A ξ-Bergman kernel proof of the concavity of minimal L^2 integrals

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Abstract

Let D be a pseudoconvex domain containing the origin, let $\varphi < 0$ be plurisubharmonic with $\varphi(0) = -\infty$, and let

$$G(t) = C_{F,\mathcal{I}(\varphi)_0}(\{\varphi < -t\} \cap D)$$

be the associated minimal L^2 integral. Guan proved that $r \mapsto G(-\log r)$ is concave. A question in the theory of ξ-Bergman kernels asks whether this theorem can be recovered from the log-convexity of the ξ-Bergman kernels together with the duality identity $B = C$.

We give such a proof. The extra ingredient that makes the argument work is the log-plurisubharmonicity theorem for fiberwise ξ-Bergman kernels with holomorphically varying functionals. On every finite-dimensional space of functionals it yields a curvature inequality for the matrix of squared dual norms. The asymptotic formula for the ξ-complex singularity exponent places the logarithmic velocity of this matrix between 0 and the identity. An elementary inverse-matrix calculation then proves Loewner concavity after the change of variables $r = \mathrm{e}^{-t}$. Finally, the finite-dimensional spaces form an upward directed family, so their scalar extremal functions converge to the full supremum without the usual obstruction that arbitrary suprema do not preserve concavity. We include the regularization needed for nonsmooth matrix metrics and an exhaustion argument for unbounded domains.

Contents

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1 Introduction

Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing the origin 0, let I be an ideal of $\mathcal{O}_0$, and let $(F, 0) \in \mathcal{O}_0$. The minimal L^2 integral is

$$C_{F,I}(D) := \inf \left\{ \int_D |\widetilde{F}|^2 : \ \widetilde{F} \in \mathcal{O}(D), \ (\widetilde{F} - F, 0) \in I \right\}.$$

If $\varphi < 0$ is plurisubharmonic on D and $\varphi(0) = -\infty$, set

$$D_t := \{\varphi < -t\} \cap D, \qquad I_\varphi := \mathcal{I}(\varphi)_0, \qquad G(t) := C_{F,I_\varphi}(D_t).$$

Guan's concavity theorem [5] states that, whenever $G(0) < +\infty$, the function

$$(0, 1] \ni r \longmapsto G(-\log r)$$

is concave.

The ξ-Bergman kernel gives a dual description of this extremal problem. For a functional ξ on holomorphic germs, let

$$K_{\xi,\Omega}(0) := \sup_{f \in A^2(\Omega) \setminus \{0\}} \frac{|(\xi \cdot f)(0)|^2}{\int_\Omega |f|^2}.$$

If ℓ_I denotes the nonzero functionals annihilating I, define

$$B(F, I, \Omega) := \sup_{\xi \in \ell_I} \frac{|(\xi \cdot F)(0)|^2}{K_{\xi,\Omega}(0)}.$$

Bao–Guan proved the identity

$$B(F, I, \Omega) = C_{F,I}(\Omega) \tag{1.1}$$

for pseudoconvex Ω [2]. On the other hand, for every fixed ξ the function

$$t \longmapsto \log K_{\xi,D_t}(0)$$

is convex [1]. This led to the question whether Guan's theorem can be deduced from these kernel-theoretic facts.

The pointwise supremum in (1.1) is a genuine obstacle: an arbitrary supremum of concave functions need not be concave. The purpose of this note is to show that the additional structure of the whole family of ξ-Bergman kernels overcomes the obstacle.

THEOREM 1.1. *Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain containing 0, let $\varphi < 0$ be plurisubharmonic on D with $\varphi(0) = -\infty$, and let $(F, 0) \in \mathcal{O}_0$. If*

$$G(0) = C_{F,\mathcal{I}(\varphi)_0}(D) < +\infty,$$

then $r \mapsto G(-\log r)$ is concave on $(0, 1]$.

More precisely, the conclusion follows from the following three kernel-theoretic results:

(i) *log-plurisubharmonicity of the fiberwise ξ-Bergman kernel for holomorphically varying functionals [3];*

(ii) *the formula for the ξ-complex singularity exponent [4];*

(iii) *the identity $B = C$ in (1.1) [2].*

The proof is organized as follows. In Section 2 we recall the precise kernel results. In Section 3 we prove an abstract finite-dimensional matrix lemma, including the nonsmooth case. In Section 4 we apply it to finite-dimensional spaces of functionals and pass to their upward directed supremum. In Section 5 we remove boundedness by exhaustion.

