytic Jordan curves. Let $H_2^{(c)}(D)$ denote the analytic Hardy class on D as the set of all analytic functions $f(z)$ on D s.t.

$$|f(z)|^2 \leq U(z) \text{ on } D, \quad \exists U \text{ harmonic}.$$

Then each $f \in H_2^c(D)$ has Fatou's nontangential boundary value a.e. on ∂D belonging to $L^2(\partial D)$.

The conjugate Hardy H^2 kernel on D is the unique kernel function $\widehat{R}_t(z, \overline{w})$ satisfying

- for any fixed $(w, t) \in D \times D$, as a function of z, $\widehat{R}_t(z, \overline{w}) \in H_2^{(c)}(D)$;
- for any $f \in H_2^{(c)}(D)$,

$$f(w) = \frac{1}{2\pi} \int_{\partial D} f(z) \overline{\widehat{R}_t(z, \overline{w})} \left(\frac{\partial G_D(z,t)}{\partial \nu_z} \right)^{-1} \mathrm{d}|z|, \quad \forall (z, w, t) \in D \times D \times D,$$

  where $G_D(z, t)$ is the Green function on D and $\frac{\partial}{\partial \nu_z}$ denotes the derivative along the outer normal unit vector ν_z.

Or equivalently, $M(z) := \dfrac{\widehat{R}_t(z, \overline{w})}{\widehat{R}_t(w, \overline{w})} \in H_2^{(c)}(D)$ is the uique solution of the following extremal problem:

$$\inf \Big\{ \int_{\partial D} |f(z)|^2 \left(\frac{\partial G_D(z,t)}{\partial \nu_z} \right)^{-1} \mathrm{d}|z| \colon f \in H_2^{(c)}(D),\ f(w) = 1 \Big\}.$$

Let

$$\widehat{R}(z, \overline{w}) := \widehat{R}_w(z, \overline{w}), \quad \text{and} \quad \widehat{R}(z) := \widehat{R}(z, \overline{z}).$$

Let $B(z, \overline{w})$ denote the Bergman kernel on D and $B(z) := B(z, \overline{z})$.

THEOREM 2.18 (Saitoh's conjecture; Guan 19'). *If $n > 1$, then*

$$\widehat{R}(z) > \pi B(z).$$

Proof. Verify for every $f \in H_2^{(2)}(D)$, and $w \in D$,

$$\lim_{t\to 0^+} \frac{\int_{\{2G_D(\,\cdot\,,w)\geq -t\}} |f|^2}{t} = \int_{\partial D} |f(z)|^2 \left(\frac{\partial 2G_D(z,w)}{\partial \nu_z}\right)^{-1} \mathrm{d}|z|,$$

which implies

$$\lim_{t\to 0^+} \frac{B_D^{-1}(w) - B^{-1}_{\{2G_D(\,\cdot\,,w)<-t\}}(w)}{t} \geq \frac{\pi}{\widehat{R}(w)}, \quad \forall\, t > 0.$$

Since the function

$$\log B(t) - t := \log B_{\{2G_D(\,\cdot\,,w)<-t\}}(w) - t$$

is convex and decreasing in $t \in (0,+\infty)$, we have

$$-1 + B(0) \lim_{t\to 0^+} \frac{B^{-1}(0) - B^{-1}(t)}{t} = \lim_{t\to 0+} \frac{\log B(t) - t - \log B(0)}{t} \leq 0,$$

and thus

$$\widehat{R}(w) \geq \pi B(z).$$

For $n > 1$, the equality will not hold: if the equality hold, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\big(\log B(t) - t\big)\Big|_{t=0^+} = 0,$$

so $\log B(t) - t$ is a constant function, and then it follows from the equality part of Suita's conjecture. ■

THEOREM 2.19 (Guan 19'). *Let $D \subsetneq \mathbb{C}^n$ be a pseudoconvex domain containing the origin o, and $\varphi \in \mathrm{Psh}^-(D)$ with $\varphi(o) = -\infty$. If $\int_D \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda < +\infty$, then for every $p > 1$ satisfying*

$$\frac{p}{p-1} > K_D(o) \int_D \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda,$$

we have $\mathrm{e}^{-p\varphi}$ is locally L^1 integrable near o.

Proof. (An optimized version of Berndtsson's proof of the openness conjecture) Let $\Phi = q\varphi$ for some $q > 1$ and $D_t = \{z \in D \colon \Phi(z) < -t\}$ for every $t \geq 0$. Using the same arguments in the above proof, we have that $\log K_{D_t}(o) - t$ is convex in t, and we can use Theorem 2.11 (only a rough version and very few calculations are needed) to get that $\log K_{D_t}(o) - t$ is also upper bounded if $(1,o) \notin \mathscr{I}(\Phi/2)_o$. Thus, $\log K_{D_t}(o) - t$ is decreasing in t, that is

$$\mathrm{e}^{-t} K_{D_t}(o) \leq K_D(o) \quad \Rightarrow \quad K_{D_t}^{-1}(o) \geq \mathrm{e}^{-t} K_D^{-1}(o), \quad \forall\, t > 0.$$

Using Fubini's theorem, when $(1,o) \notin \mathscr{I}(\Phi/2)_o$, i.e. $\mathrm{e}^{-q\varphi}$ is not L^1 integrable near o, we have

$$\begin{aligned}
\int_D \mathrm{e}^{-\varphi}\,\mathrm{d}\lambda = \int_D \mathrm{e}^{-\frac{\Phi}{q}}\,\mathrm{d}\lambda &= \int_{-\infty}^{+\infty} \left(\int_{\{\Phi<-qt\}} |1|^2\right) \mathrm{e}^t\,\mathrm{d}t \\
&\geq \lambda(D) + \int_0^{+\infty} K_{D_{qt}}^{-1}(o)\,\mathrm{d}t \\
&\geq K_D^{-1}(o) + K_D^{-1} \int_0^{+\infty} \mathrm{e}^{-qt}\,\mathrm{d}t \\
&= \frac{q}{q-1} K_D^{-1}(o).
\end{aligned}$$

Consequently, if $q > 1$ satisfies

$$\frac{q}{q-1} > K_D(o) \int_D \mathrm{e}^{-\varphi}\, \mathrm{d}\lambda,$$

then we must have that $\mathrm{e}^{-q\varphi}$ is locally L^1 integrable near o (note that $\int_D \mathrm{e}^{-\varphi} > \lambda(D) \geq K_D^{-1}(o)$).

(Bao–Guan 22'; 一些记号参见后面 "Multiplier ideal sheaf" 小节) 如果条件是 $\int_D |F|^2 \mathrm{e}^{-\varphi} < +\infty$ 对某个 $F \in \mathscr{O}(D)$, 我们考虑:

$$\xi \in \ell_1^{(n)} := \Big\{ (\xi_\alpha)_{\alpha \in \mathbb{Z}^n_{\geq 0}} \colon \sum_\alpha |\xi_\alpha| \rho^{|\alpha|} < +\infty \text{ for every } \rho > 0 \Big\},$$

以及推广的 Bergman 核

$$K_{\xi,D}(z) := \sup_{f \in \mathscr{O}(D)} \frac{|(\xi \cdot f)(z)|^2}{\int_D |f|^2},$$

其中

$$(\xi \cdot f)(z) := \sum_\alpha \xi_\alpha \frac{f^{(\alpha)}(z)}{\alpha!}.$$

则我们能得到类似的 fiberwise 的多次调和性, 并同样推出 $k(t) := \log K_{\xi,D_t}(o) - t$ 的凸性. 同样地, 由定理 2.11 可以得到如果

$$\xi \in \ell_{\mathscr{I}(\Phi)_o} := \{ \eta \in \ell_1^{(n)} \colon (\eta \cdot f)(o) = 0,\ \forall (f, o) \in \mathscr{I}(\Phi/2)_o \},$$

则 $k(t)$ 上有界, 并因此推出 $k(t)$ 的递减性. 同样我们最终会得到如果 $p > 1$ 满足

$$\frac{p}{p-1} > \frac{\int_D |F|^2 \mathrm{e}^{-\varphi}}{B(F, \mathscr{I}_+(c_o^F(\varphi)\varphi)_o, D)},$$

其中

$$B(F, \mathscr{I}(c_o^F(\varphi)\varphi)_o, D) := \sup_{\xi \in \ell_{\mathscr{I}_+(c_o^F(\varphi)\varphi)_o}} \frac{|(\xi \cdot F)(o)|^2}{K_{\xi,D}(o)},$$

则必有 $|F|^2 \mathrm{e}^{-p\varphi}$ 在 o 附近 L^1 可积. 最后为了保证良定性, 我们只需证明 $B(F, \mathscr{I}_+(c_o^F(\varphi)\varphi)_o, D) > 0$, 即, 若 $(F, o) \in \mathscr{O}_o \setminus I$, 其中 I 为 $\mathscr{O}_o$ 的一个理想, 则必存在 $\xi \in \ell_I$ 使得 $(\xi \cdot F)(o) \neq 0$ 即可 (e.g. 用泛函分析中的分离性定理). 更进一步地, 实际上我们可以证明

$$B(F, \mathscr{I}(c_o^F(\varphi)\varphi)_o, D) = C_{F, \mathscr{I}(c_o^F(\varphi)\varphi)_o}(D) > 0.$$

最终我们就会得到强开性猜想的一个证明, 并且包含一个 sharp 的有效性结果. ∎

THEOREM 2.20. *Suppose* $p \in (0, 2]$. *Let* Ω *be a pseudoconvex domain in* $\mathbb{C}^n$ *s.t.* $\sup_{z \in \Omega} |z_n| \leq 1$, *let* $\varphi \in \mathrm{Psh}(\Omega)$, *and let* $H = \{z_n = 0\}$. *Then for every* $f \in \mathscr{O}(D \cap H)$ *with*

$$\int_{H \cap \Omega} |f|^p \mathrm{e}^{-\varphi}\, \mathrm{d}\lambda < +\infty,$$

one can find a holomorphic extension F *of* f *to* Ω *satisfying*

$$\int_\Omega |F|^p \mathrm{e}^{-\varphi}\, \mathrm{d}\lambda \leq C\pi \int_{\Omega \cap H} |f|^p \mathrm{e}^{-\varphi}\, \mathrm{d}\lambda.$$

Here C *can be chosen as the* C *in the* L^2 *extension theorem.*

Proof. Berndtsson's iteration technique. See [10, pp. 61–62].

May assume Ω is bounded and φ is smooth up to boundary of Ω. We first take a holomorphic extension F_1 of f satisfying a rough estimate:

$$A_1 \coloneqq \int_\Omega |F_1|^p \mathrm{e}^{-\varphi} \,\mathrm{d}\lambda < +\infty.$$

Let

$$\varphi_1 \coloneqq \varphi + (2-p)\log|F_1| \in \mathrm{Psh}(\Omega).$$

Then the optimal L^2 extension theorem gives some $F_2 \in \mathscr{O}(\Omega)$ s.t. $F_2|_{D\cap H} = F_1|_{D\cap H} = f$ and

$$\int_\Omega |F_2|^2 \mathrm{e}^{-\varphi_1} \le \pi \int_{D\cap H} |f|^2 \mathrm{e}^{-\varphi_1} = \pi \int_{D\cap H} |f|^p \mathrm{e}^{-\varphi} =: A.$$

Note that

$$\left(\int_\Omega \frac{|F_2|^2}{|F_1|^{2-p}} \mathrm{e}^{-\varphi}\right)^{\frac{p}{2}} \cdot \left(\int_\Omega |F_1|^p \mathrm{e}^{-\varphi}\right)^{1-\frac{p}{2}} \ge \int_\Omega |F_2|^p \mathrm{e}^{-\varphi} =: A_2,$$

that is

$$A_2 \le A_1^{1-p/2} (\pi A)^{p/2}.$$

By iteration, we will finally get an extension F_∞ with $A_\infty \le \pi A$. ■

THEOREM 2.21 (Demailly's approximation theorem). *Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$, and let $\varphi \in \mathrm{Psh}(\Omega)$. For each $m \ge 1$, the* ***Demailly approximation*** *φ_m of φ is defined as*

$$\varphi_m(z) \coloneqq \frac{1}{2m} \log K_{2m\varphi}(z) \coloneqq \frac{1}{2m} \log \sup_{\substack{f \in A^2(\Omega, \mathrm{e}^{-2m\varphi}),\\ \|f\|_{2m\varphi}=1}} |f(z)|^2, \quad \forall z \in \Omega, \tag{2.5}$$

where $K_{m\varphi}$ denotes the Bergman kernel for the weight $e^{-2m\varphi}$ on Ω.

Then there are constants $C_1, C_2 > 0$ independent of m such that

$$\varphi(z) - \frac{C_1}{m} \le \varphi_m(z) \le \sup_{|\zeta - z|<r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}$$

for every $z \in \Omega$ and $r < \mathrm{dist}(z, \partial\Omega)$. In particular:

1. *φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$.*
2. *For every $z \in \Omega$,*
$$\nu(\varphi, z) - \frac{n}{m} \le \nu(\varphi_m, z) \le \nu(\varphi, z).$$

Proof. See [5, pp. 136–138].

First, applying the mean value inequality to $|f|^2$ for every $f \in A^2(\Omega, \mathrm{e}^{-2m\varphi})$, we get

$$\begin{aligned}
|f(z)|^2 &\le \frac{1}{\mathrm{Vol}\big(B(z,r)\big)} |f|^2 \,\mathrm{d}\lambda \\
&\le \frac{n!}{\pi^n r^{2n}} \left(\sup_{\zeta \in B(z,r)} \mathrm{e}^{2m\varphi(\zeta)}\right) \int_{B(z,r)} |f|^2 \mathrm{e}^{-2m\varphi} \,\mathrm{d}\lambda \\
&\le \frac{n!}{\pi^n r^{2n}} \left(\mathrm{e}^{2m \sup_{|\zeta - z|<r} \varphi(\zeta)}\right) \int_\Omega |f|^2 \mathrm{e}^{-2m\varphi} \,\mathrm{d}\lambda,
\end{aligned}$$

for every $z \in \Omega$ and $r < \mathrm{dist}(z, \partial\Omega)$, which implies

$$\varphi_m(z) = \frac{1}{2m} \log K_{2m\varphi}(z) \leq \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}.$$

Second, according to the Ohsawa–Takegoshi L^2 extension theorem, there exists a constant C_1', which only depends on n and $\operatorname{diam}\Omega$, such that for every $m \geq 1$ and $z \in \Omega$, there exists a holomorphic function F on Ω, satisfying $F(z) = 1$ and

$$\int_\Omega |F|^2 \mathrm{e}^{-2m\varphi} \, \mathrm{d}\lambda \leq C_1' \mathrm{e}^{-2m\varphi(z)}.$$

It follows that

$$\varphi_m(z) = \frac{1}{2m} \log K_{2m\varphi}(z) \geq \varphi(z) - \frac{\log C_1'}{2m} = \varphi(z) - \frac{C_1}{m}.$$

Third, we choose $r = \frac{1}{m}$ for m sufficiently large, and then we will find that φ_m converges to φ by the upper semi-continuity of φ and $\log(C_2 m^n)/m \to 0$ as $m \to +\infty$.

Finally, for the Lelong numbers, from the inequality we get

$$\frac{\sup_{|z-z_0|<2r} \varphi(z) + \frac{1}{m} \log \frac{C_2}{r^n}}{\log r} \leq \frac{\sup_{|z-z_0|<r} \varphi_m(z)}{\log r} \leq \frac{\sup_{|z-z_0|<r} \varphi(z) - \frac{C_1}{m}}{\log r}.$$

By letting $r \to 0^+$, we get

$$\nu(\varphi, z_0) - \frac{n}{m} \leq \nu(\varphi_m, z_0) \leq \nu(\varphi, z_0), \quad \forall\, z_0 \in \Omega.$$

■

Note that every φ_m has analytic singularities.

For a psh function φ near z_0, the Lelong number of φ at z_0 is defined to be

$$\nu(\varphi, z_0) = \liminf_{z \to z_0} \frac{\varphi(z)}{\log|z - z_0|} = \lim_{r \to 0^+} \frac{\sup_{B(z_0, r)} \varphi}{\log r}.$$

In particular, $\varphi = \log|f|$, with f holomorphic near z_0, then

$$\nu(\varphi, z_0) = \mathrm{ord}_{z_0}(f) = \sup\{k \colon D^\alpha f(z_0) = 0, \forall |\alpha| < k\}.$$

Corollary 2.22 (Siu's semi-continuity). *Let $\varphi \in \mathrm{Psh}(\Omega)$. Then*

$$E_c(\varphi) = \{z \in \Omega \colon \nu(\varphi, z) \geq c\}$$

is an analytic subset of Ω for every $c > 0$.

Proof. See [5, p. 138].

Let φ_m be the Demailly approximation of φ. Then by the above theorem, we have

$$E_c(\varphi) = \bigcap_{m \geq m_0} E_{c-n/m} E_c(\varphi_m)$$

for some $m_0 \gg 1$. For each $E_c(\varphi_m)$, we can see

$$\varphi_m = \Big\{z \in \Omega\colon \sigma_l^{(\alpha)}(z) = 0,\ \forall\, l \ \&\ |\alpha| < mc\Big\},$$

where (σ_l) is a complete orthogonal basis of $A^2(\Omega, \mathrm{e}^{-2m\varphi})$. Then every $E_{c-n/m}(\varphi_m)$ is an analytic set, so $E_c(\varphi)$ is an analytic set. ■

THEOREM 2.23 (Skoda 72'). *If $\nu(\varphi, z) < 1$, then $\mathrm{e}^{-2\varphi}$ is locally integrable near z.*

Proof. This can be proved by dynamically using L^2 extension and a simple lemma in complex analysis.

(Guan–Zhou) Assume $\varphi \in \mathrm{Psh}^-(\Delta^n)$, $\nu(\varphi, o) < 1$, and $\mathrm{e}^{-2\varphi}$ is not locally integrable near o.

By the definition of the Lelong number, we have that there exists a sequence of points $\{z_j\} \subseteq \Delta^n$ such that

$$z_j \to o \ \text{ and } \ |z_j|^2\mathrm{e}^{-2\varphi(z_j)} \to 0 \quad \text{when} \quad j \to +\infty.$$

For every z_j, using the Ohsawa-Takegoshi L^2 extension theorem, we can find some $F_j \in \mathscr{O}(\Delta^n)$ with $F_j(z_j) = 1$ and

$$\int_{\mathcal{L}_j \cap \Delta^n} |F_j|^2 \le C \int_{\Delta^n} |F_j|^2 \le C \int_{\Delta^n} |F_j|^2 \mathrm{e}^{-2\varphi} \le C_1 \mathrm{e}^{-2\varphi(z_j)},$$

where $\mathcal{L}_j$ is a 1-dimension plane across z_j and o, and C_1 only depends on the diameter of Δ^n. Since $\mathrm{e}^{-2\varphi}$ is not locally integrable near o, we have $F_j(o) = 0$.

A simple lemma For every $h \in \mathscr{O}(\Delta^n)$ with $h(0) = 0$ and $h(z) = 1$, where $z \in \Delta_{1/2}$, there is a constant $C_2 > 0$ independent of h and z, such that

$$\int_\Delta |h|^2 \,\mathrm{d}\lambda \ge \frac{C_2}{|z|^2}.$$

It follows that

$$|z_j|^2\mathrm{e}^{-2\varphi(z_j)} \ge \frac{C_2}{C_1} > 0, \quad \forall\, j \gg 1.$$

Contradiction! ■

2.3 Multiplier ideal sheaf

DEFINITION 2.24. Let $\varphi \in \mathrm{Psh}(\Omega)$. At each $z \in \Omega$, define the multiplier ideal

$$\mathscr{I}(\varphi)_z := \{f \in \mathscr{O}_{\Omega,z}\colon |f|^2\mathrm{e}^{-2\varphi} \text{ is locally integrable near } z\}.$$

Then $\mathscr{I}(\varphi)$ can be defined as an ideal sheaf of $\mathscr{O}_\Omega$.

DEFINITION 2.25. Let $\mathscr{A}$ be a sheaf of rings on a topological space X and $\mathscr{I}$ be a sheaf of modules over $\mathscr{A}$. Then $\mathscr{I}$ is said to be locally finitely generated if for every $x_0 \in X$, one can find a neighborhood U and sections $F_1, \ldots, F_q \in \mathscr{I}(U)$ such that for every $x \in U$ the stalks $\mathscr{I}_x$ is generated by the germs $F_{1,x}, \ldots, F_{q,x}$ as an $\mathscr{A}_x$-module.

Definition 2.26. Let U be an open subset of X, and $F_1, \dots, F_q \in \mathscr{I}(U)$. The kernel of the sheaf homeomorphism

$$\begin{aligned} F\colon (\mathscr{A}|_U)^{\otimes q} &\longrightarrow \mathscr{I}|_U \\ (g^1, \dots, g^q) &\longmapsto \sum_{1 \le j \le q} g^j F_{j,x}, \quad x \in U \end{aligned}$$

is a subsheaf $\mathscr{R}(F_1, \dots, F_q)$ of $(\mathscr{A}|_U)^{\otimes q}$, called the <u>sheaf of relations</u> between $F_1, \dots, F_q$.

A sheaf $\mathscr{I}$ of $\mathscr{A}$-module on X is said to be <u>coherent</u> if

1. $\mathscr{I}$ is locally finitely generated;
2. for any open subset U of X and any $F_1, \dots, F_q \in \mathscr{I}(U)$, the sheaf of relations $\mathscr{R}(F_1, \dots, F_q)$ is locally finitely generated.

Proposition 2.27 (Strong Noetherian property). *Let $\mathscr{F}$ be a coherent analytic sheaf on a complex manifold M (i.e. sheaf of modules over $\mathscr{O}_M$), and let $\mathscr{F}_1 \subset \mathscr{F}_2 \subset \dots$ be an increasing sequence of coherent subsheaves of $\mathscr{F}$. Then the sequence $(\mathscr{F}_k)$ is stationary on every compact subset of M.*

Proof. See [6, p. 90]. ■

Proposition 2.28 (Nadel 89'). *$\mathscr{I}(\varphi)$ is a coherent sheaf on Ω.*

Proof. See [5, p. 37].

Since the structure sheaf $\mathscr{O}_\Omega$ is coherent (Oka), it suffices to prove that $\mathscr{I}(\varphi)$ is a locally finitely generated sheaf. For each $x \in \Omega$, let $U \Subset \Omega$ which is a bounded pseudoconvex domain containing x, and by the strong Notherian property of coherent sheaves, the family of sheaves generated by finite subsets of $A^2(U, \mathrm{e}^{-2\varphi})$ has a maximal element on each relatively compact subset of U. Hence, $A^2(U, \mathrm{e}^{-2\varphi})$ generates a coherent ideal sheaf $\mathscr{I} \subset \mathscr{O}_U$. It is clear that $\mathscr{I} \subset \mathscr{I}(\varphi)$ on U.

In view of:

Krull's lemma Let $(R, \mathfrak{m})$ be a Noetherian local ring, and F be a finitely generated R-module. Then

$$\bigcap_{k \ge 0} \left(E + \mathfrak{m}^k\right) = E.$$

We only need to prove at any $x \in U$, we have

$$\mathscr{I}_x + \mathscr{I}(\varphi)_x \cap \mathfrak{m}_{U,x}^{s+1} = \mathscr{I}(\varphi)_x, \quad \forall\, s \in \mathbb{Z}_{\ge 0}.$$

Then the proof is very like what we did before. Let $f \in \mathscr{I}(\varphi)_x$, and let

$$\widetilde{\varphi}(z) = \varphi(z) + (n+s)\log|z-x| + |z|^2.$$

We will solve $\overline{\partial} u = \partial(\theta f)$ with the metric $\mathrm{e}^{-2\widetilde{\varphi}}$ and let $F = -u + \theta f$, which is holomorphic and

$$f_x - F_x = u_x \in \mathscr{I}(\varphi)_x \cap \mathfrak{m}_{U,x}^{s+1}.$$

This proves the coherence. ■

Example 2.29. If φ is a psh function near $0 \in \mathbb{C}$, then

$$\mathscr{I}(\varphi)_0 = (z^k)_0$$

for some $k \in \mathbb{N}$, and k can be determined by $\nu(\varphi, 0) \in [k, k+1)$.

Proposition 2.30 (Restriction formula). *Let H be a complex hyperplane, $Y = H \cap D$, and $\varphi \in \mathrm{Psh}(D)$. Then*

$$\mathscr{I}(\varphi|_Y) \subset \mathscr{I}(\varphi)|_Y.$$

Proof. Direct consequence of O-T L^2 extension theorem. ■

Proposition 2.31 (Subadditive property).

- *Let $\varphi_1 \in \mathrm{Psh}(D_1)$, $\varphi_2 \in \mathrm{Psh}(D_2)$, and $\pi_i \colon D_1 \times D_2 \to D_i$ be the projections. Then*

$$\mathscr{I}(\varphi_1 \circ \pi_1 + \varphi_2 \circ \pi_2) = \pi_1^* \mathscr{I}(\varphi_1) \cdot \pi_2^* \mathscr{I}(\varphi_2).$$

- *Let $\varphi, \psi \in \mathrm{Psh}(D)$. Then*

$$\mathscr{I}(\varphi + \psi) \subset \mathscr{I}(\varphi) \cdot \mathscr{I}(\psi).$$

Proof. See [5, p. 159]. ■

2.4 Openness and strong openness

Let

$$\mathscr{I}_+(\varphi) \coloneqq \bigcup_{\varepsilon > 0} \mathscr{I}((1+\varepsilon)\varphi).$$

Then $\mathscr{I}_+(\varphi)|_K = \mathscr{I}((1+\varepsilon_K)\varphi)|_K$ for some $\varepsilon_K > 0$ on compact K by the coherence (or simply by the Noetherian of $\mathscr{O}_x$, we have $\mathscr{I}_+(\varphi)_x = \mathscr{I}\big((1+\varepsilon_0)\varphi\big)_x$ for some $\varepsilon_0 > 0$). Demailly's strong openness conjecture (proved by Guan-Zhou) is:

$$\mathscr{I}_+(\varphi) = \mathscr{I}(\varphi).$$

If $\mathscr{I}(\varphi) = \mathscr{O}$, this is the openness conjecture (proved by Berndtsson).

Corollary 2.32. *If $c_o^F(\varphi) < +\infty$, then $(F, o) \notin \mathscr{I}(c_o^F(\varphi)\varphi)_o$.*

theorem 2.33. *Let φ_j be sequence of psh function near o such that φ_j increasingly converges to a psh function φ near o in Lebesgue measure. Then*

$$\mathscr{I}(\varphi)_o = \bigcup_{j \geq 1} \mathscr{I}(\varphi_j)_o.$$

Proof. Omit. See [11]. ■

Corollary 2.34 (Fornæss). *The strong openness property is also true for every $p > 0$.*

Proof. Let $k \in \mathbb{Z}$ with $2k \geq p$. Observe

$$|F|^p \mathrm{e}^{-2\varphi} = |F^k|^2 \mathrm{e}^{-\left(2\varphi + (2k-p)\log|F|\right)},$$

and for every $q > 1$,

$$|F^k|^2 \mathrm{e}^{-q\left(2\varphi + (2k-p)\log|F|\right)} = |F|^{2k-2kq+pq} \mathrm{e}^{-2q\varphi} \gtrsim |F|^p \mathrm{e}^{-2q\varphi},$$

due to $2k - 2kq + pq \leq p \Leftrightarrow q(2k-p) \geq 2k - p \Leftrightarrow q \geq 1$. ■

Corollary 2.35. *If φ, ϕ are psh functions near o, and $\nu(\phi, o) = 0$, then*

$$\mathscr{I}(\varphi + \phi)_o = \mathscr{I}(\varphi)_o.$$

Proof. Use Skoda's result.

Since $\nu(\phi, o) = 0$, we have that $\mathrm{e}^{-2q\phi}$ is integrable near o for every $q > 0$. Hölder's inequality implies

$$\int |F|^2 \mathrm{e}^{-2\varphi - 2\phi} \leq \left(\int |F|^{2(1+\varepsilon)} \mathrm{e}^{-2(1+\varepsilon)\varphi} \right)^{1/(1+\varepsilon)} \cdot \left(\int \mathrm{e}^{-2\frac{1+\varepsilon}{\varepsilon}\phi} \right)^{\varepsilon/(1+\varepsilon)}.$$

Take $\varepsilon > 0$ such that $\int |F|^2 \mathrm{e}^{-2(1+\varepsilon)\varphi} < +\infty$. ■

Corollary 2.36. *For psh function φ near o, there exists a psh function φ_A with analytic singularities such that*

$$\mathscr{I}(\varphi)_o = \mathscr{I}(\varphi_A)_o.$$

Proof. See [9].

For every $j \geq 1$, set

$$I_j = \left| \mathscr{I}(j\varphi)_o \right|.$$

Claim

$$\mathscr{I}(\varphi)_o = \mathscr{I}\left(\max\left\{ \varphi, \frac{1}{j-1} \log|I_j| \right\} \right)_o, \quad \forall\, j > 1.$$

The claim follows from, for every positive continuous function χ near o, we have

$$\begin{aligned}
0 &\leq \int_V \chi \mathrm{e}^{-2\varphi} - \int_V \chi \mathrm{e}^{-2\max\left\{\varphi, \frac{1}{j-1}\log|I_j|\right\}} \\
&= \int_{V \cap \left\{\varphi \leq \frac{1}{j-1}\log|I_j|\right\}} \chi \left(\mathrm{e}^{-2\varphi} - \mathrm{e}^{-2\frac{1}{j-1}\log|I_j|} \right) \\
&\leq \int_{V \cap \left\{\varphi \leq \frac{1}{j-1}\log|I_j|\right\}} \chi \mathrm{e}^{-2j\varphi + 2(j-1)\varphi} \\
&\leq \int_V \chi |I_j|^2 \mathrm{e}^{-2j\varphi} < +\infty.
\end{aligned}$$

Demailly's approximation theorem gives

$$\frac{1}{j}\log|I_j| \geq \varphi + O(1), \quad \forall j \geq 1.$$

According to the strong openness property, we can take some $\varepsilon_0 > 0$ such that

$$\mathscr{I}(\varphi)_o = \mathscr{I}\big((1+\varepsilon_0)\varphi\big)_o.$$

Thus,

$$\begin{aligned}\mathscr{I}\left(\max\left\{\varphi, \frac{1}{j-1}\log|I_j|\right\}\right)_o &= \mathscr{I}(\varphi)_o = \mathscr{I}\big((1+\varepsilon_0)\varphi\big)_o \\ &\subseteq \mathscr{I}\left(\frac{1+\varepsilon_0}{j}\log|I_j|\right)_o \subseteq \mathscr{I}\left(\max\left\{\varphi, \frac{1+\varepsilon_0}{j}\log|I_j|\right\}\right)_o.\end{aligned}$$

If we take j such that

$$\frac{1}{j-1} \leq \frac{1+\varepsilon_0}{j} \quad \Longleftrightarrow \quad j \geq 1 + \frac{1}{\varepsilon_0},$$

then we have

$$\mathscr{I}\left(\max\left\{\varphi, \frac{1}{j-1}\log|I_j|\right\}\right)_o \supseteq \mathscr{I}\left(\max\left\{\varphi, \frac{1+\varepsilon_0}{j}\log|I_j|\right\}\right)_o,$$

which implies that all the multiplier ideals are equal. Consequently, $\mathscr{I}(\varphi)_o = \mathscr{I}(\varphi_A)_o$, for

$$\varphi_A = \frac{1}{j-1}\log|I_j|, \quad \text{where } j \gg 1 \text{ s.t. } \quad \mathscr{I}(\varphi)_o = \mathscr{I}\left(\frac{j}{j-1}\varphi\right)_o.$$

■

The following is a sharp effectiveness result of the strong openness property, where we have proved the special case for the openness property.

THEOREM 2.37 (Guan 19'). *Let D be a pseudoconvex domain in $\mathbb{C}^n$ containing the origin o, and φ a negative psh function on D such that $\varphi(o) = -\infty$. Suppose $F \in \mathscr{O}(D)$ and $\int_D |F|^2 e^{-2\varphi} < +\infty$. Then for any $p > 1$ satisfying*

$$\frac{p}{p-1} > \frac{\int_D |F|^2 e^{-2\varphi}}{C_{F,\mathscr{I}_+(c_o^F(\varphi)\varphi)_o}(D)},$$

we have $(F, o) \in \mathscr{I}(p\varphi)_o$. Here

$$c_o^F(\varphi) = \sup\{c \geq 0\colon |F|^2 e^{-2c\varphi} \text{ is locally integrable near } o\},$$

and

$$C_{F,I}(D) = \inf\left\{\int_D |\widetilde{F}|^2\colon \widetilde{F} \in \mathscr{O}(D),\ (\widetilde{F} - F, o) \in I\right\}.$$

2.5 Skoda's L^2 division theorem

THEOREM 2.38 (Skoda, 72'). *Let $\Omega \subseteq \mathbb{C}^n$ be a complete Kähler ($\Leftarrow$ pseudoconvex) domain and $\varphi \in \mathrm{Psh}(\Omega)$. Let $g = (g_1, \ldots, g_r)$ be an r-tuple of holomorphic function on Ω s.t. $g^{-1}(0) = \varnothing$. Set $m = \min\{n, r-1\}$. Then for every holomorphy function f on Ω s.t.*

$$I = \int_\Omega |f|^2 |g|^{-2(m+1+\varepsilon)} \mathrm{e}^{-\varphi} \,\mathrm{d}\lambda < +\infty,$$

there exists holomorphic functions $(h_1, \ldots, h_r)$ on Ω s.t. $f = \sum g_j h_j$ and

$$\int_\Omega |h|^2 |g|^{-2(m+\varepsilon)} \mathrm{e}^{-\varphi} \,\mathrm{d}\lambda \leq \left(1 + \frac{m}{\varepsilon}\right) I.$$

Proof. See [6, pp. 388–392].