2 Kernel-theoretic preliminaries

Let

$$\ell_1^{(n)} := \left\{ \xi = (\xi_\alpha)_{\alpha \in \mathbb{N}^n} : \sum_\alpha |\xi_\alpha| \rho^{|\alpha|} < +\infty \text{ for every } \rho > 0 \right\}.$$

For a germ $f(z) = \sum_\alpha a_\alpha z^\alpha$ at the origin, put

$$(\xi \cdot f)(0) := \sum_\alpha \xi_\alpha a_\alpha.$$

For an ideal $I \subset \mathcal{O}_0$, we use the vector space

$$\mathscr{L}_I := \{\xi \in \ell_1^{(n)} : (\xi \cdot f)(0) = 0 \text{ for every } f \in I\}.$$

Thus $\ell_I = \mathscr{L}_I \setminus \{0\}$.

We shall use three known facts. First, the norm of a holomorphically varying functional has semipositive logarithmic curvature. We state only the form needed here.

THEOREM 2.1 (Bao–Guan–Yuan [3]). *Let $\Omega \subset \mathbb{C}^n_z \times \mathbb{C}_\tau$ be pseudoconvex and let Ω_τ be its fibers. Suppose that $\tau \mapsto \xi(\tau) \in \ell_1^{(n)}$ is holomorphic and locally uniformly bounded in the sense of [3]. Then*

$$(z, \tau) \longmapsto \log K_{\xi(\tau), \Omega_\tau}(z)$$

is plurisubharmonic wherever it is defined.

Every holomorphic family contained in the span of finitely many fixed elements of $\ell_1^{(n)}$ satisfies the locally uniform boundedness assumption. We apply Theorem 2.1 to

$$\Omega_\varphi := \{(z, \tau) \in D \times \mathbb{C} : \varphi(z) + \operatorname{Re} \tau < 0\}. \tag{2.1}$$

This domain is pseudoconvex and its fiber over τ is $D_{\operatorname{Re} \tau}$.

Second, when D is bounded, the asymptotic slope of a fixed kernel is a local invariant.

THEOREM 2.2 (Bao–Guan–Yuan [4]). *Let D be bounded and pseudoconvex. For every $\xi \in \ell_1^{(n)} \setminus \{0\}$,*

$$\gamma_\xi(\varphi) := \lim_{t \to +\infty} \frac{\log K_{\xi, D_t}(0)}{t} = \inf\{c \geq 0 : \xi \in \mathscr{L}_{\mathcal{I}(c\varphi)_0}\}. \tag{2.2}$$

In particular,

$$\xi \in \mathscr{L}_{I_\varphi} \setminus \{0\} \quad \Longrightarrow \quad \gamma_\xi(\varphi) \leq 1. \tag{2.3}$$

Finally, we use the duality theorem already mentioned in the introduction.

THEOREM 2.3 (Bao–Guan [2]). *For every pseudoconvex domain Ω containing 0, every ideal $I \subset \mathcal{O}_0$, and every $(F,0) \in \mathcal{O}_0$,*

$$C_{F,I}(\Omega) = \sup_{\xi \in \mathscr{L}_I \setminus \{0\}} \frac{|(\xi \cdot F)(0)|^2}{K_{\xi,\Omega}(0)}. \tag{2.4}$$

3 A matrix concavity lemma

For Hermitian matrices, all inequalities below are in the Loewner order. The following lemma is the main elementary ingredient.

LEMMA 3.1. *Let $A : (0,+\infty) \to \mathrm{Herm}_m^{++}$ be continuous. Assume:*

(a) *for every holomorphic map $c : S \to \mathbb{C}^m$, where S is any sufficiently small strip in $\mathbb{C}$, the function*

$$\tau \longmapsto \log\bigl(c(\tau)^* A(\operatorname{Re}\tau) c(\tau)\bigr)$$

is subharmonic, with the value $-\infty$ at zeros of c;

(b) *$A(t)$ is nondecreasing;*

(c) *for every $v \neq 0$,*

$$\limsup_{t\to+\infty} \frac{\log(v^* A(t) v)}{t} \le 1.$$

Then

$$(0,1) \ni r \longmapsto A(-\log r)^{-1}$$

is concave in the Loewner order.

Proof. We divide the proof into four steps.

Step 1: the first derivative bounds. For $v \neq 0$, set

$$q_v(t) := v^* A(t) v, \qquad u_v(t) := \log q_v(t).$$

Assumption (a), applied to the constant map $c \equiv v$, says that u_v is convex. Assumption (b) says that it is nondecreasing. For a convex function, every right derivative is bounded above by its asymptotic slope. Hence

$$0 \le u'_{v,+}(t) \le 1. \tag{3.1}$$

In particular, q_v is locally Lipschitz and

$$0 \le q'_v(t) = q_v(t) u'_v(t) \le q_v(t) \quad \text{for almost every } t. \tag{3.2}$$

Polarization shows that all entries of A are locally Lipschitz. Taking a countable dense set of vectors v in (3.2), and then using continuity in v, gives

$$0 \le A'(t) \le A(t) \quad \text{for almost every } t. \tag{3.3}$$

Step 2: regularization preserves assumption (a). Fix a compact interval $J \Subset (0,+\infty)$ and let ρ_ε be a nonnegative one-dimensional mollifier with support small enough that

$$A_\varepsilon(t) := \int_{\mathbb{R}} \rho_\varepsilon(s) A(t-s)\, ds, \qquad t \in J,$$

is defined. Then A_ε is smooth and positive definite. Moreover, (3.3) gives

$$0 \le A_\varepsilon'(t) \le A_\varepsilon(t). \tag{3.4}$$

We claim that A_ε still satisfies (a). Indeed, for every holomorphic $c(\tau)$,

$$c(\tau)^* A_\varepsilon(\operatorname{Re}\tau)c(\tau) = \int_{\mathbb{R}} \rho_\varepsilon(s)c(\tau)^* A(\operatorname{Re}\tau - s)c(\tau)\, ds. \tag{3.5}$$

For fixed s, the logarithm of the integrand is subharmonic. The logarithm of a finite positive sum of exponentials of subharmonic functions is subharmonic, because the log-sum-exp function is convex and nondecreasing in each variable. Approximating the integral in (3.5) by Riemann sums, locally uniformly away from the zeros of c, proves the claim; the extension across those zeros follows by upper semicontinuity.

Step 3: curvature and inversion. Write $B = A_\varepsilon$. Applying the subharmonicity from Step 2 to affine holomorphic vector functions, and choosing their first derivative so that the first derivative of the squared norm vanishes at a prescribed point, gives

$$B'' \ge B'B^{-1}B'. \tag{3.6}$$

For completeness, at a point τ_0 with $t_0 = \operatorname{Re}\tau_0$ and for $v \ne 0$, take

$$c(\tau) = v - \frac{\tau - \tau_0}{2} B(t_0)^{-1}B'(t_0)v.$$

A direct differentiation yields

$$\partial_\tau \partial_{\bar\tau} \log\bigl(c(\tau)^* B(\operatorname{Re}\tau)c(\tau)\bigr)\big|_{\tau=\tau_0} = \frac{v^*\bigl(B'' - B'B^{-1}B'\bigr)v}{4v^*Bv},$$

which proves (3.6).

Let $H = B^{-1}$. Then

$$H' = -HB'H, \qquad H'' = 2HB'HB'H - HB''H.$$

Using (3.6),

$$H'' + H' \le HB'HB'H - HB'H. \tag{3.7}$$

Put $X = H^{1/2}B'H^{1/2}$. By (3.4), $0 \le X \le \mathrm{I}$, and therefore

$$H'' + H' \le H^{1/2}(X^2 - X)H^{1/2} \le 0. \tag{3.8}$$

For $t = -\log r$,

$$\frac{d^2}{dr^2} H(-\log r) = \frac{H''(t) + H'(t)}{r^2} \le 0. \tag{3.9}$$

Thus $r \mapsto A_\varepsilon(-\log r)^{-1}$ is Loewner concave on every interval corresponding to J.