Consider the diagram of Hilbert spaces and linear operators:

$$\begin{array}{ccc} H_0 & \xrightarrow{T_1} & H_1 \\ \big\downarrow{\scriptstyle T_2} & & \\ H_2 & & \end{array}$$

where T_1 is continuous and T_2 is closed and densely defined. Let G_1 be a closed subspace of H_1. Suppose $T_1(\mathrm{Ker}\, T_2) \subset G_1$. We need to following lemma in functional analysis:

LEMMA 2.39. *If there exists some $c > 0$ such that*

$$\|T_1^* x_1 + T_2^* x_2\|_0 \geq c\|x_1\|_1$$

for every $x_1 \in G_1$ and $x_2 \in \mathrm{Dom}\, T_2^*$, then we have $T_1(\mathrm{Ker}\, T_2) = G_1$.

Moreover, for every $x_1 \in G_1$, there exists $x_0 \in \mathrm{Ker}\, T_2$ s.t. $T_1 x_0 = x_1$ and

$$\|x_0\|_0 \leq \frac{1}{c}\|x_1\|_1.$$

Proof. Let $G_0 = \mathrm{Ker}\, T_2$, and $G_0^\perp$ be the orthogonal complement of G_0 in H_0. Then $\mathrm{Im}\, T_2^*$ is dense in $G_0^\perp$. Note that $G_0 = H_0/G_0^\perp$ be seen as the quotient space equipped with the quotient norm. Let $\widetilde{T_1^*}\colon H_1 \to G_0$ be the induced map of $T_1^*\colon H_1 \to H_0$. Then for every $u_1 \in H_1$, with respect to the quotient norm, we have

$$\|\widetilde{T_1^*} u_1\|_{G_0}^2 = \inf_{v_0 \in G_0^\perp} \|T_1^* u_1 + v_0\|_0^2 = \inf_{v_0' \in \mathrm{Im}\, T_2^*} \|T_1^* u_1 + v_0'\|_0^2 = \inf_{v_2 \in \mathrm{Dom}\, T_2} \|T_1^* u_1 + T_2^* v_2\|_0^2 \geq c^2 \|u_1\|_1^2.$$

This implies for the given $x_1 \in G_1$,

$$\| \langle x_1, u_1 \rangle_1 \|^2 \leq \|x_1\|_1^2 \|u_1\|_1^2 \leq \frac{1}{c^2} \|x_1\|^2 \|\widetilde{T_1^*} u_1\|_{G_0}^2,$$

which shows that the functional

$$\begin{aligned} \mathcal{L}\colon \quad G_0 &\longrightarrow \mathbb{C} \\ w_1 = \widetilde{T_1^*} u_1 &\longmapsto \langle x_1, u_1 \rangle_1 \end{aligned}$$

is bounded. Thus, by the Riesz representation theorem, we can find some $x_0 \in G_0$ such that

$$\langle x_1, u_1\rangle_1 = \mathcal{L}(\widetilde{T_1^*}u_1) = \left\langle x_0, \widetilde{T_1^*}u_1\right\rangle_{G_0} = \left\langle x_0, T_1^*u_1 + G_0^{\perp}\right\rangle_0 = \langle T_1x_0, u_1\rangle, \quad \forall u_1 \in H_1.$$

Consequently, we have $x_1 = T_1x_0$, and the norm control comes from

$$\|x_0\|_0 \le \|\mathcal{L}\|_{G_0\to\mathbb{C}} \le \frac{1}{c}\|x_1\|_1.$$

■

Let

$$\begin{array}{ccc} L^{n,0}_{(2)}(\Omega, \mathrm{e}^{-\phi_0})^{\oplus r} = H_0 & \xrightarrow{\quad T_1:=g \quad} & H_1 = L^{n,0}_{(2)}(\Omega, \mathrm{e}^{-\phi_1}) \\ \Big\downarrow{\scriptstyle T_2:=\overline{\partial}} & & \\ H_2 = \operatorname{Ker}\left(L^{n,1}_{(2)}(\Omega, \mathrm{e}^{-\phi_0})^{\oplus r} \xrightarrow{\overline{\partial}} L^{n,2}_{(2)}(\Omega, \mathrm{e}^{-\phi_0})^{\oplus r}\right) & & \end{array}$$

where $\phi_0 = 2(m+\varepsilon)\log|g| + \varphi$, and $\phi_1 = 2(m+1+\varepsilon)\log|g| + \varphi$. Let $G_1 = H_1 \cap \operatorname{Ker}\overline{\partial}$. We want to prove $G_1 = T_1(\operatorname{Ker} T_2)$.

First, we can directly compute by the definition:

$$(T_1 s, u)_{H_0} = (s, T_1^* u)_{H_1}, \quad s \in H_0,\ u \in H_1,$$

to get

$$T_1^*u = u\overline{g}\mathrm{e}^{-\tau} = (u\overline{g_1}\mathrm{e}^{-\tau}, \ldots, u\overline{g_r}\mathrm{e}^{-\tau}), \quad \tau = \log|g|^2.$$

Thus, for our case, $\|T_1^*x_1\|_0^2 = \|x_1\|_1^2$ for every $x_1 \in H_1$, and especially when $x_1 \in G_1$ we have

$$T_2T_1^*x_1 = \overline{\partial}(x_1\overline{g}\mathrm{e}^{-\tau}) = x_1\overline{\partial}(\overline{g}\mathrm{e}^{\tau}) =: x_1\beta, \quad \text{where } \beta = \overline{\partial(g\mathrm{e}^{-\tau})} \in \operatorname{Ker}\overline{\partial}.$$

Then, for any real number $\alpha > 1$, and $x_1 \in G_1$, $x_2 \in \operatorname{Dom}(T_2^*) \subset H_2$, we have

$$\begin{aligned} \|T_1^*x_1 + T_2^*x_2\|_0^2 &= \|T_1^*x_1\|_0^2 + \|T_2^*x_2\|_0^2 + 2\mathrm{Re}\,\langle T_2T_1^*x_1, x_2\rangle_2 \\ &= \|x_1\|_1^2 + \|T_2^*x_2\|_0^2 + 2\mathrm{Re}\,\langle x_1\beta, x_2\rangle_2 \\ &\ge \|x_1\|_1^2 + \|T_2^*x_2\|_0^2 - \alpha\int_\Omega |g|^2\beta\cdot\overline{x_2}\mathrm{e}^{-\phi_0}\,\mathrm{d}V - \frac{1}{\alpha}\int_\Omega |g|^{-2}|x_1|^2\mathrm{e}^{-\phi_0}\,\mathrm{d}V \\ &= \left(1 - \frac{1}{\alpha}\right)\|x_1\|_1^2 + \|T_2^*x_2\|_0^2 - \alpha\Big\||g|\overline{\beta}\cdot x_2\Big\|_2^2. \end{aligned}$$

We need to estimate:

$$\|T_2^*u\|_0^2 - \alpha\Big\||g|\overline{\beta}\cdot u\Big\|_2^2 = \|\overline{\partial}^*u\|_{H_0}^2 - \alpha\Big\||g|\partial(g\mathrm{e}^{-\tau})\cdot u\Big\|_{H_2}^2 = \sum_{j=1}^r \|\overline{\partial}^*u_j\|^2 - \alpha\Big\||g|\partial(g\mathrm{e}^{-\tau})\cdot u\Big\|^2$$

for every $u = (u_1, \ldots, u_r) \in \operatorname{Dom}(T_2^*) = \operatorname{Dom}(\overline{\partial}^*) \cap H_2$. In fact, by the Bochner–Kodaira–Nakano identity and the approximation result, we have

$$\|\overline{\partial}^*u_j\|^2 = \|\overline{\partial}^*u_j\|^2 + \|\overline{\partial}u_j\|^2 \ge (\mathrm{i}\Theta_{\mathrm{e}^{-\phi_0}}\Lambda u_j, u_j), \quad j = 1, \ldots, r,$$

for every $u =\in \operatorname{Dom}(\overline{\partial}^*) \cap H_2$. We also need the following

Skoda's basic inequality

$$m\sum_{j=1}^{r}(\mathrm{i}\partial\overline{\partial}\tau)(v_j,v_j)\geq|g|^2|\partial(g\mathrm{e}^{-\tau})\cdot v|^2,$$

for every $v=(v_1,\cdots,v_r)\in\big(T_\Omega\big)^{\oplus r}$ at any point in Ω. In other words, for every $\eta=(\eta_j)_{1\leq j\leq r}\in\mathbb{C}^r$, we have

$$m\sum_{j=1}^{r}\sum_{\mu,\nu=1}^{n}\frac{\partial^2\log|g|^2}{\partial z_\mu\partial\overline{z_\nu}}\eta_{j\mu}\overline{\eta_{j\nu}}\geq|g|^2\left|\sum_{j=1}^{r}\sum_{\mu=1}^{n}\frac{\partial(g_j\mathrm{e}^{-\tau})}{\partial\overline{z_{j\mu}}}\eta_\mu\right|^2.$$

Some calculations: On the one hand,

$$\partial\tau=\frac{\langle\partial g,g\rangle}{|g|^2},\quad\overline{\partial}\tau=\frac{\langle g,\partial g\rangle}{|g|^2}.$$

$\Rightarrow$

$$\mathrm{i}\partial\overline{\partial}\tau=\mathrm{i}\frac{\langle\partial g,\partial g\rangle}{|g|^2}-\mathrm{i}\frac{\langle\partial g,g\rangle\wedge\langle g,\partial g\rangle}{|g|^4},$$

$\Rightarrow$

$$\mathrm{i}\partial\overline{\partial}\tau(\xi,\xi)=\frac{|g|^2|\partial g\cdot\xi|^2-|\left\langle\partial g\cdot\xi,g\right\rangle|^2}{|g|^4}=|g|^{-4}\sum_{1\leq j<l\leq r}\sum_{\nu=1}^{n}\Big|(g_j\partial_\nu g_l-g_l\partial_\nu g_j)\xi_\nu\Big|^2,$$

for every $\xi=\sum\limits_{\nu=1}^{n}\xi_\nu\dfrac{\partial}{\partial z_\nu}\in T_\Omega$, by Lagrange's equality. On the other hand,

$$|g|^2|\partial(g\mathrm{e}^{-\tau})\cdot\eta|^2=|g|^{-2}\left|\Big(|g|^2\partial g-g\left\langle\partial g,g\right\rangle\Big)\cdot\eta\right|^2=|g|^{-6}\left|\sum_{j,k=1}^{r}\sum_{\nu=1}^{n}\overline{g}_j\big(g_j\partial_\nu g_k-g_k\partial_\nu g_j\big)\eta_{k,\nu}\right|^2.$$

Lemma 2.40. *Let $n,r\in\mathbb{Z}_{>0}$, and (a_k), (b_k^ν), (c_ν^k) be complex numbers for $1\leq k\leq r$, $1\leq\nu\leq n$. Then, with $m=\min\{n,r-1\}$, and $|a|^2=\sum_k|a_k|^2$, we have*

$$\left|\sum_{j,k,\nu}\overline{a_j}(a_jb_k^\nu-a_kb_j^\nu)c_\nu^k\right|^2\leq m|a|^2\sum_{k,\nu}\sum_{j<l}\left|(a_jb_l^\nu-a_lb_j^\nu)c_\nu^k\right|^2.$$

Proof. The case $m=r-1$ is a direct consequence of the Cauchy–Schwarz inequality, but the case $n<r-1$ is a bit complicated. See [12, pp. 282–284]. ■

Now by Bochner–Kodaira–Nakano identity and Skoda's basic inequality, we have

$$\sum_{j=1}^{r}\|\overline{\partial}^*u_j\|^2\geq(m+\varepsilon)\sum_{j=1}^{r}(\mathrm{i}\partial\overline{\partial}\tau)(u_j,u_j)\geq\left(1+\frac{\varepsilon}{m}\right)\Big\||g|\partial(g\mathrm{e}^{-\tau})\cdot u\Big\|^2.$$

Take $\alpha=1+\frac{\varepsilon}{m}$. Then we have $1-\frac{1}{\alpha}=\frac{\varepsilon}{m+\varepsilon}$, and

$$\|T_1^*x_1+T_2^*x_2\|_0^2\geq\frac{\varepsilon}{m+\varepsilon}\|x_1\|_1^2,\quad x_1\in G_1,\ x_2\in\mathrm{Dom}(T_2^*).$$

Thus, for every $x_1 \in G_1 = H_1 \cap \operatorname{Ker} \bar{\partial}$, there exists $x_0 \in \operatorname{Ker} T_2$ s.t. $T_1 x_0 = x_1$ and

$$\|x_0\|_0^2 \le \frac{m+\varepsilon}{\varepsilon} \|x_1\|_1^2,$$

which is the desired result exactly ($f = x_1$ holomorphic, and $h = x_0$ holomorphic r-tuple, with $g \cdot h = f$ and the L^2 estimate). ∎

实际上不必要求 $g^{-1}(0) = \varnothing$. 如果我们进一步假设 $|g|$ 在 Ω 上有上界以及正的下界, 那么可以得到 $A^2(\Omega, \mathrm{e}^{-\varphi})$ 中的元素都可以被 g 做除法, 并且被除之后的都仍然落在 $A^2(\Omega, \mathrm{e}^{-\varphi})$ 中 (即 $\mathbb{C}^n$ 中的弱拟凸域上的关于 psh 函数的加权 Bergman 空间中的 Corona 问题均是可解的).

Definition 2.41. Let $k \in \mathbb{R}_+$, and $\mathscr{J} = (g_1, \dots, g_r) \neq (0)$ be an ideal of $\mathscr{O}_{n,o}$. Define:

- the ideal
$$\overline{\mathscr{J}}^{(k)} := \big\{ f \in \mathscr{O}_{n,o} \colon |u| \le C|g|^k \text{ near } o,\ \exists C > 0 \big\},$$
where $|g|^2 = |g_1|^2 + \cdots + |g_r|^2$;
- the ideal
$$\widehat{\mathscr{J}}^{(k)} := \Big\{ \int_V |f|^2 |g|^{-2(k+\varepsilon)} \, \mathrm{d}\lambda < +\infty,\ \exists V \ni o \ \&\ \varepsilon > 0 \Big\}.$$
In fact, we know $\varepsilon = 0$ is enough (e.g. by strong openness property), that is, $\widehat{\mathscr{J}}^{(k)} = \mathscr{I}(k \log |g|)_o$.

Corollary 2.42 (Briançon–Skoda, 74’). *If set $p = \min\{n-1, r-1\}$, we have*

1. $\widehat{\mathscr{J}}^{(k+1)} = \mathscr{J} \widehat{\mathscr{J}}^{(k)} = \overline{\mathscr{J}} \widehat{\mathscr{J}}^{(k)}$, *for $k \ge p$;*
2. $\overline{\mathscr{J}}^{(k+p)} \subset \widehat{\mathscr{J}}^{(k+p)} \subset \mathscr{J}^k$, *for all $k \in \mathbb{Z}_+$.*

Proof. Obviously, $\mathscr{J} \widehat{\mathscr{J}}^{(k)} \subset \overline{\mathscr{J}} \widehat{\mathscr{J}}^{(k)} \subset \widehat{\mathscr{J}}^{(k+1)}$. We only need to prove $\widehat{\mathscr{J}}^{(k+1)} \subset \mathscr{J} \widehat{\mathscr{J}}^{(k)}$.

If $r \le n$, it immediately follows from the L^2 division theorem. If $r > n$, we need the following lemma:

Lemma 2.43. *If $\mathscr{J} = (g_1, \dots, g_r)$ and $r > n$, we can find elements $\widetilde{g_1}, \dots, \widetilde{g_n}$ in $\mathscr{J}$ which are linear combinations of g_i:*

$$\widetilde{g_j} = a_j \cdot g = \sum_{1 \le k \le r} a_{jk} g_k, \quad a_j \in \mathbb{C}^r \setminus \{0\},$$

such that $C^{-1}|g| \le |\widetilde{g}| \le C|g|$ near o. In particular, the ideal $J = (\widetilde{g_1}, \dots, \widetilde{g_n}) \subset \mathscr{J}$ satisfies $\overline{J}^{(k)} = \overline{\mathscr{J}}^{(k)}$ and $\widehat{J}^{(k)} = \widehat{\mathscr{J}}^{(k)}$ for all k.