Step 4: removal of the regularization. Since A is continuous, $A_\varepsilon \to A$ locally uniformly. Positive definiteness then gives $A_\varepsilon^{-1} \to A^{-1}$ locally uniformly. Passing to the limit in the defining three-point inequality for concavity proves the conclusion. ■

Remark 3.2. *Assumption (a) is stronger than convexity of $t \mapsto \log(v^*A(t)v)$ for constant vectors v. The difference is exactly the matrix curvature term $A'A^{-1}A'$ in (3.6). Thus the result for holomorphically varying functionals, rather than only the fixed functional version, is essential to the proof.*

4 The bounded-domain case

In this section D is bounded. Put $I = I_\varphi$ and $\mathscr{L} = \mathscr{L}_I$.

Let $E \subset \mathscr{L}$ be a finite-dimensional vector subspace, and choose a basis $\xi_1, \dots, \xi_m$. Since the ξ-Bergman kernel is the squared dual norm of the functional $f \mapsto (\xi \cdot f)(0)$, there is a unique positive definite Hermitian matrix $A_E(t)$ such that

$$K_{\sum_{j=1}^m c_j \xi_j, D_t}(0) = c^* A_E(t) c, \qquad c \in \mathbb{C}^m. \tag{4.1}$$

The matrix is positive definite because a nonzero element of $\ell_1^{(n)}$ does not vanish on all polynomials. Its entries are continuous: for each constant $c \neq 0$, the logarithm of the right-hand side of (4.1) is a finite convex function of t, and polarization then gives continuity of every matrix entry.

Proposition 4.1. *For every finite-dimensional $E \subset \mathscr{L}$, the function*

$$\mathcal{G}_E(r) := \sup_{\xi \in E \setminus \{0\}} \frac{|(\xi \cdot F)(0)|^2}{K_{\xi, D_{-\log r}}(0)}, \qquad 0 < r < 1,$$

is concave.

Proof. We verify the assumptions of Theorem 3.1 for A_E.

If $c(\tau)$ is holomorphic, then

$$\xi(\tau) := \sum_{j=1}^m c_j(\tau) \xi_j$$

is a holomorphic, locally uniformly bounded family in $\ell_1^{(n)}$. Applying Theorem 2.1 to (2.1) gives assumption (a).

If $t_2 > t_1$, then $D_{t_2} \subset D_{t_1}$. Domain monotonicity of the Bergman kernel gives

$$K_{\xi, D_{t_2}}(0) \geq K_{\xi, D_{t_1}}(0)$$

for every ξ, hence $A_E(t_2) \geq A_E(t_1)$. This is assumption (b).

Finally, every nonzero $\xi \in E$ belongs to $\mathscr{L}_{I_\varphi}$. By (2.3),

$$\lim_{t \to +\infty} \frac{\log K_{\xi, D_t}(0)}{t} \leq 1.$$

Thus assumption (c) holds. The matrix lemma shows that

$$r \longmapsto A_E(-\log r)^{-1}$$

is Loewner concave.

Let $b \in \mathbb{C}^m$ represent the linear functional $\xi \mapsto (\xi \cdot F)(0)$ in the chosen basis. The elementary dual-norm formula gives

$$\mathcal{G}_E(r) = b^* A_E(-\log r)^{-1} b. \tag{4.2}$$

The right-hand side is concave by Loewner concavity. ∎

The remaining point is to pass from finite-dimensional subspaces to the full space $\mathscr{L}$. The following elementary observation is the mechanism that avoids the usual supremum obstruction.

Lemma 4.2. *Let $\{h_\alpha\}$ be an upward directed family of concave real-valued functions on an interval: for every α, β there is γ such that $h_\gamma \geq h_\alpha$ and $h_\gamma \geq h_\beta$ pointwise. Then $h := \sup_\alpha h_\alpha$ is concave, provided it is finite.*

Proof. Fix r_0, r_1, $0 < \lambda < 1$, and $\varepsilon > 0$. Choose α and β such that

$$h_\alpha(r_0) > h(r_0) - \varepsilon, \qquad h_\beta(r_1) > h(r_1) - \varepsilon.$$