Proof. See [5, pp. 119–120]. ∎

Let $J \subset \mathscr{J}$ be as in the lemma. Then

$$\widehat{\mathscr{J}}^{(k+1)} = \widehat{J}^{(k+1)} \subset J \widehat{J}^{(k)} \subset \mathscr{J} \widehat{\mathscr{J}}^{(k)}, \quad \forall\, k \ge n-1.$$

The first result implies

$$\widehat{\mathscr{J}}^{(k+p)} = \mathscr{J}^k \widehat{\mathscr{J}}^{(p)}, \quad \forall\, k \in \mathbb{Z}_+,$$

by induction, which gives the second result. ∎

Corollary 2.44. *The ideal $\overline{\mathscr{J}}$ is the integral closure of $\mathscr{J}$, i.e. the set of germs $f \in \mathscr{O}_{n,o}$ which satisfies*

$$f^d + a_1 f^{d-1} + \cdots + a_d = 0, \quad a_s \in \mathscr{J}^s, \quad 1 \le s \le d.$$

Proof. See [5, pp. 120–121].

If f is in the integral closure of $\mathscr{J}$, using $|a_s| \le C|g|^s$ and

$$|\text{roots}| \le 2 \max_{1\le s\le d} |a_s|^{1/s},$$

we can get $|f| \le C|g|$.

Let $f \in \overline{\mathscr{J}}$. Let $v_1, \dots, v_N$ be generators of $\widehat{\mathscr{J}}^{(p)}$. Since $f v_j \in \overline{\mathscr{J}}\,\widehat{\mathscr{J}}^{(p)} = \mathscr{J}\,\widehat{\mathscr{J}}^{(p)}$, there exists elements $b_{jk} \in \mathscr{J}$ such that

$$f v_j = \sum_{1\le k\le N} b_{jk} v_k.$$

Thus, f satisfies the equation

$$\det\left(f\delta_{jk} - b_{jk}\right) = 0.$$

■

The L^2 division theorem can also be used to prove Levi's problem.

Proof of Levi's problem. Let Ω be a bounded pseudoconvex domain, and let $a = (a_1, \dots, a_n) \in \partial\Omega$ be any boundary point. We prove that there exists a holomorphic function in Ω which can not be analytically extended across a.

Let $\varphi(z) = -2\log\delta(z) \in \mathrm{Psh}(\Omega)$. Then $\varphi(z) \ge -2\log|z-a|$, and thus for every $\varepsilon > 0$,

$$\int_\Omega |z-a|^{-2(n+\varepsilon)} \mathrm{e}^{-\varphi(z)}\,\mathrm{d}\lambda < +\infty.$$

In addition, $z_1 - a_1, \dots, z_n - a_n$ have no common zeros in Ω. So we can divide 1 by $z_1 - a_1, \dots, z_n - a_n$, which gives $1 = \sum_{j=1}^n (z_j - a_j) h_j(z)$ with $h_j \in \mathscr{O}(\Omega)$ for every j. Then we can see that there must be one of h_j which can not be analytically extended across a. ■

3 Pluripotential Theory

The "weakly convergence" in this section are in fact "weak*-convergent" what we usually called.

3.1 Preliminaries: Distribution

Let Ω be an open subset of $\mathbb{R}^n$. Denote by $\mathscr{D}(\Omega)$ the set of smooth real functions on Ω compactly supported in Ω and by $\mathscr{D}_K$ the functions in $\mathscr{D}(\Omega)$ supported in K. Then for every $s \in \mathbb{Z}_{\geq 0}$, we can define a seminorm:

$$p^s(u) \coloneqq \sup_{x\in\Omega} \max_{|\alpha|\leq s} |u^{(\alpha)}(x)|, \quad \forall\, u \in \mathscr{D}(\Omega).$$

Besides, for every compact subset K of Ω, and $s \in \mathbb{Z}_{\geq 0}$, we may also define

$$p^s_K(u) \coloneqq \sup_{x\in K} \max_{|\alpha|\leq s} |u^{(\alpha)}(x)|, \quad \forall\, u \in \mathscr{D}_K.$$

We can equip $\mathscr{D}(\Omega)$ with the topology defined by the family of seminorms $\left(p^s\right)_s$, or equivalently $\left(p^s_K\right)_{K,s}$, which makes $\mathscr{D}(\Omega)$ to be a locally convex space.

Definition 3.1. A distribution Λ on Ω, denoted by $\Lambda \in \mathscr{D}'(\Omega)$, is a continuous linear functional on $\mathscr{D}(\Omega)$ (with respect to the above topology), i.e. $\Lambda\colon \mathscr{D}(\Omega) \to \mathbb{R}$ is linear and for every compact subset K of Ω, there exists $N \in \mathbb{Z}_{\geq 0}$ and $C > 0$ such that

$$|\Lambda(u)| \leq C p^N_K(u), \quad \forall\, u \in \mathscr{D}_K.$$

Example 3.2.

- Let $p \in \Omega$, and

$$\delta_p(u) \coloneqq u(p), \quad \forall\, u \in \mathscr{D}(\Omega).$$

  Then the Dirac function (measure) $\delta_p \in \mathscr{D}'(\Omega)$.

- Suppose $f \in L^1_{\text{loc}}(\Omega)$. Define

$$\Lambda_f(u) \coloneqq \int_\Omega u(x)f(x)\,\mathrm{d}x, \quad \forall\, u \in \mathscr{D}(\Omega).$$

  Then $\Lambda_f \in \mathscr{D}'(\Omega)$.

- If μ is a positive measure on Ω with $\mu(K) < +\infty$ for every compact $K \subset \Omega$, then

$$\Lambda_\mu(u) = \int_\Omega u\,\mathrm{d}\mu, \quad \forall\, u \in \mathscr{D}(\Omega)$$

  defines a distribution $\Lambda_\mu \in \mathscr{D}'(\Omega)$.

Remark 3.3.

- *Let $\Lambda \in \mathscr{D}'(\Omega)$ and $\alpha \in \mathbb{Z}^n_{\geq 0}$. Then*

$$(D^\alpha\Lambda)(u) = (-1)^{|\alpha|}\Lambda(D^\alpha u), \quad \forall\, u \in \mathscr{D}(\Omega)$$

  defines a distribution $D^\alpha\Lambda \in \mathscr{D}'(\Omega)$.

- *Suppose* $\Lambda \in \mathscr{D}'(\Omega)$ *and* $f \in C^\infty(\Omega)$. *Then*
$$(f\Lambda)(u) = \Lambda(fu), \quad \forall\, u \in \mathscr{D}(\Omega)$$
defines a distribution $f\Lambda \in \mathscr{D}'(\Omega)$. *The Lebniz rule can also be applied.*

- *Suppose* $\Lambda_i \in \mathscr{D}'(\Omega)$, $i = 1, 2, \ldots$, *and*
$$\Lambda u = \lim_{i\to\infty} \Lambda_i u, \quad \forall\, u \in \mathscr{D}(\Omega).$$
Then $\Lambda \in \mathscr{D}'(\Omega)$.

Definition 3.4. The order of $\Lambda \in \mathscr{D}'(\Omega)$ is the smallest $N \in \mathbb{Z}_{\geq 0}$ such that there exists $C > 0$ such that
$$|\Lambda(u)| \leq Cp^N(u), \quad \forall\, u \in \mathscr{D}(\Omega).$$
The order of a distribution can be $+\infty$.

Example 3.5. For every $p \in \Omega$ and $\alpha \in \mathbb{Z}^n_{\geq 0}$, the distribution $D^\alpha \delta_p$ is of order $N = |\alpha|$.

Definition 3.6. The support of a distribution $\Lambda \in \mathscr{D}'(\Omega)$ (denoted by $\operatorname{Supp}\Lambda$) is the smallest closed subset A of Ω, such that $\Lambda(u) = 0$ for all $u \in \mathscr{D}(\Omega \setminus A)$.

Proposition 3.7. *If* $\Lambda \in \mathscr{D}'(\Omega)$ *has compact support, then* Λ *has finite order.*

Proof. See [17, pp. 164–165]. Take a cut-off function θ of the support, and notice $\theta\Lambda = \Lambda$:
$$|\Lambda u| = |(\theta\Lambda)u| = |\Lambda(\theta u)| \leq Cp^N_{K'}(\theta u) \leq C'p^N(u).$$
■

Example 3.8.
- $\operatorname{Supp}\delta_p = \operatorname{Supp} D^\alpha \delta_p = \{p\}$.
- Suppose $\Lambda \in \mathscr{D}'(\Omega)$, $p \in \Omega$, $\operatorname{Supp}\Lambda = \{p\}$, and Λ has order N. Then there are constants c_α such that
$$\Lambda = \sum_{|\alpha|\leq N} c_\alpha D^\alpha \delta_p.$$

theorem 3.9. *Suppose* $\Lambda \in \mathscr{D}'(\Omega)$, *and* $K \subset \Omega$ *compact. Then there exist a continuous function* f *on* Ω *and* $\alpha \in \mathbb{Z}^n_{\geq 0}$ *s.t.*
$$\Lambda u = (-1)^{|\alpha|} \int_\Omega f(x)(D^\alpha u)(x)\,\mathrm{d}x, \quad \forall\, u \in \mathscr{D}_K.$$
In particular, if Λ *has compact support and of order* N, *then there exist finitely many compactly supported continuous functions* f_β *s.t.*
$$\Lambda u = \sum_\beta (-1)^{|\beta|} \int_\Omega f_\beta(x)(D^\beta u)(x), \quad \forall\, u \in \mathscr{D}(\Omega).$$

Proposition 3.10 (Approximation of distributions). *Let* $\Lambda \in \mathscr{D}'(\Omega)$, $\varepsilon > 0$ *and* $\rho_\varepsilon \in \mathscr{D}(\mathbb{R}^n)$ *the smoothing kernel with support in* $B(o, \varepsilon)$. *Then*
$$(\Lambda * \rho_\varepsilon)(x) := \Lambda\big(\rho_\varepsilon(x - \cdot)\big)$$
defines a smooth function $\Lambda * \rho_\varepsilon$ *on* Ω_ε *s.t.* $\Lambda * \rho_\varepsilon$ *converges to* Λ *in* $\mathscr{D}'(\Omega)$ *as* $\varepsilon \to 0$. *As current,*
$$(\Lambda * \rho_\varepsilon)(u) = \Lambda(\rho_\varepsilon * u), \quad \forall\, u \in \mathscr{D}_K, \quad \textit{(somehow nontrivial)}.$$

Proof. See [17, p. 173]. ■

EXAMPLE 3.11. If $\Lambda = \delta_o$, then $(\delta_o * \rho_\varepsilon)(x) = \rho_\varepsilon(x - \cdot)(o) = \rho_\varepsilon(x)$.

EXAMPLE 3.12. The following is an extremly wrong calculation, which shows that it is not always appropriate to define the products of distributions (so for currents): Let $\varepsilon \geq 0$ and $H_\varepsilon \in C^\infty(\mathbb{R})$ be increasing with

$$H_\varepsilon(x) = \begin{cases} 1 & x \geq 0, \\ 0 & x \leq -\varepsilon. \end{cases}$$

Then $H_\varepsilon \to H = \chi_{[0,+\infty)}$ and $\mathrm{d}H_\varepsilon \to \delta_0$ as distributions. We have (really?) $(H_\varepsilon \delta_0)(u) = \delta_0(H_\varepsilon u) = \delta_0(u) \Rightarrow H\delta_0 = \delta_0$ by letting $\varepsilon \downarrow 0$, while $\delta_0 \leftarrow \mathrm{d}(H_\varepsilon^2) = 2H_\varepsilon \, \mathrm{d}H_\varepsilon \to 2H\delta_0 \Rightarrow \delta_0 = H\delta_0 = 0$.

3.2 Current

DEFINITION 3.13. A current on a domain Ω in $\mathbb{C}^n$ of bidegree $(n-p, n-q)$ (or bidimension (p,q)) is a differential form

$$T = \sum_{|I|=n-p,\, |J|=n-q} T_{IJ} \, \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

whose coefficients T_{IJ} are distributions. The space of bidegree $(n-p, n-q)$ currents on Ω is denoted by $\mathscr{D}'_{p,q}(\Omega)$ or $\mathscr{D}'^{n-p,n-q}(\Omega)$.

Equivalently, a current of bidegree $(n-p, n-q)$ on Ω is a continuous linear functional on the space $\mathscr{D}^{p,q}(\Omega)$ of smooth (p,q) forms with compact support.

PROPOSITION 3.14 (Approximation of currents). *Let $T \in \mathscr{D}'_{p,q}(\Omega)$ and $\rho_\varepsilon \in \mathscr{D}(\mathbb{R}^n)$ the smoothing kernel. Then*

$$T * \rho_\varepsilon := \sum_{I,J} (T_{I,J} * \rho_\varepsilon) \, \mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$$

is a smooth $(n-p, n-q)$ form Ω_ε s.t. $T * \rho_\varepsilon$ converges to T in $\mathscr{D}'_{p,q}(\Omega)$ as $\varepsilon \to 0$.

DEFINITION 3.15. A current $T \in \mathscr{D}'_{p,p}(\Omega)$ is positive ($T \geq 0$) if $\langle T, u \rangle \geq 0$ for all test forms $u \in \mathscr{D}^{p,p}(\Omega)$ that are strongly positive at each point, i.e.,

$$u(x) = \sum \gamma_s \mathrm{i}\alpha_{s,1} \wedge \overline{\alpha}_{s,1} \wedge \ldots \wedge \mathrm{i}\alpha_{s,p} \wedge \overline{\alpha}_{s,p}$$

where $\alpha_{s,j} \in T^*_{\Omega,x}$ and $\gamma_s \geq 0$. Denote by $T \in \mathscr{D}'^{+}_{p,p}(X)$.

REMARK 3.16.
- *Positive (p,p) forms are forms with L^1_{loc} coefficients that are positive as (p,p) currents.*
- *A form $u = \mathrm{i} \sum_{j,k} u_{j,k} \, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$ of bidegree $(1,1)$ is positive if and only if (u_{jk}) is semi-positive definite.*
- *The notions of positive and strongly positive $(p,p)-$forms coincide for $p = 0, 1, n-1, n$. But this is not true for other cases in general.*
- *If $u_1, \ldots, u_s$ are positive forms, and strongly positive expect one, then $u_1 \wedge \ldots \wedge u_s$ is positive.*
- *There is example shows: α, β positive forms but $\alpha \wedge \beta < 0$.*

- *$T \in \mathscr{D}'_{p,p}(\Omega)$ is positive iff $T \wedge u \in \mathscr{D}'_{(0,0)}(\Omega)$ is a positive measure for all strongly positive smooth (p,p) forms u on Ω (constant coefficients strongly positive forms are enough).*

PROPOSITION 3.17. *Every positive current T is real and of order 0, i.e. if*

$$T = \mathrm{i}^{(n-p)^2} \sum T_{IJ}\,\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J,$$

then the coefficients T_{IJ} are complex measures and $\overline{T_{IJ}} = T_{JI}$.

DEFINITION 3.18. If T is order 0, define the mass measure of T by

$$\|T\| := \sum_{I,J} |T_{I,J}|.$$

(This depends on the choice of coordinates).

PROPOSITION 3.19. *If $T \in \mathscr{D}'^{+}_{p,p}(X)$ and $v \in C_c^{s,s}(X)$ is strongly positive, then $T \wedge v$ is a positive current.*

Proof. See [6, pp. 133–134] or [2, p. 3]. ■

DEFINITION 3.20. A current $T \in \mathscr{D}'_{p,p}(\Omega)$ is closed if it is d-closed, i.e. $\mathrm{d}T = 0$, where

$$\langle\, \mathrm{d}T, u\rangle := (-1)^{q+1}\langle T, \mathrm{d}u\rangle, \quad \forall\, u \in \mathscr{D}^{(p-1,p-1)}(\Omega).$$

EXAMPLE 3.21. Let $\phi \in \mathrm{Psh} \cap L^1_{\mathrm{loc}}(\Omega)$. Then

$$T = \mathrm{i}\partial\overline{\partial}\phi = \mathrm{i} \sum_{1\le j,k\le n} \frac{\partial^2 \phi}{\partial z_j \partial \bar{z}_k}\,\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$$

is a closed positive current of bidegree $(1,1)$.

Using Dolbeault-Grothendieck's lemma, conversely we have

PROPOSITION 3.22. *If $\Theta \in \mathscr{D}'^{+}_{n-1,n-1}(\Omega)$ is a closed positive current of bidegree $(1,1)$, then for every point $z_0 \in \Omega$ there exists a neighborhood D of z_0 and $\phi \in \mathrm{Psh}(D)$ s.t. $\Theta = \mathrm{i}\partial\overline{\partial}\phi$.*

Proof. See [6, p. 135]. ■

EXAMPLE 3.23 (see [5, p. 10], [6, p. 135]). Let $A \subset \Omega$ be a closed analytic subset of pure dimension p. Define

$$\langle [A], \alpha\rangle := \int_{A_{\mathrm{reg}}} \alpha, \quad \alpha \in \mathscr{D}^{p,p}(\Omega).$$

The current $[A] \in \mathscr{D}'^{+}_{p,p}(\Omega)$ is a closed positive current (Lelong 57'; see [6, p. 140]).

若 A 是子流形, 这个 current 的良定性是直接的, 一般情形可以通过局部参数化定理来得到. 闭性在正则部分由 Stokes 定理得到, 奇异部分要用到 Skoda–El Mir 延拓定理.

DEFINITION 3.24. For every $T \in \mathscr{D}'^{+}_{p,p}(\Omega)$, the trace measure of T w.r.t. ω is the positive measure

$$\sigma_T = \frac{1}{2^p} T \wedge \omega^p.$$

When $\omega = \frac{\mathrm{i}}{2}\sum_j \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j$, and $T = \mathrm{i}^{(n-p)^2}\sum_{I,J} T_{IJ}\,\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J$,

$$\sigma_T = 2^{-p}\left(\sum_{|I|=n-p} T_{I,I}\right)\mathrm{i}\,\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \mathrm{i}\,\mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n.$$

EXAMPLE 3.25. Let Z be a p-dimensional complex analytic submanifold of Ω. Then

$$\sigma_{[Z]} = \text{Riemannian volume measure on } Z.$$

Proof. See [6, pp. 136–137]. ■

若没有 complex analytic 条件, 则 $\sigma_{[Z]} = \alpha\,\mathrm{d}V_Z$ 且 $|\alpha| \leq 1$ 以及等号成立当且仅当 complex analytic. 这说明解析子流形是极小子流形. 实际上 current theory 非常普遍应用在几何测度论的研究中.

PROPOSITION 3.26. *For positive current $T \in \mathscr{D}'^{+}_{p,p}(\Omega)$, the mass measure is dominated by its trace measure, that is*

$$\|T\| \leq C\sigma_T, \quad C \text{ is a constant.}$$

Proof. See [6, pp. 133–134]. ■

REMARK 3.27. *With the weak topology defined by seminorms $T \longmapsto |\langle T, f_\nu\rangle|$, where (f_ν) dense in $\mathscr{D}_{p,p}(\Omega)$, $\mathscr{D}'^{+}_{p,p}(\Omega)$ is a metrizable subspace of $\mathscr{D}'_{p,p}(\Omega)$.*

THEOREM 3.28 (Skoda-El Mir). *Let $E \subset \Omega$ be a closed complete pluripolar set and $T \in \mathscr{D}'^{+}_{p,p}(\Omega, E)$ a closed positive current. Assume that T has finite mass in a neighborhood of every point in E. Then the trivial extension $\widetilde{T} \in \mathscr{D}'^{+}_{p,p}(\Omega)$ (obtained by extending measure T_{IJ} by 0 on E) is closed on Ω.*

Proof. See [6, pp. 138–140]. ■

DEFINITION 3.29. A current Θ is said to <u>normal</u> if Θ and $\mathrm{d}\Theta$ are currents of order 0.

实际上我们这里不关心 normal current, 只关心其中的 closed positive current, 但因为这个概念太有名了于是提一下.

PROPOSITION 3.30. *Let $\Theta \in \mathscr{D}'_{p,p}(\Omega)$ be a normal current. If $\operatorname{Supp}\Theta$ is contained in an analytic subset A of dimension $< p$, then $\Theta = 0$.*

Proof. See [6, p. 141].

On A_{reg}, observe that $\bar{\partial}g_k \wedge \Theta = \bar{\partial}(g_k\Theta) - g_k\bar{\partial}\Theta = 0$ for every generator g_k of A. Use dimensional assumption to deduce each $\Theta_{IJ} = 0$. Then $\operatorname{Supp}\Theta \subset A_{\text{sing}}$ and conclude by induction on $\dim A$. ■

PROPOSITION 3.31. *Let A be an analytic subset of Ω with global irreducible components A_j of pure dimension p. Then any closed current $\Theta \in \mathscr{D}'_{p,p}(\Omega)$ of order 0 with support in A is of the form*

$$\Theta = \sum_j \lambda_j [A_j], \quad \lambda_j \in \mathbb{C}.$$

Moreover, Θ is positive iff all $\lambda_j \geq 0$.

这地方就类似于除子了. 但是除子我们可不会让系数取复数值, 这也说明 current 作为一个更灵活的工具可以作为应用.

Proof. See [6, p. 143]. ■

THEOREM 3.32 (Lelong-Poincaré equation). *Let $f \in \mathcal{M}(\Omega)$ be a meromorphic function, and let $\sum m_j Z_j$ be the divisor of f. Then*

$$\frac{\mathrm{i}}{\pi}\partial\overline{\partial}\log|f| = \sum m_j[Z_j]$$

in the space $\mathscr{D}'_{n-1,n-1}(X)$.

Proof. See [6, p. 143].

Let $Z = \bigcup Z_j$ be its support. Near $a \in Z_j \cap Z_{\mathrm{reg}}$, using Weierstrass' preparation theorem, write $f(w) = u(w)w_1^{m_j}$. Note that $\mathrm{i}\partial\overline{\partial}\log|w_1|^2 = \pi[\{w_1 = 0\}]$. At last, use Proposition 3.30. ■

3.3 Complex Monge-Ampère operator

3.3.1 Definition: Case of locally bounded psh function

Denote

$$\mathrm{d} = \partial + \overline{\partial}, \quad \mathrm{d}^c = \frac{1}{2\pi\mathrm{i}}(\partial - \overline{\partial}).$$

Then $\mathrm{dd^c} = \frac{\mathrm{i}}{\pi}\partial\overline{\partial}$.

PROPOSITION 3.33. *Let Ω be a smoothly bounded open set and let f, g be forms of class C^2 on $\overline{\Omega}$ of pure bidegree (p,p) and (q,q) with $p+q = n-1$. Then*

$$\int_\Omega f \wedge \mathrm{dd^c}\, g - \mathrm{dd^c}\, f \wedge g = \int_{\partial\Omega} f \wedge \mathrm{d^c} g - \mathrm{d^c} f \wedge g.$$

Proof. See [6, p. 144]. Using Stokes' formula. ■

Let u be a locally bounded psh function and T a closed positive current.

DEFINITION 3.34 (Bedford-Taylor 82'). Then uT is well-defined: u is locally Borel function and T has measure coefficients. Define

$$\mathrm{dd^c}\, u \wedge T := \mathrm{dd^c}(uT).$$

REMARK 3.35. *Generally, we can not define the product of two currents (Kiselman 83').*

PROPOSITION 3.36. $\mathrm{dd^c}\, u \wedge T$ *is a closed positive current.*

Proof. See [6, p. 145].

Let smooth $u_k \to u$ decreasing convergent. Note that $\mathrm{dd^c}$ is weakly continuous. ■

DEFINITION 3.37. Given locally bounded psh functions $u_1, \ldots, u_q$, define inductively

$$\mathrm{dd^c}\, u_1 \wedge \mathrm{dd^c}\, u_2 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T := \mathrm{dd^c}\left(u_1\, \mathrm{dd^c}\, u_2 \wedge \ldots \wedge \mathrm{dd^c}\, u_1 \wedge T\right).$$

In particular, for locally bounded psh function u, the bidegree (n,n) current $(\mathrm{dd^c}\, u)^n$ is well-defined and is a positive measure.

Remark 3.38. *If u is of class C^2, then*

$$(\mathrm{dd^c}\, u)^n = \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right) \cdot \frac{n!}{\pi^n} \mathrm{i}\, \mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 \wedge \ldots \wedge \mathrm{i}\, \mathrm{d}z_n \wedge \mathrm{d}\bar{z}_n.$$

All operators $(\mathrm{dd^c})^q$ are called Monge-Ampère operators.

theorem 3.39 (Chern–Levine–Nirenberg inequality, 69'). *For all compact subsets K, L of Ω with $L \subset K^\circ$, there exists a constant $C_{K,L} \geq 0$ s.t.*

$$\| \mathrm{dd^c}\, u_1 \wedge \mathrm{dd^c}\, u_2 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T\|_L \leq C_{K,L} \|u_1\|_{L^\infty(K)} \cdots \|u_q\|_{L^\infty(K)} \|T\|_K,$$

where $\|\Theta\|_K = \|\Theta\|(K) = \int_K \sum_{I,J} |T_{I,J}|$.

Proof. See [6, p. 146].

Note that $\|T\|$ and σ_T are dominated by each other when T is positive. ∎

Corollary 3.40. *Let $u_1, \ldots, u_q$ be continuous psh functions, and $u_1^k, \ldots, u_q^k$ be sequences of psh functions s.t. $u_j^k \to u_j$ locally uniformly for each j. If a sequence of closed positive currents $T_k \to T$ weakly, then*

1. $u_1^k \,\mathrm{dd^c}\, u_2^k \wedge \ldots \wedge \mathrm{dd^c}\, u_q^k \wedge T_k \longrightarrow u_1 \,\mathrm{dd^c}\, u_2 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T$ *weakly;*
2. $\mathrm{dd^c}\, u_1^k \wedge \ldots \wedge \mathrm{dd^c}\, u_q^k \wedge T_k \longrightarrow \mathrm{dd^c}\, u_k \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T$ *weakly.*

Proof. See [6, p. 147].

(2) implies (1) due to the weak continuity of $\mathrm{dd^c}$.

By induction, only need to prove (1) when $q = 1$. Let u_ε be regularization of u, and write

$$u^k T_k - uT = (u^k - u)T_k + (u - u_\varepsilon)T_k + u_\varepsilon(T_k - T).$$

Then apply the Chern–Levine–Nirenberg inequality to the first two terms and $u_\varepsilon(T_k - T)$ converges weakly to 0 since u_ε is smooth. ∎

theorem 3.41 (Bedford-Taylor 82'). *Let $u_1, \ldots, u_q$ be locally bounded psh functions, and $u_1^k, \ldots, u_q^k$ be decreasing sequences of psh functions s.t. $u_j^k \to u_j$ pointwise for each j. Then*

1. $u_1^k \,\mathrm{dd^c}\, u_2^k \wedge \ldots \wedge \mathrm{dd^c}\, u_q^k \wedge T \longrightarrow u_1 \,\mathrm{dd^c}\, u_2 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T$ *weakly;*
2. $\mathrm{dd^c}\, u_1^k \wedge \ldots \wedge \mathrm{dd^c}\, u_q^k \wedge T \longrightarrow \mathrm{dd^c}\, u_k \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T$ *weakly.*

Proof. See [6, pp. 147–149]. ∎

Corollary 3.42 ([6, p. 149]). *The product $\mathrm{dd^c}\, u_1 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T$ is symmetric w.r.t. $u_1, \ldots, u_q$.*

3.3.2 Definition: Case of unbounded psh function

Assume that Ω is pseudoconvex. Let $u \in \operatorname{Psh}(\Omega)$ and $T \in \mathscr{D}'^{+}_{p,p}(\Omega)$ a clsoed positive current on Ω of bidimension (p, p).

Definition 3.43. The unbounded locus $L(u)$ is defined as:

$$L(u) \coloneqq \{x \in \Omega \colon u \text{ is unbounded in every neighborhood of } x\} \subseteq u^{-1}(-\infty).$$

Proposition 3.44. *If*

1. *T has nonzero bidimension (p, p) (i.e. degree of $T < 2n$),*
2. *the boundary $\partial\Omega \cap L(u) \cap \operatorname{Supp} T = \varnothing$,*

then the current uT has locally finite mass in Ω.

Proof. See [6, pp. 150–152].
(1) is necessary: $T = \delta_0$, $u = \log|z|$. ■

Definition 3.45. Let $u_1, \ldots, u_q \in \operatorname{Psh}(\Omega)$, and $\partial\Omega \cap L(u_j) \cap \operatorname{Supp} T = \varnothing$. Define inductively

$$\mathrm{dd^c}\, u_1 \wedge \mathrm{dd^c}\, u_2 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T \coloneqq \mathrm{dd^c} \left(u_1\, \mathrm{dd^c}\, u_2 \wedge \ldots \wedge \mathrm{dd^c}\, u_1 \wedge T\right).$$

If $u_1^k, \ldots, u_q^k$ are decreasing sequence of psh functions with $u_j^k \to u_j$ pointwise, then Theorem 3.41 (1,2) hold.

Proof. See [6, p. 152]. ■

Remark 3.46. *[6, Proposition 4.4, p. 152] Moreover, we can define $V\, \mathrm{dd^c}\, u_1 \wedge \ldots \wedge \mathrm{dd^c}\, u_q$ for $V \in \operatorname{Psh}(\Omega)$, when $\partial\Omega \cap L(u_j) = \varnothing$ (V can be unbounded near $\partial\Omega$, but it is OK due to V is L^1_{loc}).*

Remark 3.47. *[6, Theorem 4.5, pp. 152-156] The currents $u_1\, \mathrm{dd^c}\, u_2 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T$ and $\mathrm{dd^c}\, u_1 \wedge \ldots \wedge \mathrm{dd^c}\, u_1 \wedge T$ are also well-defined and has locally finite mass in Ω if*

1. *$q \leq p$;*
2. *and the Hausdorff measure*

$$\mathscr{H}_{2p-2m+1}\left(L(u_{j_1}) \cap \ldots \cap L(u_{j_m}) \cap \operatorname{Supp} T\right) = 0$$

for all choices of indices $j_1 < \ldots < j_m$ in $\{1, \ldots, q\}$. Moreover, the analogue of the monotone convergence Theorem 3.41 (1,2) hold.

Remark 3.48. *[6, Proposition 4.12, pp. 156-157] Let $u_j = \log|f_j|$ for $f_j \in \mathscr{O}^*(\Omega)$, and $[Z_j] = \mathrm{dd^c}\, u_j$, $1 \leq j \leq q$. If the supports satisfy*

$$\operatorname{codim}(|Z_{j_1}| \cap \ldots |Z_{j_m}|) = m, \quad \forall\, m,$$

then it follows from the above remark that $[Z_1] \wedge \ldots \wedge [Z_q]$ is well-defined.

Moreover, if $(C_k)_{k \geq 1}$ is the irreducible components of the point set intersection $|Z_1| \cap \ldots \cap |Z_q|$, then there exists integers $m_k > 0$ s.t.

$$[Z_1] \wedge \ldots \wedge [Z_q] = \sum m_k [C_k].$$

The integer m_k is the multiplicity of intersection of $Z_1, \ldots, Z_q$ along the component C_k.

3.4 Lelong Number

3.4.1 Demailly's generalized Lelong number

Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and $\varphi\colon \Omega \to [-\infty, +\infty)$ a continuous (here means e^φ is continuous) psh function. Set

$$\begin{aligned} S(r) =& \{x \in \Omega\colon \varphi(x) = r\}, \\ B(r) =& \{x \in \Omega\colon \varphi(x) < r\}, \\ \overline{B}(r) =& \{x \in \Omega\colon \varphi(x) \leq r\}. \end{aligned}$$

DEFINITION 3.49. We say φ is semi-exhaustive if $\exists\, R$ s.t. $B(R) \Subset \Omega$; semi-exhaustive on $A \subset \Omega$ closed if $A \cap B(R) \Subset \Omega$, $\exists\, R$.

We now let $T \in \mathscr{D}'^{+}_{p,p}(\Omega)$, and φ semi-exhaustive on Ω. Assume $B(R) \cap \operatorname{Supp} T \Subset \Omega$.

DEFINITION 3.50. Set for $r \in (-\infty, R)$,

$$\begin{aligned} \nu(T, \varphi, r) &= \int_{B(r)} T \wedge (\mathrm{dd^c}\, \varphi)^p, \\ \nu(T, \varphi) &= \int_{S(-\infty)} T \wedge (\mathrm{dd^c}\, \varphi)^p = \lim_{r \to -\infty} \nu(T, \varphi, r). \end{aligned}$$

PROPOSITION 3.51. *For convex increasing $\chi\colon \mathbb{R} \to \mathbb{R}$,*

$$\int_{B(r)} T \wedge (\mathrm{dd^c}\, \chi \circ \varphi)^p = \chi'(r - 0)\nu(T, \varphi, r).$$

Proof. See [6, p. 158]. ∎

EXAMPLE 3.52.
- Let $\chi(r) = e^{2r}$:

$$\nu(T, \varphi, r) = e^{-2pr} \int_{B(r)} T \wedge (\mathrm{dd^c}\, e^{2\varphi})^p.$$

- Let $\varphi(z) = \log|z - a|$:

$$\mu(T, \varphi, \log r) = \frac{1}{r^{2p}} \int_{|z-a|<r} T \wedge (\mathrm{dd^c}\, |z|^2)^p = \frac{\sigma_T(B(a, r))}{\pi^p r^{2p}/p!}.$$

- Denote $\nu(T, a) = \nu(T, \log|z - a|)$, which is the (ordinary) Lelong number.
- When $T = 1 = [\Omega]$, $\varphi = \log|z - a|$, we have

$$\int_{B(a,r)} (\mathrm{dd^c} \log|z - a|)^n = \nu(T, \varphi) = \frac{|B(a, r)|}{\pi^n r^{2n}/n!} = 1.$$

Thus, $(\mathrm{dd^c} \log|z - a|)^n = \delta_a$.

- When $T = [A]$, and A is an analytic subset of pure dimension p,

$$\nu([A], x) = \begin{cases} 0 & x \notin A, \\ 1 & x \in A \text{ is a regular point.} \end{cases}$$

Since $\sigma_{[A]}$ is the volume measure on A near regular points.