Choose γ dominating both. Concavity of h_γ gives

$$\begin{aligned} h(\lambda r_0 + (1-\lambda) r_1) &\geq h_\gamma(\lambda r_0 + (1-\lambda) r_1) \\ &\geq \lambda h_\gamma(r_0) + (1-\lambda) h_\gamma(r_1) \\ &\geq \lambda h(r_0) + (1-\lambda) h(r_1) - \varepsilon. \end{aligned}$$

Letting $\varepsilon \downarrow 0$ proves the assertion. ■

Proof of Theorem 1.1 for bounded D. For finite-dimensional $E \subset \mathscr{L}$, the functions $\mathcal{G}_E$ from Theorem 4.1 form an upward directed family: given E_1, E_2, the space $E_1 + E_2$ contains both and therefore

$$\mathcal{G}_{E_1+E_2} \geq \mathcal{G}_{E_j}, \qquad j = 1, 2.$$

Moreover, every $\xi \in \mathscr{L}$ belongs to a one-dimensional subspace, so

$$\sup_{\substack{E \subset \mathscr{L} \\ \dim E < \infty}} \mathcal{G}_E(r) = \sup_{\xi \in \mathscr{L} \setminus \{0\}} \frac{|(\xi \cdot F)(0)|^2}{K_{\xi, D_{-\log r}}(0)}. \tag{4.3}$$

By Theorem 2.3, the right-hand side is $G(-\log r)$. It is finite because $G(0) < +\infty$ and restriction of an admissible function from D to D_t shows $G(t) \leq G(0)$. The conclusion now follows from Theorem 4.2. Finally, $D_t \nearrow D$ as $t \downarrow 0$, and the standard exhaustion property of minimal integrals gives $G(t) \nearrow G(0)$. Hence the concavity extends from $(0,1)$ to $(0,1]$. ■

5 Exhaustion of an unbounded domain

We now remove the boundedness assumption. Choose bounded pseudoconvex domains

$$D^{(1)} \Subset D^{(2)} \Subset \cdots \Subset D, \qquad \bigcup_{j=1}^{\infty} D^{(j)} = D,$$

all containing a fixed neighborhood of the origin, and put

$$D_t^{(j)} := D^{(j)} \cap D_t, \qquad G_j(t) := C_{F, I_\varphi}(D_t^{(j)}).$$

The bounded-domain case shows that $G_j(-\log r)$ is concave.

We recall the standard exhaustion property

$$G_j(t) \nearrow G(t) \qquad (j \to \infty). \tag{5.1}$$

For completeness, monotonicity of the minimal norm gives existence of a limit $L \leq G(t)$. Choose functions on $D_t^{(j)}$ whose squared norms tend to $G_j(t)$. Their restrictions to each fixed $D_t^{(k)}$ have uniformly bounded L^2 norms. A diagonal normal-family argument gives a holomorphic limit on D_t. The condition that its germ differs from F by an element of I_φ is preserved under local uniform convergence, since a finitely generated ideal in $\mathcal{O}_0$ is closed in the germ topology. Fatou's lemma then gives an admissible function on D_t with squared norm at most L. Thus $G(t) \leq L$, proving (5.1).

Passing to the limit in the concavity inequality for $G_j(-\log r)$ proves Theorem 1.1 for arbitrary pseudoconvex D.

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References

[1] S. J. Bao and Q. A. Guan. L^2 extension and effectiveness of strong openness property. Acta Math. Sin. (Engl. Ser.), 38(11):1949–1964, 2022.

[2] S. J. Bao and Q. A. Guan. Equivalence of the sharp effectiveness results of strong openness property. arXiv:2408.16372, 2024.

[3] S. J. Bao, Q. A. Guan, and Z. Yuan. The log-plurisubharmonicity of fiberwise ξ-Bergman kernels for variant functionals. arXiv:2303.16525, 2023.

[4] S. J. Bao, Q. A. Guan, and Z. Yuan. A note on ξ-Bergman kernels. Front. Math., 20:481–506, 2025. arXiv:2212.01561.

[5] Q. A. Guan. A sharp effectiveness result of Demailly's strong openness conjecture. Adv. Math., 348:51–80, 2019. doi:10.1016/j.aim.2019.03.017.