PROPOSITION 3.53. *For every $c > 0$, the set $E_c = \{x \in \Omega \colon \nu(T, x) \geq c\}$ is a closed set of locally finite $\mathscr{H}_{2p}$ Hausdorff measure in Ω.*

Proof. See [6, p. 160]. ■

REMARK 3.54.

- *$T_k \to T$ weakly $\Rightarrow$*

$$\limsup_{k \to \infty} \nu(T_k, \varphi) \leq \nu(T, \varphi).$$

- $e^{\varphi_k} \to e^{\varphi}$ *compactly uniform $\Rightarrow$*

$$\limsup_{k \to \infty} \nu(T, \varphi_k) \leq \nu(T, \varphi).$$

Set $\varphi_{\geq r} = \max\{\varphi, r\}$. Then

$$(\mathrm{dd^c}\,\varphi_{\geq r})^n = \begin{cases} (\mathrm{dd^c}\,\varphi)^n & \text{on } \Omega \setminus \overline{B}(r), \\ 0 & \text{on } B(r). \end{cases}$$

DEFINITION 3.55. Define the measure

$$\mu_r = (\mathrm{dd^c}\,\varphi_{\geq r})^n - \mathbb{I}_{\Omega \setminus B(r)} (\mathrm{dd^c}\,\varphi)^n, \quad r \in (-\infty, R).$$

Then the measure μ_r is supported on $S(r)$ and $\mu_r(S(r)) = \mu_r(B(s)) = \int_{B(r)} (\mathrm{dd^c}\,\varphi)^n$ for $s \geq r$.

THEOREM 3.56 (Jensen-Lelong formula). *Let $V \in \mathrm{Psh}(\Omega)$. Then*

$$\mu_r(V) - \int_{B(r)} V (\mathrm{dd^c}\,\varphi)^n = \int_{-\infty}^{r} \nu(\mathrm{dd^c}\,V, \varphi, t)\,\mathrm{d}t.$$

Proof. See [6, pp. 163–164]. ■

For $r < r_0 < R$, we then have

$$\mu_r(V) - \mu_{r_0}(V) + \int_{B(r_0) \setminus B(r)} V (\mathrm{dd^c}\,\varphi)^n = \int_{r_0}^{r} \nu(\mathrm{dd^c}\,V, \varphi, t)\,\mathrm{d}t.$$

COROLLARY 3.57. *If $(\mathrm{dd^c}\,\varphi)^n = 0$ on $\Omega \setminus S(-\infty)$, then $r \longmapsto \mu_r(V)$ is convex increasing, and*

$$\mu(\mathrm{dd^c}\,V, \varphi) = \lim_{r \to -\infty} \frac{\mu_r(V)}{r}.$$

Proof. See [6, p. 164].
Because $\nu(\mathrm{dd^c}\,V, \varphi, t)$ is increasing and non-negative. ■

EXAMPLE 3.58. If $\varphi(z) = \log|z-a|$, then

$$\nu(\mathrm{dd^c}\, V, \log|z-a|, \log t) = \frac{(n-1)!}{\pi^{n-1}t^{2n-2}} \int_{B(a,t)} \frac{1}{2\pi}\Delta V.$$

Thus,

$$\nu(\mathrm{dd^c}\, V, a) = \lim_{r\to 0} \frac{\int_{S(a,r)} V \frac{\mathrm{d}S}{\lambda(S(a,r))}}{\log r} = \lim_{r\to 0} \frac{\sup_{S(a,r)} V}{\log r} = \sup\{\gamma \colon V \le \gamma \log|z-a| + O(1) \text{ near } a\},$$

by Harnack's inequality and the convexity of $\log r \longmapsto \sup_{S(a,r)V}$.

THEOREM 3.59. *Let $\varphi, \psi \in C(\Omega) \cap \mathrm{Psh}(\Omega)$, and semi-exhaustive on $\mathrm{Supp}\, T$. If*

$$\ell := \limsup \frac{\psi(x)}{\varphi(x)} < +\infty, \qquad \text{as } x \in \mathrm{Supp}\, T \text{ and } \varphi(x) \to -\infty,$$

then $\nu(T, \psi) \le \ell^p \nu(T, \varphi)$, and the equality holds if $\ell = \lim \psi/\varphi$.

Proof. See [6, p. 166].

Set $u_c = \max\{\psi - c, \varphi\}$ for c large enough. ■

REMARK 3.60.
- *This shows the Lelong numbers $\nu(T, x)$ are independent of the choice of local coordinates.*
- *For $\lambda_1, \dots, \lambda_n > 0$, it implies*

$$\left(ddc \log \max_{1\le j\le n} |z_j|^{\lambda_j}\right)^n = \left(ddc \log \sum_{1\le j\le n} |z_j|^{\lambda_j}\right)^n = \lambda_1 \cdots \lambda_n \delta_0.$$

- *The Kiselman number is*

$$\nu(T, x, \lambda) := \nu\left(T, \log \max_{1\le j\le n} |z_j - x_j|^{\lambda_j}\right),$$

and $\nu(T, x) = \nu(T, x, (1, \dots, 1))$.

THEOREM 3.61 (Thie 67'). *Let A be an analytic set of dimension p in $\mathbb{C}^n$. The **Local parametrization theorem** shows we can take coordinates $(z'; z'') = (z_1, \dots, z_d; z_{d+1}, \dots, z_n)$ near x such that (A, x) is contained in $|z''| \le C|z'|$, and the projection map $\pi \colon A \cap (\Delta' \times \Delta'') \to \Delta'$ is a ramified covering with m sheets. The $m \in \mathbb{Z}_{\ge 0}$ is called the multiplicity of A at x. Then*

$$\nu([A], x) = m.$$

Proof. See [6, p. 168]. ■

THEOREM 3.62. *Let $u_1, \dots, u_q$ and $v_1, \dots, v_q$ be psh functions s.t. each q-tuple satisfies the conditions in Definition 3.45 or Remark 3.47. Assume $u_j = -\infty$ on $\mathrm{Supp}\, T \cap \varphi^{-1}(-\infty)$ and*

$$\ell_j := \limsup \frac{v_j(z)}{u_j(z)} \quad \text{when} \quad z \in \mathrm{Supp}\, T \setminus u_j^{-1}(-\infty),\ \varphi(z) \to -\infty.$$

Then

$$\nu(\mathrm{dd^c}\, v_1 \wedge \ldots \wedge \mathrm{dd^c}\, v_q \wedge T, \varphi) \le \ell_1 \cdots \ell_q \nu(\mathrm{dd^c}\, u_1 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T, \varphi).$$

Proof. See [6, p. 169].

Set $w_{j,c} = \max\{v_j - c, u_j\}$ for c large. ■

COROLLARY 3.63. *If* $\mathrm{dd^c}\, u_1 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T$ *is well-defined, then at each* $x \in \Omega$,

$$\nu(\mathrm{dd^c}\, u_1 \wedge \ldots \wedge \mathrm{dd^c}\, u_q \wedge T, x) \geq \nu(\mathrm{dd^c}\, u_1, x) \cdots \nu(\mathrm{dd^c}\, u_q, x)\nu(T, x).$$

Proof. See [6, p. 169]. Take $\varphi(z) = v_1(z) = \ldots = v_q(z) = \log|z - x|$. ■

3.5 Siu's theorems

3.5.1 Siu's slicing theorem

THEOREM 3.64 (Siu's slicing theorem). *For almost* $S \in G(q, n)$ *with* $q \geq n-p$, *the slice* $T|_S$ *satisfies* $\nu(T|_S, 0) = \nu(T, 0)$, *where* $G(q, n)$ *is the Grassmannian of* q*-dimensional subspace in* $\mathbb{C}^n$.

Proof. See [6, pp. 172–173].

The slice of current is somehow a little complicated, but we can think about the psh functions.

One important ingredient in the proof is **Crofton's formula**:

$$\int_{S \in G(q,n)} [S]\, \mathrm{d}v(S) = (\mathrm{dd^c} \log|z|)^{n-q},$$

where $\mathrm{d}v$ is the unique $U(n)$-invariant measure of mass 1 on $G(q, n)$.

Think that ($p = n - 1, q = 1$ and $T = \mathrm{dd^c}\, \psi$ smooth)

$$\nu(\psi|_S, 0, r) = \int_{S \cap B(0,r)} T|_S = \int_{B(0,r)} T \wedge S,$$

so

$$\int_{S \in \mathbb{CP}^{n-1}} \nu(\psi|_S, 0, r)\, \mathrm{d}v(S) = \int_S \int_{B(0,r)} T \wedge [S] = \int_{B(0,r)} T \wedge (\mathrm{dd^c} \log|z|)^{n-1} = \nu(\psi, 0, r).$$

In this special case, we also have $\nu(\psi|_S, 0) \leq \nu(\psi, 0)$ for every S. ■

EXAMPLE 3.65. Let $\psi = \log \max_{i=1,2} |z_i|^{\lambda_i}$, where $0 < \lambda_1 < \lambda_2$. Then $\nu(\psi|_S, 0) = \lambda_2$ except for $S = \{z_2 = 0\}$.

3.5.2 Siu's semi-continuity theorem

Let $\varphi\colon \Omega \times D \to [-\infty, +\infty)$ be a continuous psh function, and $T \in \mathscr{D}'^{+}_{p,p}(\Omega)$. Assume for every compact subset $K \subset D$, there exists $R = R(K) < 0$ s.t.

$$\{(x, y) \in \operatorname{Supp} T \times K \colon \varphi(x, y) \leq R\} \Subset \Omega \times D.$$

For every $y \in D$, let $\varphi_y(x) = \varphi(x, y)$. Assume that φ is locally Hölder continuous w.r.t. y, i.e. locally we have

$$|\varphi(x, y_1) - \varphi(x, y_2)| \leq M|y_1 - y_2|^\gamma, \quad \gamma \in (0, 1].$$

THEOREM 3.66 (Demailly 87'). *The upperlevel sets*

$$E_c(T,\varphi) = \{y \in D \colon \nu(T,\varphi_y) \geq c\}$$

are analytic subsets of Y.

Proof. See [6, pp. 173–180].

The proof is an application of Kiselman's minimum principle and Corollary 2.7. The proof is complicated, but we will give a proof for the semi-continuity of relative types (another generalization of Lelong number) with the similar methods later. ■

COROLLARY 3.67 (Siu's semi-continuity theorem). *If $T \in \mathscr{D}'^{+}_{p,p}(\Omega)$, then*

$$E_c(T) = \{x \in \Omega \colon \nu(T,x) \geq c\}$$

of Lelong numbers are analytic subsets of dimension $\leq p$.

Proof. Let $\varphi(x,y) = \log|x-y|$. ■

3.5.3 Siu's decomposition theorem

THEOREM 3.68. *If $T \in \mathscr{D}'^{+}_{p,p}(\Omega)$, there is a unique decomposition of T as a (finite or infinite) weakly convergent series*

$$T = \sum_{j\geq 1} \lambda_j[A_j] + R, \quad \lambda_j > 0,$$

where A_j are irreducible p-dimensional analytic subset, and R is a closed positive current with the property that $\dim E_c(R) < p$ *for every $c > 0$.*

Proof. See [6, p. 181].

Existence: Let A_j be the p-dimensional irreducible components occuring in one of $E_c(T)$, $c \in \mathbb{Q}_+$, and $\lambda_j > 0$ be the generic Lelong number

$$\lambda_j \coloneqq \inf\{\nu(T,x) \colon x \in A_j\}.$$

The rest is to show $R_N = T - \sum_{1\leq j\leq N} \lambda_j[A_j]$ is positive. ■

EXAMPLE 3.69. Let $\psi \in \mathrm{Psh}^-(\Delta)$ in $\mathbb{C}$, and $\nu(\psi,0) = c \in (0,+\infty)$. Then $0 \in E_c(\psi)$. Since the 0-dimensional irreducible analytic subset must be a point, by Siu's decomposition, we have

$$0 \leq R \coloneqq T - c[0] = \mathrm{dd^c}\,\psi - c\,\mathrm{dd^c}\log|z| = \mathrm{dd^c}(\psi - c\log|z|),$$

that is, $\phi \coloneqq \psi - c\log|z| \in \mathrm{Psh}(\Delta)$. Note that $\phi \leq 0$ near $\partial\Delta$, so $\phi \in \mathrm{Psh}^{-1}(\Delta)$. Thus, $\psi \leq c\log|z|$ on Δ. From Siu's decomposition (or by definition), we can also find that $\nu(\phi,0) = 0$.

Now if $c \in [k,k+1)$ for $k \in \mathbb{Z}_{\geq 0}$, then

$$\int_{\Delta(r)} |f|^2 \mathrm{e}^{-2\psi} \approx \int_{\Delta(r)} |z|^{2m}\mathrm{e}^{-2c\log|z|-2\phi} = \int_{\Delta(r)} |z|^{2m-2c}\mathrm{e}^{-2\phi}, \quad m = \mathrm{ord}_0(f).$$

Since $\nu(\phi,0) = 0$, by Skoda's result, $\mathrm{e}^{-2\phi}$ is locally integrable. We can then conclude that $\mathscr{I}(\psi)_0 = (z^k)_0$.

3.6 Cegrell's class

Definition 3.70. A domain $\Omega \subset \mathbb{C}^n$ is said to be hyperconvex if there exists a bounded psh exhaustion function, i.e. $\exists\, \varphi \in \mathrm{Psh}^-(\Omega)$, s.t. $\{\varphi < -t\} \Subset \Omega$ for any $t < 0$.

In this "subsection", we will always assume that Ω is a bounded hyperconvex domain in $\mathbb{C}^n$.

Proposition 3.71. *Suppose $u \in \mathrm{Psh}^-(\Omega)$. Then there is a decreasing sequence of functions $u_j \in \mathrm{Psh}(\Omega) \cap C(\overline{\Omega})$ with $u_j|_{\partial\Omega} = 0$ for every $j \in \mathbb{Z}_{>0}$, $\lim\limits_{j\to\infty} u_j(z) = u(z)$ for every $z \in \Omega$, and $\int_\Omega (\mathrm{dd^c}\, u_j)^n < +\infty$.*

Proof. See [4, p. 3]. ■

Definition 3.72. $\mathcal{E}_0(\Omega)$ is the convex cone of bounded psh function u with $\lim\limits_{z\to\xi} u(z) = 0$ for any $z \in \partial\Omega$, and $\int_\Omega (\mathrm{dd^c}\, u)^n < +\infty$.

Remark 3.73. *For $u \in \mathrm{Psh}^-(\Omega) \setminus \{0\}$ with $\lim\limits_{z\to\xi} u(z) = 0$ for any $z \in \partial\Omega$, and T a closed positive current on Ω, the current $\mathrm{dd^c}\, u \wedge T$ is well-defined.*

Definition 3.74. Let $u \in \mathrm{Psh}^-(\Omega)$. We say $u \in \mathcal{E}(\Omega)$ if for every $z_0 \in \Omega$ there is a neighborhood ω of z_0 in Ω and a decreasing sequence $h_j \in \mathcal{E}_0(\Omega)$ s.t. $h_j \to u$ on ω and $\sup_j \int_\Omega (\mathrm{dd^c}\, h_j)^n < +\infty$.

For $u_1, \ldots, u_n \in \mathcal{E}(\Omega)$, define $\mathrm{dd^c}\, u_1 \wedge \mathrm{dd^c}\, u_2 \wedge \ldots \wedge \mathrm{dd^c}\, u_n$ to be the (weakly convergence) limit measure obtained by the decreasing sequences (independent of the choice in fact, see [4, pp. 7–8]).

Definition 3.75 (Cegrell's class). We denote by $\mathscr{F}(\Omega)$ the subclass of functions u in $\mathcal{E}(u)$ s.t. there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ s.t. $u_j \to u$ on Ω and $\sup_j \int_\Omega (\mathrm{dd^c}\, u_j)^n < +\infty$.

Remark 3.76.
- *The integration by parts is allowed in $\mathscr{F}(\Omega)$.*
- *Every $u \in \mathcal{E}(\Omega)$ is locally in $\mathscr{F}(\Omega)$.*
- *In the unit ball $\mathbb{B}$ of $\mathbb{C}^2$, $u = \log|z_2| \notin \mathcal{E}(\mathbb{B})$. Moreover, $-(-\log|z_2|)^\alpha \in \mathcal{E}(\mathbb{B})$ iff $0 < \alpha < 1/2$.*

Theorem 3.77. *Suppose $u_1, \ldots, u_n \in \mathscr{F}(\Omega)$ and $h \in \mathcal{E}_0(\Omega)$. Then*

$$\int_\Omega -h\, \mathrm{dd^c}\, u_1 \wedge \ldots \wedge \mathrm{dd^c}\, u_n \le \prod_{j=1}^n \left(\int_\Omega -h (\mathrm{dd^c}\, u_j)^n \right)^{1/n}.$$

Especially, we have

$$\int_\Omega \mathrm{dd^c}\, u_1 \wedge \ldots \wedge \mathrm{dd^c}\, u_n \le \prod_{j=1}^n \left(\int_\Omega (\mathrm{dd^c}\, u_j)^n \right)^{1/n},$$

and

$$\int_\Omega (\mathrm{dd^c}\, u_1)^j \wedge (\mathrm{dd^c}\, u_2)^{n-j} \le \left(\int_\Omega (\mathrm{dd^c}\, u_1)^n \right)^{j/n} \cdot \left(\int_\Omega (\mathrm{dd^c}\, u_2)^n \right)^{1-j/n}.$$

Proof. See [4, pp. 169–170]. ■

Corollary 3.78. *Suppose $u, v \in \mathscr{F}(\Omega)$. Then*

$$\left(\int_\Omega \big(\mathrm{dd^c} (u+v) \big)^n \right)^{1/n} \le \left(\int_\Omega \big(\mathrm{dd^c}\, u \big)^n \right)^{1/n} + \left(\int_\Omega \big(\mathrm{dd^c}\, v \big)^n \right)^{1/n}.$$

Proof. Actually we need to show $u + v \in \mathscr{F}(\Omega)$, but directly we compute

$$\begin{aligned}
\mathrm{MA}(u+v)^{1/n} &= \left(\int_\Omega \left(\mathrm{dd^c}\, u + \mathrm{dd^c}\, v\right)^n\right)^{1/n} \\
&= \left(\int_\Omega \sum_{j=1}^{n} (\mathrm{dd^c}\, u)^j \wedge (\mathrm{dd^c}\, v)^{n-j}\right)^{1/n} \\
&\le \left(\sum_{j=1}^{n} \int_\Omega \mathrm{MA}(u)^{j/n} \cdot \mathrm{MA}(v)^{1-j/n}\right) \\
&= \left(\left(\mathrm{MA}(u)^{1/n} + \mathrm{MA}(v)^{1/n}\right)^n\right)^{1/n} = \mathrm{MA}(u)^{1/n} + \mathrm{MA}(v)^{1/n},
\end{aligned}$$

where $\mathrm{MA}(u) = \mathrm{MA}_\Omega(u) = \int_\Omega (\mathrm{dd^c}\, u)^n$. ■

Corollary 3.79. *Suppose $u \in \mathcal{E}(\Omega)$, and $0 \in \Omega$. Then the (higher) Lelong numbers*

$$e_j(u) = \int_{\{0\}} (\mathrm{dd^c}\, u)^j \wedge (\mathrm{dd^c} \log |z|)^{n-j},$$

satisfy

1. $e_j(u)^2 \le e_{j-1}(u) e_{j+1}(u)$, *for* $1 \le j \le n-1$;
2. $e_j(u) \ge e_1(u)^j$, *for* $0 \le j \le n$;
3. $e_k(\varphi) \le e_j(u)^{(l-k)/(l-j)} e_l(u)^{(k-j)/(l-j)}$, *for* $0 \le j < k < l \le n$.

theorem 3.80. *Let $u, v \in L^\infty \cap \mathrm{Psh}(\Omega)$, and $\liminf_{z \to \partial\Omega} \big(u(z) - v(z)\big) \ge 0$. Then*

$$\int_{\{u<v\}} (\mathrm{dd^c}\, v)^n \le \int_{\{u<v\}} (\mathrm{dd^c}\, u)^n.$$

Especially, if $(\mathrm{dd^c}\, u)^n \le (\mathrm{dd^c}\, v)^n$, then $v \le u$ in Ω.

Proof. See [2, p. 10].
For $u, v \in C(\overline{\Omega})$ this case, take $v_\varepsilon = \max\{v, u + \varepsilon\}$. ■

3.7 Extremal plurisubharmonic function

We continue letting Ω bounded hyperconvex.

3.7.1 Maximal weight and relative type

Assume $0 \in \Omega$. Let $\varphi \in L^\infty_{\mathrm{loc}}(\Omega \setminus \{0\}) \cap \mathrm{Psh}^-(\Omega)$ which satisfies $\varphi(0) = -\infty$ and $(\mathrm{dd^c}\, \varphi)^n = 0$ on $\Omega \setminus \{0\}$, i.e. φ is maximal on $\Omega \setminus \{0\}$. We call φ a *maximal weight* near 0.

Definition 3.81 (Rashkovskii). For every psh germ u at 0, the relative type of u to φ is defined by
$$\sigma(u,\varphi) := \liminf_{z\to 0}\frac{u(z)}{\varphi(z)} = \sup\{c\geq 0\colon u\leq c\varphi + O(1) \text{ near } 0.\}$$

Proposition 3.82. *Assume $u\in \mathrm{Psh}(\Omega)$, and $u(0)=-\infty$. Then*
$$\Lambda(u,\varphi,r) := \sup_{z\in B(r)} u(z)$$
is convex and decreasing in $\log r$, where $B(r)=\{\varphi<r\}$.

Proof. See [16, p. 8]. ■

Corollary 3.83. *We have*
$$u\leq \sigma(u,\varphi)\varphi + O(1)$$
near 0.

Proof. See [16, p. 9]. ■

theorem 3.84. *Let $\Phi\colon \Omega\times\Omega\to[-\infty,0)$ be a negative psh function s.t.*

1. $\{x\colon \Phi(x,\zeta)=-\infty\}=\{\zeta\}$;
2. *for any $\zeta\in\Omega$ and $r<0$, there exists a neighborhood U of ζ s.t.*
$$\{(x,\zeta)\colon \Phi(x,\zeta)<r,\ \zeta\in U\}\Subset \Omega\times\Omega;$$
3. $(\mathrm{dd^c}\,\Phi_\zeta)^n=0$ *on $\Omega\setminus\{\zeta\}$ for every fixed ζ, where $\Phi_\zeta(x):=\Phi(x,\zeta)$;*
4. $\mathrm{e}^{\Phi(x,\zeta)}$ *is locally Hölder continuous in ζ, i.e. $\exists\beta>0$ s.t.*
$$\left|\mathrm{e}^{\Phi(x,\zeta_1)}-\mathrm{e}^{\Phi(x,\zeta_2)}\right|\leq C|\zeta_1-\zeta_2|^\beta.$$

Let $u\in\mathrm{Psh}(\Omega)$, and denote
$$\mathcal{S}_c(u,\Phi;\Omega) := \big\{\zeta\in\Omega\colon \sigma(u,\Phi_\zeta)\geq c\big\},\quad c>0.$$
Then $\mathcal{S}_c(u,\Phi;\Omega)$ is an analytic subset of Ω for each $c>0$.

Proof. See [16, pp. 22–23]. ■

Corollary 3.85. *Let φ be a maximal weight near 0 which is Hölder continuous. Then for every $u\in\mathrm{Psh}(\Omega)$ and $c>0$,*
$$E_c(u;\varphi) := \{z\in\Omega\colon \sigma_z(u,\tau_z\varphi)\geq c\}$$
is an analytic set.

Proof. Let $R>0$ such that $\varphi\in\mathrm{Psh}^-(B(0,2R))$, and let $\Phi(x,\zeta)=\varphi(x-\zeta)$, which is a negative psh function on $B(0,R)\times B(0,R)$. ■

3.7.2 Pluricomplex Green function

DEFINITION 3.86. A locally bounded psh function u is maximal if $(\mathrm{dd^c}\, u)^n = 0$.

In fact, it is the maximal psh function among those have same boundary values.

DEFINITION 3.87. For every $z_0 \in \Omega$, the function

$$G_\Omega(z, z_0) = \sup\left\{v(z)\colon v \in \mathrm{Psh}^-(\Omega),\ v(z) \le \log|z - z_0| + O(1) \text{ near } z_0\right\}$$

is the solution to the Dirichlet problem

$$\begin{cases} u \in \mathrm{Psh}(\Omega) \cap C(\overline{\Omega}) \\ (\mathrm{dd^c}\, u)^n = 0 \text{ on } \Omega \setminus \{z_0\} \\ u|_{\partial\Omega} = 0 \\ u(z) = \log|z - z_0| + O(1) \text{ near } z_0, \end{cases}$$

where $u \in C(\overline{\Omega})$ means $\mathrm{e}^u \in C(\overline{\Omega})$. We call $G_\Omega(\cdot, z_0)$ the pluricomplex Green function on Ω with a logarithm pole at z_0.

REMARK 3.88.
- *Actually we only need Ω bounded or hyperconvex to define the pluricomplex Green function, but we may have $G_\Omega \equiv -\infty$ when unbounded.*
- *$G_\Omega(z, w)$ is continuous on $\overline{\Omega} \times \Omega \setminus \{z = w\}$.*
- *If Ω is strongly pseudoconvex with $C^{2,1}$ boundary, then $G_\Omega(\cdot, z_0)$ is $C^{1,1}$ in $\overline{\Omega} \setminus \{z_0\}$.*
- *Even Ω is strongly pseudoconvex with C^∞ boundary, the pluricomplex Green function can be not C^2 up to boundary.*
- *When $n \ge 2$, in general, $G_\Omega(z, w)$ is not symmetric in z and w.*

DEFINITION 3.89. A bounded domain $D \subset \mathbb{C}^n$ is said to be strictly hyperconvex if there exists a bounded domain Ω and a function $\varrho\colon \Omega \to (-\infty, 1)$ such that $\varrho \in C(\Omega) \cap \mathrm{Psh}(\Omega)$, $D = \{z \in \Omega\colon \varrho(z) < 0\}$, ϱ is exhaustive for Ω and for any real number $c \in [0, 1]$, the open set $\{z \in \Omega\colon \varrho(z) < c\}$ is connected.

We have the following approximation theorem of pluricomplex Green function:

THEOREM 3.90. *Let D be a bounded strictly hyperconvex domain and $0 \in D$. For any $m \in \mathbb{N}_+$, define*

$$\mathscr{E}_m(D) \coloneqq \left\{f \in \mathscr{O}(D)\colon \sup_{z\in D}|f(z)| \le 1,\ \mathrm{ord}_0(f) \ge m - 1\right\},$$

and

$$\phi_m(z) \coloneqq \sup_{f\in\mathscr{E}_m(D)} \frac{1}{m}\log|f(z)|, \quad \forall z \in D.$$

Then ϕ_m converges to $G_\Omega(\,\cdot\,, 0)$ pointwise.

Proof. See [13].

Use Demailly's approximation theorem. ■

COROLLARY 3.91. *If $0 \in D$ is a bounded strictly hyperconvex domain, then*

$$G_\Omega(w, 0) = \sup\left\{\frac{1}{\nu(\log|f|, 0)}\log|f(w)|\colon f \in \mathscr{O}(D),\ \sup_D |f| \le 1,\ f(o) = 0,\ f \not\equiv 0\right\}.$$

3.7.3 Capacity function

4 Some applications

4.1 Asymptotic behavior of Bergman kernel

Suppose Ω is an open set in $\mathbb{C}^n$, and $\phi\colon \Omega \to \mathbb{R}$ is a C^2 psh function. Consider the weighted Bergman space $\mathscr{H}_k \coloneqq A^2(\Omega, \mathrm{e}^{-k\phi})$ and Let B_k be the Bergman kernel on the diagonal for $\mathscr{H}_k$.

Lemma 4.1. *We have that*

$$\limsup_{k\to\infty} \frac{1}{k^n} B_k(z)\mathrm{e}^{-k\phi(z)}\,\mathrm{d}\lambda(z) \le \frac{1}{\pi^n n!}(\mathrm{i}\partial\overline{\partial}\phi)^n(z).$$

Proof. Let $h \in \mathscr{H}_k$ with

$$\int_\Omega |h|^2 \mathrm{e}^{-k\phi} \le 1.$$

WLOG, say $z = 0$. We first assume ϕ has the Taylor expansion

$$\phi(\zeta) = \phi(0) + \sum \gamma_j|\zeta_j|^2 + o(|\zeta|^2),$$

where each $\gamma_j > 0$. Then for every $A < +\infty$,

$$1 \ge \int_{\sum \gamma_j|\zeta_j|^2 < A/k} |h|^2\mathrm{e}^{-k\phi}\,\mathrm{d}\lambda \ge (1-\varepsilon_k)\mathrm{e}^{-k\phi(0)} \int_{\gamma_j|\zeta_j|^2<A/k} |h|^2 \mathrm{e}^{-k\sum\gamma_j|\zeta_j|^2}\,\mathrm{d}\lambda,$$

where $\varepsilon_k \to 0$ as k tends to ∞. By the mean value inequality,

$$\begin{aligned}\int_{\gamma_j|\zeta_j|^2<A/k} |h|^2 \mathrm{e}^{-k\sum\gamma_j|\zeta_j|^2}\,\mathrm{d}\lambda &\ge |h(0)|^2 \int_{\gamma_j|\zeta_j|^2<A/k} \mathrm{e}^{-k\sum\gamma_j|\zeta_j|^2}\,\mathrm{d}\lambda \\ &= |h(0)|^2(\gamma_1\cdots\gamma_n)^{-1}\int_{B(0,A/k)} \mathrm{e}^{-k|\zeta|^2}\,\mathrm{d}\lambda \\ &= |h(0)|^2(\gamma_1\cdots\gamma_n k^n)^{-1}\pi^n(1-\delta_A),\end{aligned}$$

where $\delta_A \to 0$ as A tends to ∞. Hence,

$$|h(0)|^2\mathrm{e}^{-k\phi(0)}/k^n \le \pi^{-n}\gamma_1\cdots\gamma_n(1+2\delta_A)(1+2\varepsilon_k)$$

for A and k large. Taking the supremum over h of norm at most 1 we then get

$$B_k(0)\mathrm{e}^{-k\phi(0)}/k^n \le \pi^{-n}\gamma_1\cdots\gamma_n(1+2\delta_A)(1+2\varepsilon_k).$$

It follows that (e.g. taking $A = C\sqrt{k}$)

$$\limsup_{k\to\infty}\frac{1}{k^n}B_k(0)\mathrm{e}^{-k\phi(0)}\,\mathrm{d}\lambda \le \frac{1}{\pi^n}\gamma_1\cdots\gamma_n = \frac{1}{\pi^n n!}(\mathrm{i}\partial\overline{\partial}\phi)^n(0).$$

It is easy to see that this also holds when some of γ_j's are 0. A general ϕ has the Taylor expansion

$$\phi(\zeta) = \phi(0) + 2\mathrm{Re}\,(p(\zeta)) + q(\zeta,\bar\zeta) + o(|\zeta|^2).$$

We can reduce this case by change of variables and the substitution $h \mapsto h\mathrm{e}^{-p}$. ∎

THEOREM 4.2. *Additionally assume that D is bounded pseudoconvex. We have that*

$$\frac{1}{k^n}B_k(z)\mathrm{e}^{-k\phi(z)}\,\mathrm{d}\lambda(z)\to\frac{1}{\pi^n n!}(\mathrm{i}\partial\overline{\partial}\phi)^n(z)$$

pointwise.

Proof. We only need to show the reverse estimate. We may assume $z=0$, and near 0,

$$\phi(\zeta)=\phi(0)+|\zeta|^2+o(|\zeta|^2).$$

Let $\chi\in C_c^\infty$ be compactly supported in $B(0,1)$ and $\chi\equiv 1$ in $B(0,1/2)$ and δ_k be a sequence of positive numbers s.t. $\delta_k\to 0$ and $\tau_k:=\sqrt{k}\delta_k\to\infty$. Set

$$g_k(\zeta)=k^{n/2}\mathrm{e}^{k\phi(0)/2}\chi(\tau_k\zeta).$$

Then

$$\begin{aligned}\int_\Omega |g_k|^2\mathrm{e}^{-k\phi}\,\mathrm{d}\lambda &\le \int_\Omega k^n\mathrm{e}^{k\phi(0)}|\chi(\tau_k\zeta)|^2\mathrm{e}^{-k\phi}\,\mathrm{d}\lambda\\ &\le k^n\mathrm{e}^{k\phi(0)}\int_{|\zeta|^2<\tau_k^{-2}}\mathrm{e}^{-k\phi}\,\mathrm{d}\lambda\\ &\le \mathrm{e}^{k\phi(0)}\delta_k^{-2n}\int_{B(0,1)}\mathrm{e}^{-k\phi(\zeta/\tau_k)}\,\mathrm{d}\lambda\\ &\le (1+\epsilon_k)\mathrm{e}^{k\phi(0)}\delta_k^{-2n}\mathrm{e}^{-k\phi(0)}\int_{B(0,1)}\mathrm{e}^{-|\zeta/\delta_k|^2}\,\mathrm{d}\lambda\\ &\le (1+\epsilon_k)\int_{B(0,\delta_k^{-1})}\mathrm{e}^{-|\zeta|^2}\,\mathrm{d}\lambda\to\pi^n\quad(k\to\infty).\end{aligned}$$

We can also see

$$|\overline{\partial}g_k(z)|^2\le Ck^n\mathrm{e}^{k\phi(0)}\tau_k^2,$$

and $\overline{\partial}g_k=0$ except when $\tau_k^{-1}/2<|\zeta|<\tau_k^{-1}$. So we can use Hörmander's L^2 estimate to solve $\overline{\partial}u_k=\overline{\partial}g_k$ with

$$\begin{aligned}\int_\Omega |u_k(z)|^2\mathrm{e}^{-k\phi}\,\mathrm{d}\lambda &\le C\frac{1}{k}\int_{\tau_k^{-1}/2<|\zeta|<\tau_k^{-1}}k^n\mathrm{e}^{k\phi(0)}\tau_k^2\mathrm{e}^{-k\phi}\,\mathrm{d}\lambda\\ &\le Ck^n\delta_k^2\int_{\tau_k^{-1}/2<|\zeta|<\tau_k^{-1}}\mathrm{e}^{-k|\zeta|^2}\,\mathrm{d}\lambda\\ &\le C\delta_k^2\int_{\delta_k^{-1}/2<|\zeta|<\delta_k^{-1}}\mathrm{e}^{-|\zeta|^2}\,\mathrm{d}\lambda\to 0\quad(k\to\infty).\end{aligned}$$

Set $F_k=g_k-u_k$, which is holomorphic. Then $F_k(0)=g_k(0)=k^{n/2}\mathrm{e}^{k\phi(0)/2}$, and

$$\int_\Omega |F_k|^2\mathrm{e}^{-k\phi}\,\mathrm{d}\lambda\le\pi^n+o(1).$$

Thus,

$$\liminf_{k\to\infty}\frac{1}{k^n}B_k(z)\mathrm{e}^{-k\phi(0)}\ge\liminf_{k\to\infty}\frac{1}{k^n}\cdot\frac{k^n\mathrm{e}^{k\phi(0)}}{\pi^n}\mathrm{e}^{-k\phi(0)}=\frac{1}{\pi^n}.$$

The rest is similar as above. ∎

4.2 Spectrum of Toeplitz operator

Let Ω be a bounded pseudoconvex subset in $\mathbb{C}^n$ and φ a C^2 psh function on Ω. Let χ be a compactly suported bounded function on Ω. Define the Toeplitz concentraction operator $T_{\chi,\varphi}$ by

$$T_{\chi,\varphi}f(z) = \int_\Omega \chi(\zeta)f(\zeta)B_\varphi(z,\bar{\zeta})\mathrm{e}^{-\varphi(\zeta)},$$

where $B_\varphi(z,\bar{\zeta})$ is the off-diagonal Bergman kernel of $A^2(\Omega,\mathrm{e}^{-\varphi})$. It can be seen that $T_{\chi,\varphi}$ is of trace class, where

$$\mathrm{Tr}(T_{\chi,\varphi}) = \int_D \chi(z)B_\varphi(z,\bar{z})\mathrm{e}^{-\varphi(z)}$$

and

$$\mathrm{Tr}(T_{\chi,\varphi}^2) = \int_\Omega\int_\Omega |\chi(\zeta)B_\varphi(z,\bar{\zeta})|^2\mathrm{e}^{-\varphi(z)}\mathrm{e}^{-\varphi(\zeta)} = \int_\Omega |\chi(z)|^2 B_\varphi(z,\bar{z})\mathrm{e}^{-\varphi(z)}.$$

If f is an eigenfunction of $T_{\chi,\varphi}$ with eigenvalue λ, then

$$\lambda\|f\|_\varphi^2 = (T_{\chi,\varphi}f,f) = (\chi f, P_\varphi f) = (\chi f, f) = \int_\Omega \chi(z)|f(z)|^2\mathrm{e}^{-\varphi(z)}, \tag{4.6}$$

where P_φ is the Bergman projection.

Now let χ be a characteristic of a relatively compact open subset K, and denote the eigenvalues of $T_{\chi,\varphi}$ be $\lambda_m(D,\varphi) = \lambda_m(\chi,\varphi)$, ordered in a non-increasing sequence (counted with multiplicity)

$$\lambda_0(D,\varphi) \geq \lambda_1(D,\varphi) \geq \lambda_2(D,\varphi) \geq \cdots.$$

It is well-known [Weyl–Courant lemma]

$$\begin{aligned}\lambda_m(\chi,\varphi) &= \min_{\dim E = m-1}\ \max_{\substack{f\perp E\\ \|f\|_\varphi=1}} (T_{\chi,\varphi}f,f)\\ &= \max_{\dim F=m}\ \min_{\substack{f\in F\\ \|f\|_\varphi=1}} (T_{\chi,\varphi}f,f),\end{aligned}$$

where E, F are subspaces of $A^2(\Omega,\mathrm{e}^{-\varphi})$.

THEOREM 4.3. *Let D be a relatively compact smooth open subset of Ω. Then for any $\gamma\in(0,1)$ we have*

$$\lim_{k\to\infty} k^{-n}\sharp\big\{m\in\mathbb{Z}_{\geq 0}\colon \lambda_m(D,k\varphi) > \gamma\big\} = \frac{1}{\pi^n n!}\int_D (\mathrm{i}\partial\bar{\partial}\varphi)^n.$$

Proof. According to Theorem 4.2, we have

$$\frac{1}{k^n}\chi(z)B_{k\varphi}(z)\mathrm{e}^{-k\varphi(z)}\,\mathrm{d}\lambda(z) \to \frac{1}{\pi^n n!}\chi(z)(\mathrm{i}\partial\bar{\partial}\varphi)^n(z)$$

pointwise, which implies ($T_k = T_{\chi,k\varphi}$)

$$\lim_{k\to\infty} k^{-n}\,\mathrm{Tr}(T_k) = \lim_{k\to\infty} k^{-n}\,\mathrm{Tr}(T_k^2) = \frac{1}{\pi^n n!}\int_D (\mathrm{i}\partial\bar{\partial}\varphi)^n. \tag{4.7}$$

By definition, we have ($\lambda_m(k) = \lambda_m(T_{\chi,k\varphi})$)

$$\mathrm{Tr}(T_k) = \sum_m \lambda_m(k), \quad \mathrm{Tr}(T_k) = \sum_m \lambda_m(k)^2.$$

It follows from (4.6) that $\lambda_m(k) \le 1$. Note that

$$\lambda_1(k) = \sup \frac{(T_k f, f)}{\|f\|_{k\varphi}^2} \ge \frac{(T_k 1, 1)}{\|1\|_{k\varphi}^2} > 0.$$

Thus, we have

$$0 < \sum_m \lambda_m(k)^2 \le \sum_m \lambda_m(k).$$

By (4.7), for any $\delta > 0$, there exists k_δ s.t. when $k \ge k_\delta$,

$$1 - \delta \le \frac{\sum_m \lambda_m(k)}{\sum_m \lambda_m(k)} \le 1.$$

Now fix k_δ and for every $\beta \in (0,1)$ denote

$$S_\beta = \frac{\sum_{\lambda_m \le \beta} \lambda_m(k)}{\sum_m \lambda_m(k)}.$$

Then

$$\begin{aligned}
(1-\delta)\sum_m \lambda_m(k) &\le \sum_{\lambda_m > \gamma} \lambda_m(k)^2 + \sum_{\lambda_m \le \gamma} \lambda_m(k)^2 \\
&\le \sum_{\lambda_m > \gamma} \lambda_m(k) + \gamma \cdot \sum_{\lambda_m \le \gamma} \lambda_m(k) \\
&\le (1 - S_\gamma) \sum_m \lambda_m(k) + \gamma S_\gamma \sum_m \lambda_m(k),
\end{aligned}$$

which shows

$$S_\gamma \le \frac{\delta}{1-\gamma}.$$

Hence,

$$\sharp\{m \colon \lambda_m(k) > \gamma\} \ge \sum_{\lambda_m > \gamma} \lambda_m(k) \ge \left(1 - \frac{\delta}{1-\gamma}\right) \sum_m \lambda_m(k). \tag{4.8}$$

Note that for $\gamma < \gamma' < 1$ we have

$$\begin{aligned}
\sharp\{m \colon \lambda_m(k) > \gamma\} &= \sharp\{m \colon \lambda_m(k) > \gamma'\} + \sharp\{m \colon \gamma' \ge \lambda_m(k) > \gamma\} \\
&\le \frac{1}{\gamma'} \sum_{\lambda_m > \gamma'} \lambda_m(k) + \frac{1}{\gamma} \sum_{\gamma' \ge \lambda_m > \gamma} \lambda_m(k) \\
&\le \frac{1}{\gamma'} \sum_m \lambda_m(k) + \frac{1}{\gamma} S_{\gamma'} \sum_m \lambda_m(k).
\end{aligned} \tag{4.9}$$

Now combining (4.7), (4.8) and (4.9), we get

$$\liminf_{k\to\infty} \sharp\{m\colon \lambda_m(k) > \gamma\} \geq \left(1 - \frac{\delta}{1-\gamma}\right) \frac{1}{\pi^n n!} \int_D (\mathrm{i}\partial\overline{\partial}\varphi)^n,$$

and

$$\limsup_{k\to\infty} \sharp\{m\colon \lambda_m(k) > \gamma\} \leq \left(\frac{1}{\gamma'} + \frac{\delta}{\gamma(1-\gamma')}\right) \frac{1}{\pi^n n!} \int_D (\mathrm{i}\partial\overline{\partial}\varphi)^n.$$

The result follows since $\delta > 0$ and $\gamma' \in (\gamma, 1)$ are arbitrary. ■

4.3 Riemann–Roch; Holomorphic Morse Inequality

Lemma 4.4. *Let $\mathscr{H}$ be a subspace of $L^2(X,\mu)$ consisting of continuous functions, and denote by $K(x,y)$ the reproducing kernel of $\mathscr{H}$. Then*

$$\dim \mathscr{H} = \int_X K(x,x)\,\mathrm{d}\mu.$$

Proof. Let (h_j) be an orthonormal basis for $\mathscr{H}$. Then

$$\dim \mathscr{H} = \int_X \sum |h_j|^2\,\mathrm{d}\mu = \int_X K(x,x)\,\mathrm{d}\mu.$$

■

Proposition 4.5 (Bouche, Tian, ...). *Let $L \to X$ be a hermitian holomorphic line bundle over a compact complex manifold. Assume $h = \mathrm{e}^{-\phi}$ is a smooth metric on L with $\mathrm{i}\partial\overline{\partial}\phi \geq 0$ and positive somewhere. Let X_0 be the open subset where $\mathrm{i}\partial\overline{\partial}\phi$ is positive. Then*

$$\lim_{k\to\infty} \frac{1}{k^n} h^0(X, kL) = \frac{1}{\pi^n n!} \int_{X_0} (\mathrm{i}\partial\overline{\partial}\phi)^n.$$

Proof. Since h is smooth, by the above lemma,

$$h^0(X, kL) = \int_X B_k \mathrm{e}^{-k\phi}.$$

The proposition then follows from Theorem 4.2. ■

Now let X be a compact complex manifold, E a holomorphic vector bundle of rank r and L a line bundle over X. If L is equipped with a smooth metric h of curvature form $\Theta_{L,h}$, we define the q-index set of L to be the open subset $X(L,h,q)$ where $\mathrm{i}\Theta_{L,h}$ has exactly q negative eigenvalues and $(n-q)$ positive eigenvalues for $0 \leq q \leq n$. Denote

$$\Delta = \{x \in X\colon\ \det(\Theta_{L,h}(x)) = 0\}.$$

Then $X = \Delta \cup \bigcup_q X(L,h,q)$. Set

$$X(L,h,\leq q) = \bigcup_{0\leq j\leq q} X(L,h,j).$$

THEOREM 4.6 (Demailly, 85'; Holomorphic Morse Inequalities). *The cohomology groups $H^q(X, E \otimes \mathscr{O}(kL))$ satisfy the following asymptotic inequalities:*

1. *Weak Morse inequalities*

$$h^q\big(X, \mathscr{O}(E) \otimes \mathscr{O}(kL)\big) \leq r\frac{k^n}{n!}\int_{X(L,h,q)} (-1)^q \left(\frac{\mathrm{i}}{2\pi}\Theta_{L,h}\right)^n + o(k^n).$$

2. *Strong Morse inequalities*

$$\sum_{0\leq j\leq q} (-1)^{q-j} h^j\big(X, \mathscr{O}(E) \otimes \mathscr{O}(kL)\big) \leq r\frac{k^n}{n!}\int_{X(L,h,\leq q)} (-1)^q \left(\frac{\mathrm{i}}{2\pi}\Theta_{L,h}\right)^n + o(k^n).$$

5 Some open problems

5.1 Zero mass conjecture

5.2 Sublevel sets of pluricomplex Green function

References

[1] Bo Berndtsson and László Lempert. "A Proof of the Ohsawa–Takegoshi Theorem with Sharp Estimates". In: *Journal of the Mathematical Society of Japan* 68.4 (Oct. 2016), pp. 1461–1472. DOI: 10.2969/jmsj/06841461.

[2] Zbigniew Blocki. "Minicourse on Pluripotential Theory". 2012. URL: https://gamma.im.uj.edu.pl/~blocki/publ/ln/ln-vienna.pdf.

[3] Zbigniew Blocki. "The Complex Monge-Ampère Operator in Pluripotential Theory". 1997. URL: https://gamma.im.uj.edu.pl/~blocki/publ/ln/wykl.pdf.

[4] Urban Cegrell. "The General Definition of the Complex Monge-Ampère Operator". In: *Annales de l'institut Fourier* 54.1 (2004), pp. 159–179. ISSN: 0373-0956. DOI: 10.5802/aif.2014. (Visited on 12/26/2024).

[5] Jean-Pierre Demailly. *Analytic Methods in Algebraic Geometry*. 2nd ed. Surveys of Modern Mathematics 1. Somerville: International press, 2012. ISBN: 978-1-57146-234-3. URL: https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/analmeth_book.pdf.

[6] Jean-Pierre Demailly. *Complex Analytic and Differential Geometry*. URL: https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf.

[7] Qi'an Guan. "A Sharp Effectiveness Result of Demailly's Strong Openness Conjecture". In: *Advances in Mathematics* 348 (2019/05/25/), pp. 51–80. DOI: 10.1016/j.aim.2019.03.017.

[8] Qi'an Guan and Zhitong Mi. "Concavity of Minimal L^2 Integrals Related to Multiplier Ideal Sheaves on Weakly Pseudoconvex Kähler Manifolds". In: *Science China Mathematics* 65.5 (May 2022), pp. 887–932. ISSN: 1674-7283, 1869-1862. DOI: 10.1007/s11425-021-1930-2.

[9] Qi'an Guan and Xiangyu Zhou. "A Proof of Demailly's Strong Openness Conjecture". In: *Annals of Mathematics* 182.2 (2015), pp. 605–616. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2015.182.2.5.

[10] Qi'an Guan and Xiangyu Zhou. "A Solution of an L^2 Extension Problem with an Optimal Estimate and Applications". In: *Annals of Mathematics* 181.3 (2015), pp. 1139–1208. ISSN: 0003-486X,1939-8980. DOI: 10.4007/annals.2015.181.3.6.

[11] Qi'an Guan and Xiangyu Zhou. "Effectiveness of Demailly's Strong Openness Conjecture and Related Problems". In: *Inventiones mathematicae* 202.2 (Nov. 2015), pp. 635–676. ISSN: 0020-9910, 1432-1297. DOI: 10.1007/s00222-014-0575-3.

[12] Jeffery D. McNeal and Mircea Mustaţă. *Analytic and Algebraic Geometry: Common Problems, Different Methods*. IAS/Park City Mathematics Series Volume 17. Providence (R.I.): American mathematical society, 2010. ISBN: 978-0-8218-4908-8.

[13] Stéphanie Nivoche. "The Pluricomplex Green Function, Capacitative Notions, and Approximation Problems in $\mathbb{C}^n$". In: *Indiana University Mathematics Journal* 44.2 (1995), pp. 489–510. ISSN: 0022-2518,1943-5258. DOI: 10.1512/iumj.1995.44.1998.

[14] Takeo Ohsawa. *Analysis of Several Complex Variables*. Trans. by Shu Gilbert Nakamura. Translations of Mathematical Monographs Volume 211. Providence, Rhode Island: American Mathematical Society, 2002. ISBN: 978-0-8218-2098-8 978-1-4704-4636-9.

[15] Takeo Ohsawa. *L^2 Approaches in Several Complex Variables: Towards the Oka–Cartan Theory with Precise Bounds*. Springer Monographs in Mathematics. Tokyo: Springer Japan, 2018. ISBN: 978-4-431-56851-3 978-4-431-56852-0. DOI: 10.1007/978-4-431-56852-0.

[16] Alexander Rashkovskii. “Relative Types and Extremal Problems for Plurisubharmonic Functions”. In: *International Mathematics Research Notices* (Jan. 2006), p. 76283. ISSN: 1073-7928, 1687-0247. DOI: 10.1155/IMRN/2006/76283.

[17] Walter Rudin. *Functional Analysis*. 2. ed., [Nachdr.] International Series in Pure and Applied Mathematics. Boston, Mass.: McGraw-Hill, 2007. ISBN: 978-0-07-054236-5.